Invent. math. 95, 25–30 (1989)

Inventiones mathematicae
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Invariants of plane algebraic curves via representations of the braid groups

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1. Introduction

Recently new remarkable polynomial invariants of knots in 3-sphere were introduced using some representations of the braid groups which were shown to be effective in distinguishing knots ([J], [FYHLMO]). The purpose of this note is to introduce, using representation of the braid groups, an invariant of continuous equisingular families of plane algebraic curves.

A pleasant feature is that this invariant is defined for any representation of the braid group (no coherence of representations for different number of strings is required), though for many representations this invariant is trivial. In the case of the reduced Burau representation, this invariant essentially coincides with the Alexander polynomial of plane curves introduced in [L 1]. This fact is the counterpart of the well-known relation between the Alexander polynomial of closed braids and the reduced Burau representation.

The definition of this invariant depends on braid monodromy associated with the curve ([M]). In particular we obtain a direct (not involving calculations of the fundamental group, though computationally cumbersome) method of computation of the Alexander polynomial via braid monodromy. In the next section we shall recall the background on continuous equisingular families of plane algebraic curves and their braid monodromies. In the Sect. 3 we introduce the invariant and in the last section we consider the case of reduced Burau representations.

I would like to thank P. Deligne for his comments on this work which allowed substantial simplification of the arguments and the NSF for its support.

2. Preliminaries

We shall be concerned with the equisingular families of plane curves of a fixed degree, i.e. such that for any two curves in a family there is a $1-1$ correspondence between the singularities of these curves so that the corresponding singularities are topologically equivalent. The typical and the most interesting case is the case of the families of curves of degree d which have a fixed number δ of nodes (i.e. the singularities having $x^2=y^2$ as the local equation) and a fixed number

κ of cusps (i.e. the singularities having $x^2=y^3$ as the local equation). Denote such a family by $C(d, \delta, \kappa)$. J. Harris showed ([H]) that the irreducible curves in $C(d, \delta, 0)$ form a single connected component. If $\kappa \neq 0$ then $C(d, \delta, \kappa)$ has in general several connected components. The first example of disconnected families of irreducible curves in $C(d, \delta, \kappa)$ is due to O. Zariski. He showed that the fundamental group of the complement to the curve of degree 6 with six cusps (which lie on the conic) given by $(x^2+y^2)^3+(y^3+z^3)^2=0$ is $\mathrm{PSL}_2(\mathbb{Z})$. On the other hand, he constructed an irreducible sextic with six cusps (not on a conic) for which the fundamental group of the complement is $\mathbb{Z}/6\mathbb{Z}$. These two sextics belong to different connected components of $C(6,0,6)$ because the fundamental group is unchanged in a continuous equisingular family. The Alexander polynomial (cf. [L1]) does distinguish the connected components of $C(6,0,6)$ as well. This work was motivated by an attempt to find other invariants which distinguish the connected components of equisingular families.

Fundamental groups of the complements can be found from a more subtle object associated with plane curves, namely, from their braid monodromy (cf. [M]). Recall its definition. Let $\mathscr{C}$ be a curve in $\mathbb{CP}^2$ transversal to the line in infinity. Let $p\colon \mathbb{C}^2 \to \mathscr{C}$ be a linear projection of the affine portion of $\mathbb{CP}^2$ from a point in infinity such that a) the fibers of p are transversal to $\mathscr{C}$ except for a finite set $\mathrm{Cr}(\mathscr{C})$: $P_1, \ldots, P_N$; b) fibers over $\mathrm{Cr}(\mathscr{C})$ have simple tangency with $\mathscr{C}$ or pass through the singularities of $\mathscr{C}$ so that these fibers are transversal to the tangent cones of singularities of $\mathscr{C}$; c) the center of projection p (on the line in infinity) does not belong to $\mathscr{C}$. A projection satisfying a), b), c) we call a generic projection. A choice of a base point $P_0 \in \mathbb{C}-\mathrm{Cr}(\mathscr{C})$ and trivializations of the restrictions of $p^{-1}(\mathbb{C}-\mathrm{Cr}(\mathscr{C}))$ over loops in $\mathbb{C}-\mathrm{Cr}(\mathscr{C})$ based in P_0, defines the map θ of $\pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}), P_0)$ into the group of the isotopy classes of homeomorphisms of $p^{-1}(P_0)$ preserving the set of $d=\deg(\mathscr{C})$ points $p^{-1}(P_0)\in\mathscr{C}$. These homomorphisms can be chosen to preserve a circle of a sufficiently large radius. Therefore in fact one has a homomorphism θ from $\pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}))$ into the braid group on d strings B_d. This homomorphism is the braid monodromy of $\mathscr{C}$.

