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ON DIVISIBILITY PROPERTIES OF BRAIDS ASSOCIATED WITH ALGEBRAIC CURVES.

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0. Introduction.

Let $f(x,y)$ be a complex polynomial in two variables and let C be the curve in $\mathbf{C}^2$ given by $f(x,y) = 0$. Let us assume that C has an isolated singularity at the origin 0, i.e. 0 is an isolated solution of $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = f(x,y) = 0$. Let B_ε be a ball of sufficiently small radius ε about 0. The intersection of C with the boundary ∂B_ε of B_ε is a link in the 3-sphere ∂B_ε. Links which can be obtained by such procedure are known as algebraic links. Topologically algebraic links can be characterized as iterated torus links for which parameters describing torus links which are iterated satisfy certain inequalities (cf. for example [EN]). Using this R. Williams was able to derive the following beautiful theorem.

THEOREM 1 (R. Williams). Any algebraic link can be represented as a closed braid which is product of a positive braid and the positive generator Δ^2 of the center of the braid group. The number of strands in this braid is equal to the multiplicity of C at the origin (i.e. to the minimal degree of monomials in $f(x,y)$).

This theorem combined with the results of J. Franks and R. Williams ([FW]) and H. Morton ([Mor]) also implies that the braid index of an algebraic link is equal to the multiplicity of the singularity of defining polynomial $f(x,y)$.

R. Williams suggested that there should be an algebro-geometric proof of the theorem 1. The purpose of this note is to provide such a proof.

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To describe our second result let us consider a curve D of degree n in $\mathbf{CP}^2$ and let L be a line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ which we shall call the line in infinity. Let S be the boundary of a sufficiently small tubular neighborhood of L. The intersection of D and S is a link in the 3-sphere S which is called the link in infinity (cf. [L1] and [NR]). This link is naturally a closed braid on n strands.

THEOREM 2. The braid corresponding to the link in infinity associated with an algebraic curve is a divisor of the positive generator Δ^2 of the center of the braid group B_n on n strands.

In the case when D is transversal to L the link in infinity is the closed braid corresponding to Δ^2 (cf. [Ch], [Mo1] or survey [L2]).

I would like to thank R. Williams for bringing my attention to these questions and L. Ein for the version of the proof of the lemma given below.

§1. PROOF OF THEOREM 1.

First recall that the braid corresponding to the link of a singularity can be constructed as follows. Let us choose a coordinate system near the origin 0 in such a way that a line $x = t$ for any t such that $0 < |t| \leq \varepsilon$ intersects the curve C given by $f(x,y) = 0$ at exactly m points $y_1(t),\dots,y_m(t)$ (where m is the multiplicity of f at the origin) such that for sufficiently small one has $|y_i(t)| < \varepsilon$ $(i=1,\dots,m)$. The set $(t,y_i(t))$ $(i=1,\dots,m)$ for $|t|= \varepsilon$ is a collection of closed curves in the torus $|x| = \varepsilon$, $|y| \leq \varepsilon$ i.e. is a closed braid which is the link of $f = 0$ in the 3-sphere identified with the boundary of the polydisk $|x| \leq \varepsilon$, $|y| \leq \varepsilon$.

Next let us consider a polynomial $\tilde{f}$ which is a small perturbation of f e.g. $\tilde{f} = f+h$ where coefficients of h are sufficiently close to zero. We still can assume that the curve $\tilde{C}$ which is given by $\tilde{f}(x,y) = 0$ is transversal to all lines $x = t$ where $|t| = \varepsilon$. However, there will be finitely many points, let us say $t_1,\dots,t_N$, inside the disk $|t| < \varepsilon$ such that lines $x = t_i$ intersect $\tilde{C}$ inside the polydisk $|x| < \varepsilon$, $|y| < \varepsilon$

in less then m points. The points t_i are those for which either the lines $x = t_i$ are tangent to $\tilde{C}$ or $\tilde{C}$ has a singular point on the line $x = t_i$ near the origin.

Let P be a point inside $|x| < \varepsilon$ different from all t_i's and let γ_i $(i=1,\dots,N)$ be an ordered system of closed curves in the disk $|x| < \varepsilon$ having P as a base point which do not intersect each other and each bounding a disk D_i (cf. Fig. 1). Let us also assume that there is a small disk D_o about P such that $D_o \cap \gamma_i$ consists of two radii of D_o. For each i $(i=1,\dots,N)$ the intersection of C and the torus $T_i = \{(x,y) \mid x \in \gamma_i, |y| < \varepsilon\}$ is a closed braid which we shall denote β_i.

