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On the homology of finite Abelian coverings

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Abstract

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We calculate the first Betti number of an Abelian covering of a CW-complex X as the number of points with cyclotomic coordinates (of orders determined by the Galois group) which belong to a certain subvariety of a torus constructed from the fundamental group of X. This generalizes the classical formulas for the cyclic coverings due to Zariski and Fox. We also describe certain properties of these subvarieties of tori in the case when X is a complement to an algebraic curve in $\mathbb{CP}^2$ which are analogs of Traldi–Torres relations from the link theory and the divisibility theorem for Alexander polynomials of plane algebraic curves [8].

Keywords: Abelian coverings, fundamental group, reducible algebraic curves.

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Introduction

Let X be a finite 2-dimensional complex such that $H_1(X, Z) = Z^k$. Every homomorphism from $H_1(X, Z)$ onto $G_a = Z/a_1Z + \cdots + Z/a_kZ$ defines an Abelian covering X_a of X. The purpose of this paper is to calculate $H_1(X_a, \mathbb{C})$ in terms of the fundamental group of X. Our first result calculates $H_1(X_a, \mathbb{C})$ in terms of the quotient of the commutator of the fundamental group $\pi_1(X)$ by its second commutator considered as the module over the group ring of $H_1(X, Z)$. The module structure is coming from the exact sequence

$$0 \to \pi_1'(X)/\pi_1''(X) \to \pi_1(X)/\pi_1''(X) \to H_1(X, Z) \to 0. \qquad (0.1)$$

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The group ring in question is just the ring Λ_k of finite Laurent polynomials of k variables. The left term in (0.1) can be identified with $H_1(\tilde{X}, Z)$ where $\tilde{X}$ is the universal Abelian cover of X and the same module structure can be obtained from the action of the group Z^k of deck transformations on this homology group. Actually the major role in what follows is played by the tensor product with $\mathbb{C}$ of the left term in (0.1) (or $H_1(\tilde{X}, \mathbb{C})$) as a module over the ring of Laurent polynomials with $\mathbb{C}$ coefficients. The latter by abuse of notation we will denote as Λ_k or Λ as well. The calculation of H_1 is the only nontrivial part in finding the Betti numbers of an Abelian cover in terms of a data from the base because of the multiplicative property of Euler characteristic. On the other hand one can use these results to obtain the first Betti number of an Abelian cover of any finite complex by applying these results to the 2-skeleton.

Recall that for a finitely generated module M over a commutative ring R one can define the ith Fitting ideal $F_i(M)$ (cf. [3]) (respectively ith determinantal ideal $E_i(M)$ [1]) as the ideal generated by the $(n-i+1)$ by $(n-i+1)$ (respectively $(n-i)$ by $(n-i)$) minors of the matrix of a map $\Phi: R^m \to R^n$ such that $M = \text{Coker } \Phi$. These ideals depend only on M and are independent of a choice of Φ. For a point P in $(\mathbb{C}^*)^k$ and a Λ-module M let $f(P, M)$ (respectively $e(P, M)$) be the largest integer i such that all polynomials from $F_i(M)$ (respectively determinantal ideal $E_i(M)$) vanish at P. Clearly $f(P, M) = e(P, M) + 1$.

Theorem 0.1. *The rank of $H_1(X_a, \mathbb{C})$ is equal to*

$$k + \sum f((\omega_{a_1}, \ldots, \omega_{a_k}), H_1(\tilde{X}, C))$$

where the sum is taken over all vectors of the form $(\omega_{a_1}, \ldots, \omega_{a_k})$ such that ω_{a_i} is a root of unity of degree a_i and which are different from $(1, \ldots, 1)$.

In practice it is more convenient to find a presentation of a closely related to $H_1(\tilde{X}, \mathbb{C})$ module $A(X) = H_1(X, pt, \Lambda_k)$. The latter is called by knot theorists as the Alexander module (cf. [6]) and has presentation $\Phi: \Lambda^s \to \Lambda^t$ where the matrix of Φ is the Jacobian matrix of the Fox derivatives $\partial r_i / \partial x_j$ where x_j $(j = 1, \ldots, s)$ are the generators and $r_i = 1$ $(i = 1, \ldots, t)$ are the defining relators of a presentation of $\pi_1(X)$ (cf. [6]).