It is convenient to describe a braid monodromy by its values on paths from "ordered good" system of generators of $\pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}, P_0)$ (cf. [M]). The latter can be obtained by fixing a small circle α_i (counterclockwise oriented) about each point $P_1, \ldots, P_N$ from $\mathrm{Cr}(\mathscr{C})$, a system of non intersecting paths β_i in $\mathbb{C}-\bigcup\alpha_i$ connecting P_0 and $\partial\alpha_i$ and by letting $\gamma_i=\beta_i^{-1}\alpha_i\beta_i$. Recall that if $\mathscr{C}$ is in general position relative to the line in infinity then (product is taken in the natural ordering of γ_i's) $\prod_{i=1}^{N}\gamma_i=\Delta^2$ where Δ^2 is the generator of the center of B_d and that one can obtain any ordered good system of generators from any other by a sequence of the following moves:

$$(\gamma_1, \ldots, \gamma_N) \to (\gamma\gamma_1\gamma^{-1}, \ldots, \gamma\gamma_N\gamma^{-1}) \tag{1}$$

$$(\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_N) \to (\gamma_1, \ldots, \gamma_i\gamma_{i+1}\gamma_i^{-1}, \gamma_i, \ldots, \gamma_N) \tag{2}$$

cf. [M], [L2]).

Lemma. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two curves belonging to the same connected component of an equisingular family of plane curves and p_i $(i=1, 2)$ be two generic linear projection of $\mathbb{C}^2-\mathscr{C}_i$ $(i=1,2)$. Then there is a homeomorphism j:*

$$\mathbb{C}-\mathrm{Cr}(\mathscr{C}_1)\to\mathbb{C}-\mathrm{Cr}(\mathscr{C}_2)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}_1)) & \searrow & \\ {\scriptstyle j_*}\downarrow & & B_d \\ \pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}_2)) & \nearrow & \end{array} \tag{3}$$

Proof. Let O_i be the center of projection p_i $(i=1,2)$. By abuse of notation we denote by p_i also the corresponding projection of the projective plane. The target of any linear projection can be identified with the uniquely determined by the center of projection line in the dual plane. Let $\mathbf{I}$ be the set of pairs $(0, \mathscr{C})$ where 0 is a point in the plane and $\mathscr{C}$ is a curve from the irreducible component, say F, of an equisingular family of curves in question such that projection of $\mathscr{C}$ from 0 is generic. $\mathbf{I}$ is a Zariski open set in an irreducible variety $F\times(\mathbb{CP}^2)^*$ and hence is connected. The projection of a path in $\mathbf{I}$ connecting O_1 and O_2 on the first factor of $\mathbf{I}$ defines an isotopy of the targets of projections p_i inducing the vertical isomorphism in (3).

Finally note that according to the van Kampen theorem $\pi_1(\mathbb{C}^2-\mathscr{C})$ has the following presentation:

$$\pi_1(\mathbb{C}^2-\mathscr{C})=\{e_1,\dots,e_d\,|\,\theta(\gamma_i)\,e_j=e_j,\ i=1,\dots,N, j=1,\dots,d\} \tag{4}$$

where the action of the braid group is the standard action of the braid group on the free group (cf. [B]). The braid monodromy seems to carry, however, more information, e.g. the homotopy type of the complement is determined by it ([L3]), and it is unlikely that the object from the next section will depend only on π_1.

3. The invariant

Let ρ be an n dimensional linear representation of B_d over the ring A of Laurant polynomials $\mathbb{Q}[t,t^{-1}]$. We shall consider $M=H_0(\pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}), \rho(\theta))$ as a module over A. Let $M=\bigoplus A^k\oplus\bigoplus_{i=1}^{l}A/\lambda_i$ be its cyclic decomposition. We shall denote its order $\Pi\lambda_i$, which is a Laurant polynomial defined up to a unit in A, by $P(\mathscr{C}, \rho)$. This homology group is the largest quotient of A^n on which $\pi_1(\mathbb{C}-\mathrm{Cr}(\mathscr{C}))$ acts trivially via the composition of the braid monodromy and chosen representation. In down to earth terms, $P(\mathscr{C}, \rho)$ can be described as the greatest common divisor of the minors of the order n in the $N\cdot n$ by n matrix of the map $\bigoplus(\rho(\theta(\gamma_i))-\mathrm{Id})$ which takes $(A^n)^N$ into A^n.

The lemma above implies that $P(\mathscr{C}, \rho)$ depends only on $\mathscr{C}$ and not on the choices on which the definition of the braid monodromy depends and moreover this polynomial is the same for all $\mathscr{C}$ in a connected component of an equisingular family.

4. Alexander polynomial of plane curves and Burau representations

Now we shall calculate $P(\mathscr{C}, \rho)$ where ρ is the reduced Burau representation. Recall first that the Alexander polynomial of $\mathscr{C}$ can be defined as follows ([L1]). If $\mathscr{C} \in \mathbb{C}^2$ is an algebraic curve, then the map which assigns to any loop from $\pi_1(\mathbb{C}^2 - \mathscr{C})$ its linking number with $\mathscr{C}$ defines a surjective map ψ of $\pi_1(\mathbb{C}^2 - \mathscr{C})$ onto $\mathbb{Z}$. The first homology group of the infinite cyclic cover of $\mathbb{C}^2 - \mathscr{C}$ corresponding to the kernel of this homomorphism is a torsion $A = \mathbf{Q}[t, t^{-1}]$ module ([L1]). Its order as an A-module is the Alexander polynomial of $\mathscr{C}$.