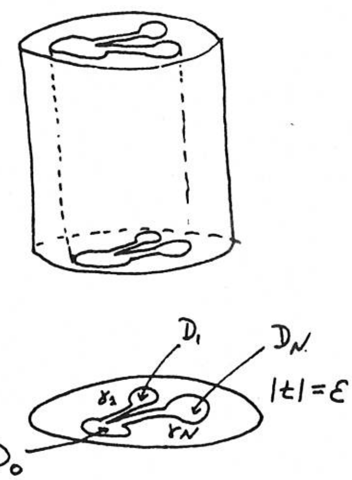

Fig. 1

A retraction of the boundary of $\bigcup_{i=1}^{N} D_i \cup D_o$ onto $|x| = \varepsilon$ defines an isotopy of $\bigcup_{i=1}^{N} T_i$ onto torus $\{(x,y) \mid |x| = \varepsilon, |y| < \varepsilon\}$ which carries the product $\prod_{i=1}^{N} \beta_i$ (in the order defined by the ordering of γ_i's) into braid β i.e. one has

$$\beta = \prod_{i=1}^{N} \beta_i \qquad (1)$$

To explain the idea in the proof of the Theorem 1 we start with

Lemma 1. Let $g(x,y)$ be a polynomial having at the origin a singular point of multiplicity m. If the homogeneous polynomial which is the sum of monomials

1) several double points (i.e. locally given by $x^2 = y^2$) all of which are real and projects on the axis x into interval $(-\varepsilon, \varepsilon)$.

2) several critical points of projections on the axis x, distinct from double points, which all are real and moreover this projection has exactly two critical values $\pm\varepsilon$.

3) $\tilde{f}$ has one ordinary singular point of multiplicity m at the origin all branches of which are real (i.e. locally given by $\prod_{i=1}^{m} (x-\alpha_i y) = 0$ $(\alpha_i \in \mathbb{R})$.

The examples of deformations of the type described in the proposition given on fig. 2.

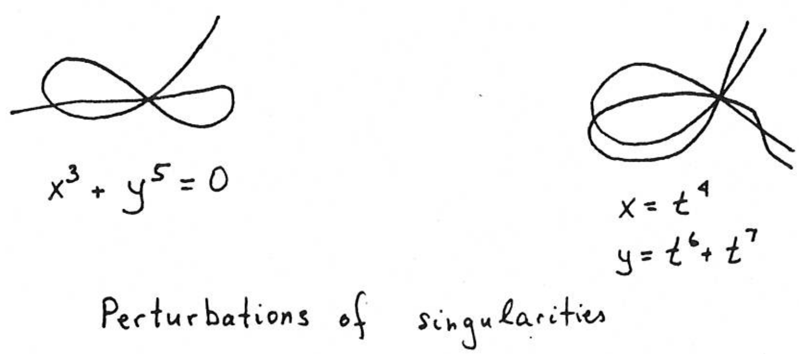

Figure 2

Assuming the proposition Theorem 1 can be deducted as follows. Let us consider the path (cf. fig. 3) in x-plane starting at some noncritical positive value $x = c$ of projection on axis x going in negative direction circumventing along small half circle below real axis each projection of the double point and point $x = 0$, going around point $x = -\varepsilon$, then continuing toward point $x = +\varepsilon$ each time circumventing projection of any double points above real axis of x-plane and finally returning to the initial point circumventing $x = \varepsilon$ and projections of double points on axis x

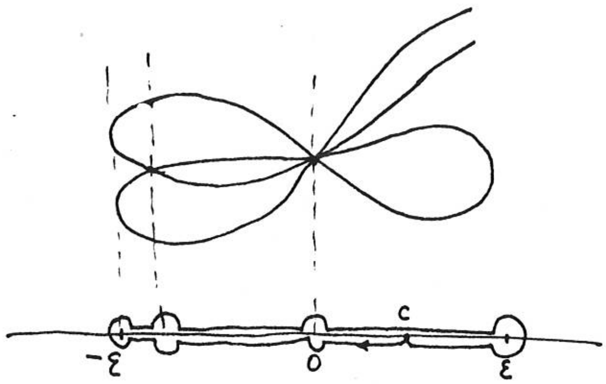

Figure 3

Each line $x = c$ with $|c|$ small intersects $\tilde{f} = 0$ on m points with exception of lines passing through critical values of projection on x-axis. Let us choose as generators of the braid group (of homeomorphisms of the line $x = c$ taking into itself the set $\tilde{f}(c,y) = 0$) the Dehn twists about segments of the real axis of the line $x = c$ connecting consecutive points. One can easily verify by local calculation that circumventing a value of projection on x-axis of a double point below real axis along half circle going in positive direction amounts to one half twist in positive directions about segment of the real axis of the line $x = c$ connecting points merging into double points considered. Moreover circumventing ordinary m-fold point along half circle amounts to homeomorphism which is just Δ. Full circle around ε or $-\varepsilon$ is equivalent to the product of half twists in positive direction corresponding to pairs of points merging in each critical point. Hence the theorem follows.