Proposition 0.2. *If $P \neq (1, \ldots, 1)$, then $e(P, A(X)) = f(P, (\pi_1'(X)/\pi_1''(X)) \otimes \mathbb{C})$.*

Theorem 0.1 and the proposition imply:

Corollary 0.3. *The rank of $H_1(X_a, \mathbb{C}) = k + \sum e(\omega_a, A(X))$ where the summation is over all vectors $\omega_a = (\omega_1, \ldots, \omega_k)$ such that $\omega_a \neq (1, \ldots, 1)$ and $\omega_i^{a_i} = 1$.*

Proposition 0.2 is essentially contained in the Hillman's exposition of Crowell-Strauss-Traldi results (cf. [6, 13]). The reason that we prove this proposition here is to show the usefulness of the following "geometric" invariant suggested by Theorem 0.1. Recall that for a module M over a commutative ring R one can define the support Supp M as the subvariety of Spec R consisting of prime ideals p of R such that $M \otimes (R/p) \neq 0$ (cf. [11]). Let us define the ith characteristic variety of M as $\text{Supp}(\Lambda^i M)$ and denote it as $V_i(M)$. It follows from [3, 1, 11] that $V_i(M) = \text{Supp}(R/F_i(M)) = \text{Supp}(R/E_{i-1}(M)) = \text{Supp}(R/\text{Ann}\, \Lambda^i(M))$. For modules over Λ_k the characteristic varieties are subschemes of the tori $(\mathbb{C}^*)^k = \text{Spec}(\Lambda_k)$. It follows from Theorem 0.1 and Corollary 0.3 that the homology of Abelian coverings depends only on the number of points with cyclotomic coordinates on (reduction of) characteristic varieties of $A(X)$ (and the homology of the base).

The applications which we have in mind are concerned with 2-complexes which are either complements to links in S^3 or complements to algebraic curves in $\mathbb{C}^2$ (cf. [8]). In the case when these complexes have cyclic first homology group (i.e., knots in 3-sphere and irreducible plane algebraic curves) Theorem 0.1 and Corollary 0.3 above are equivalent to well-known results (cf. [5, 17, 9, 8] and Example 4.1 below).

The paper is organized as follows. Sections 1 and 2 contain a proof of Theorem 0.1 and Proposition 0.2. Section 3 describes an analog to the case of plane algebraic curves of the Traldi–Torres relations concerning the change of determinantal ideals (or characteristic varieties) with removing a component of the curve. Section 3 contains an analog to the divisibility properties of Alexander polynomials of algebraic curves (cf. [8]) for characteristic varieties and the last section contains examples.

For further calculations with characteristic varieties we refer to Hironaka's thesis (cf. [7]). Sarnak [10] studied the homology of Abelian covers from an analytic point of view. He also obtained polynomial periodicity of the first Betti numbers in certain towers of coverings as result from his proof of a conjecture of Lang on torsion points on algebraic subvarieties of tori. Finally note that independent use of the geometry of $\text{Supp}(H_1(\tilde{X}, C))$ was made in the work of Freed and Dwyer (cf. [5]) which studies the finiteness of complexes which are infinite Abelian covers of finite complexes and that the closely related case of locally reducible hypersurfaces singularities was studied by Sabbah (cf. [9]).