The Alexander polynomial of $\mathscr{C}$ can be found in terms of Fox derivatives in a way similar to the well-known procedure from the knot theory. Namely if $\pi_1(\mathbb{C}^2 - \mathscr{C})$ has $e_1, \ldots, e_d$ as generators and $R_1 = 1, \ldots, R_N = 1$ are relators then the Alexander polynomial is equal to the g.c.d. of the minors of the order $d-1$ in the matrix $(\psi(\partial R_i / \partial e_j))$, $i = 1, \ldots, N, j = 1, \ldots, d$ where by abuse of notation we denote by ψ the homomorphism of the group rings induced by the homomorphism ψ above ([L1]).

Finally recall that the Burau representation of B_d can be described as the $\mathbb{Z}[t, t^{-1}]$ representation which maps the standard generators as follows:

$$\sigma_1 \to \begin{pmatrix} -t & 1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & & \vdots & \\ 0 & & \ldots & 1 \end{pmatrix}, \quad \sigma_i \to \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & 0 & 0 & & \\ & & & t & -t & 1 & & \\ & & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{matrix} \\ i\text{-th row} \\ 2 \leqq i \leqq d-1. \end{matrix} \tag{5}$$

The reduced Burau representation is the quotient of this representation by the obvious 1-dimensional invariant subspace. If one views B_d as the automorphism group of a free group with generators $e_1, \ldots, e_d$ acting by the formulas:

$$\sigma_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_i \, e_{i+1} \, e_i^{-1} & j = 1 \\ e_i & j = i+1 \end{cases} \tag{6}$$

and if $g_i = e_1 \ldots e_i$, then the Burau representation is just $\pi(\sigma) = \psi(\partial \sigma(g_i)/g_j)$, i, $j = 1, \ldots, d$ where $\psi(e_i) = t$.

Theorem. *If $\bar{\pi}$ is the reduced Burau representation then $P(\mathscr{C}, \bar{\pi})$ is equal to the Alexander polynomial of $\mathscr{C}$ multiplied by $(1 + t + \ldots + t^{d-1})$.*

Proof. From the fact that (4) is a presentation of $\pi_1(\mathbb{C}^2-\mathscr{C})$ it follows that

$$\{g_1,\ldots,g_d \mid \theta(\gamma_i)\, g_j = g_j,\ i=1,\ldots,N, j=1,\ldots,d\} \tag{7}$$

is a presentation of $\pi_1(\mathbb{C}^2-\mathscr{C})$ as well. Therefore

$$\bigoplus_{i=1}^{N}((\partial\theta(\gamma_i)\, g_k/\partial g_l)-I) \tag{8}$$

is a matrix of the Fox derivatives of $\pi_1(\mathbb{C}^2-\mathscr{C})$. Hence the Alexander polynomial of $\mathbb{C}^2-\mathscr{C}$ is the g.c.d. of the minors of the order $(d-1)$ in (8).

It follows from the facts about Burau representations reviewed above that (8) is the same as

$$\bigoplus_{i=1}^{N}(\bar{\pi}(\theta(\gamma_i)-I)). \tag{9}$$

Let $\Delta_j^{i_1\ldots i_{d-1}}$ denote the minor of the matrix (9) obtained by delecting column j and containing rows $i_1,\ldots,i_{d-1}$ $(1<i_k<Nd)$ and let $\Delta^{i_1,\ldots,i_{d-1}}$ be the g.c.d. of $\Delta_j^{i_1,\ldots,i_{d-1}}$ for $j=1,\ldots,d$. From the "fundamental formula of the free calculus" ([CF], Ch. 7, (2, 11)) $\Sigma_j(\partial R_i/\partial g_j)(t^j-1)=0$ we obtain

$$\frac{\Delta_j^{i_1,\ldots,i_{d-1}}}{t^j-1}=\frac{\Delta_K^{i_1,\ldots,i_{d-1}}}{t^k-1}.$$

Therefore $\Delta_d^{i_1,\ldots,i_{d-1}}=\Delta^{i_1,\ldots,i_d}(1+t+\ldots+t^{d-1})$ and our theorem follows.

Example. Let $\mathscr{C}$ be a non-singular cubic curve. Then the braid monodromy can be written as $\Delta^2=\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$ (cf. [M] Th. 1 p. 120). We have $\bar{\pi}(\sigma_1)-I=\begin{pmatrix}-t-1 & 1\\ 0 & 0\end{pmatrix}$, $\bar{\pi}(\sigma_2)=\begin{pmatrix}0 & 0\\ t & -t-1\end{pmatrix}$. The g.c.d. of the minors of order 2 in the matrix

$$\begin{pmatrix}-t-1 & 1\\ 0 & 0\\ t & -t-1\\ 0 & 0\end{pmatrix}$$

is equal to $1+t+t^2$. Hence the Alexander polynomial is 1. Of course in the this case $\pi_1(\mathbb{C}^2-\mathscr{C})$ is just $\mathbb{Z}$.

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Oblatum 24-XI-1985 & 20-IV-1988