Remark: It is necessary that there will be only two critical values of projection on x-axis corresponding to the tangency points. For example in situation when the curve of degree 4 consists of two branches $x = y^2$, $x = y^2 - 1/2$ the braid corresponding to going full circle of radius 1 about the origin is $\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2$.

<u>Proof of the proposition</u>. Let us first assume that our singular curve is unibranched (i.e. corresponding algebraic link is a knot). It is well known (cf. [EN] for further references) that after change of coordinates the curve can be parametrized (Puiseaux expansion) as

$$x = t^m$$

$$y = \sum_{i=1}^{b} \lambda_i t^{q_c} \qquad (m<q_1<\ldots<q_b) \tag{2}$$

Let $T_\ell(t) = \cos(\ell \arccos t)$ be the ℓ-th Tchebychev polynomial. Recall that $T_\ell(t)$ has ± 1 as the only critical values and it has ℓ real zeroes on the interval $[-1,1]$. Let $T^\varepsilon_\ell(t) = \varepsilon^\ell T_\ell(t/\varepsilon)$. We claim that the curve

$$x = T^\varepsilon_m(t)$$

$$y = \sum_{i=1}^{b} \lambda_i T^\varepsilon_m(t) T^\varepsilon_{q_i-m}(t) \tag{3}$$

has required properties 1)-3).

Indeed if $t_1,\ldots,t_m$ are the zeroes of T^ε_m then $x(t_i) = 0$, $y(t_i) = 0$ and we have m-fold ordinary singular point at the origin. Projection of the curve (3) has $\pm\varepsilon$ as the only critical values at non-double points. Now we are going to show that all double points of the curve (3) are real. The double points of this curve correspond to solutions of the system

$$T^\varepsilon_m(t) = T^\varepsilon_m(s)$$

$$\sum_{l=1}^{b} \lambda_i T^\varepsilon_m(t) T^\varepsilon_{q_i-m}(t) = \sum_{i=1}^{b} \lambda_i T^\varepsilon_m(s) T^\varepsilon_{q_i-m}(s) \tag{4}$$

such that $t \neq s$. Assuming $T^\varepsilon_m(t) \neq 0$ (i.e. considering the points of the curve outside the origin where (3) has m-fold ordinary point), replacing the first equation by $\arccos t = \arccos s + \frac{2\pi k}{m}$ $(k=0,\ldots,m-1)$ and introducing $z = \ell^{2\pi i \arccos}$, $a = \ell^{(2\pi i k/m)}$ $(k=0,\ldots,m-1)$ one can replace the system (4) by

$$\sum_{i=1}^{b} \lambda_i \varepsilon^{q_i - q_1} z^{q_i - m} (1-a^{q_i}) - \sum_{i=1}^{b} \lambda_i \varepsilon^{q_i - q_1} z^{-(q_i - m)} (1-a^{-q_i}) = 0$$

The condition that all solutions of (4) are real is equivalent to the statement that the solutions of (5) are on unit circle. This follows from the following lemma due to I. Shur applied to $P(z) = z_1^{q_1+q_b-2m}(1-a^{q_1}) + \varepsilon^{q_2-q_1}(z^{q_2+q_b-2m})(1-a^{q_3}) + \ldots$

Lemma 2. Let P be a polynomial having roots inside the unit circle, $\bar{P}$ obtained by conjugation of the coefficients of P and $P^*(z) = z^n\bar{P}(z^{-1})$. Then all zeroes of $P + P^*$ are on the unit circle.

Proof of the lemma is given in [PS], vol. 1, p. 302.

In the case of several branches C_i with common tangent the equations of C_i can be chosen as

$$\begin{aligned} x &= t^{a_i} \\ y &= \Sigma \lambda_{ij} t^{a_{ij}}. \end{aligned} \tag{6}$$

One can easily see that we can choose

$$\begin{aligned} x &= T^{\varepsilon}_{a_i}(t) \\ y &= \Sigma \lambda_{ij} T^{\varepsilon}_{a_{ij}}(t) \end{aligned} \tag{7}$$

as perturbation satisfying proposition.

In order to treat the case of distinct tangents note that after appropriate linear changes of coordinates one can get perturbations of collection of branches with the same tangent for which projection parallel to the line $\delta y = x$ has only 2 critical values. Rotation taking this line into a line parallel to y-axis produces perturbation of collection of branches with the same tangent $\delta y = x$ projection of which on x-axis has 2 critical values and satisfying all conditions of proposition. Q.E.D.

Remark. Use of Tshebyshev polynomials above is a modification of ideas of Gusein-Zade [GZ].

§. PROOF OF THE THEROEM 2.

We shall use the following description of the braid corresponding to the link in infinity. Let us assume that the curve D does not pass through the intersection of $x = 0$ and the line in infinity. Let B_o be a disk of a sufficiently large radius in the x-plane such that any line $x = b$ with b outside of B intersects D transversally (cf. Fig. 2). Let $T_o = \{(x,y) | x \in \partial B, |y| < R, R$ sufficiently large$\}$. Then $T_o \cap D$ is a closed braid β_o in the boundary of the polydisk $\{(x,y) | x \in B_o, |y| \leq R\}$ which is the link in infinity of D.

Let $\tilde{D}$ be a curve in $\mathbf{CP}^2$ sufficiently closed to D which intersects the line in infinity and the torus T_o transversally. Let $\tilde{B}$ be a disk in the x-plane containing B_o and such that all lines $x = b$ with b outside of $\tilde{B}$ intersect $\tilde{D}$ transversally. Let $b_1, \ldots, b_k$ be the set of

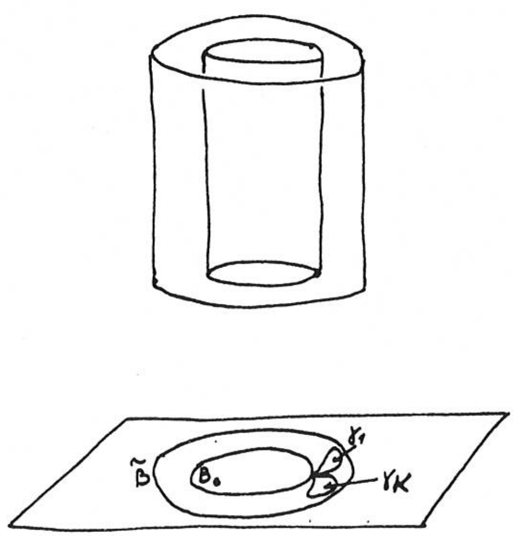

Fig. 4

points in $\tilde{B} - B_o$ such that the line $x = b_i$ is not transversal to $\tilde{D}$. Let γ_i $(i=1,\ldots,K)$ be an ordered system of loops in $\tilde{B} - B_o$ without self-intersections, each containing exactly one of the points b_i inside, having a point $p \in \partial B_o$ as a base point and which do not intersect each other. Let Γ denote the boundary of the region bounded by ∂B_o and γ_i's. The intersection of $\tilde{D}$ and the torus $T_i = \{(x,y) \mid x \in \gamma_i, |y| < R\}$ is a closed braid which we shall denote by β_i. The retraction of $\prod_{i=1}^{K} \gamma_i$ ∂B_o onto $\partial \tilde{B}$ defines an isotopy of $\bigcup_{i=1}^{K} T_i$ T onto the torus $\tilde{T} = \{(x,y) \mid x \in \partial B, |y| < R\}$ which takes $\beta_o \prod_{i=1}^{K} \beta_i$ with the natural ordering defined by the order of γ_i's, into the braid $\tilde{T} \cap \tilde{D}$. The latter braid is Δ^2. Braids β_i are conjugates of positive braids. To avoid conjugation we note that the arguments used in the proof of the theorem 1 show that one can choose perturbation $\tilde{D}$ transversed to line in infinity in such a way that all critical points of projection on x-axis are real with only 2 critical values, say R and S, at non-singular points of $\tilde{D}$, and such that double points of $\tilde{D}$ are real and have projections in interval (R,S). This can be done using perturbations (3) and (7) in local coordinates about points in infinity (y-axis $x = 0$ becomes the line in infinity) which are singular points of D and by using Tshebyshev polynomial $x = T_m^{\varepsilon}(y)$ at the tangency points of order m to the line infinity of D, combined with shifts $x \to x + 2\varepsilon$ eliminating intersection of the interval $[-\varepsilon,\varepsilon]$ with y-axis.

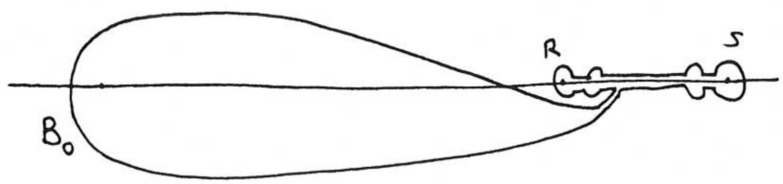

Fig. 5

Taking a non-critical point of projection on x-axis in the interval (R,S) (which is shifted interval $[-\varepsilon,\varepsilon]$ in local coordinates used for construction of perturbations) as a base point and going in x-plane around interval (R,S) defines positive braid which is $\prod_{i=1}^{k} \beta_i$ in description of factorization of Δ^2 in which braid of D in infinity appears as a factor. (cf. Fig. 5). Q.E.D.

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