1. Proof of Theorem 0.1

Let H_a denote the kernel of the map $H_1(X, Z) = Z^k \to G_a$ defining the Abelian covering in question. The homology Leray spectral sequence of the group H_a acting freely on the universal Abelian covering $\tilde{X}$ has the following form (cf. [2]):

$$E^2_{p,q} = H_p(H_a, H_q(\tilde{X}, C)) \Rightarrow H_{p+q}(X_a, C). \tag{1.1}$$

From this spectral sequence one can derive the following exact sequence of the low degree terms:

$$\begin{aligned} H_2(X_a, C) \to H_2(H_a, C) \to H_1(\tilde{X}, C) \otimes_{C[H_a]} Z \\ \to H_1(X_a, C) \to H_1(H_a, C) \to 0. \end{aligned} \tag{1.2}$$

Let γ be the left-most homomorphism in this sequence. The proof below consists of four steps.

Step 1. We are going to show that if K is the kernel of the cup product map $\Lambda^2 H^1(X, C) \to H^2(X, C)$, then $\dim \operatorname{Im} \gamma + \dim K = k(k-1)/2$.

First notice that because $\dim H^2(H_a, C) = k(k-1)/2$ the linear algebra implies

$$\dim \operatorname{Im} \gamma = k(k-1)/2 - \dim \operatorname{Ker}(H_2(H_a, C)^* \to H_2(X_a, C)^*) \tag{1.3}$$

or restating (1.3)

$$\dim \operatorname{Im} \gamma = k(k-1)/2 - \dim \operatorname{Ker}(H^2(H_a, C) \to H^2(X_a, C)). \tag{1.4}$$

On the other hand we have the following diagram of spaces:

$$\begin{array}{ccc} X_a & \longrightarrow & X \\ \downarrow & & \downarrow \\ BH_a & \longrightarrow & BZ^k \end{array} \tag{1.5}$$

where the horizontal arrows are the quotients of the action of the covering group $G_a = Z/a_1 \oplus \cdots \oplus Z/a_k$ and the vertical arrows are the classifying maps of the universal Abelian covers. (1.5) implies the following commutative diagram:

$$\begin{array}{ccc} H^2(X_a, C) & \longleftarrow & H^2(X, C) \\ \downarrow & & \downarrow \\ H^2(BH_a, C) & \longleftarrow & H^2(BZ^k, C) \end{array} \tag{1.6}$$

The upper horizontal arrow is injective as one can see using transfer (or spectral sequence for the action of the group G_a which degenerates because $H^i(G_a, C) = 0$ for $i > 0$). The lower horizontal arrow is an isomorphism for the same reason. Therefore

$$\operatorname{Ker}(H^2(BH_a, C) \to H^2(X_a, \mathbb{C})) = \operatorname{Ker}(H^2(BZ^k, C) \to H^2(X, C)).$$

But $H^2(BZ^k) = \Lambda^2 H^1(BZ^k)$ and this isomorphism is given by the cup product. Hence $H^1(BZ^k, C) = H^1(X, C)$ implies that $\operatorname{Im}(H^2(BZ^k, C) \to H^2(X, C))$ is the image of the cup product on $H^1(X, C)$ and the claim of Step 1 follows.

Step 2.

$$\dim H_1(X_a, C) = \dim H_1(X, C) \otimes_{C[H_a]} C + k - \dim \operatorname{Ker}(\Lambda^2 H^1(X) \to H^2(X))$$

where $\Lambda^2 H^1(X) \to H^2(X)$ is induced by the cup product.

This is an immediate consequence of the sequence (1.2) and Step 1.

Step 3. $\dim H^1(X, C) \otimes_{C[H_a]} C = \sum f(P, H_1(\tilde{X}, C))$ where

$$P = (\omega_{a_1}, \ldots, \omega_{a_k}) \tag{1.7}$$

runs through all points in $(\mathbb{C}^*)^k$ with $\omega_{a_i}^{a_i} = 1$.

Indeed let

$$\Lambda^s \to \Lambda^t \to H_1(\tilde{X}, C) \to 0 \tag{1.8}$$

be a presentation of the Λ module $H_1(\tilde{X}, C)$. Let I_a be the ideal of Λ generated by the polynomials $t_1^{a_1}-1, \ldots, t_k^{a_k}-1$. After tensoring (1.7) with C over $C[H_a]$ and using $\Lambda \otimes_{C[H_a]} C = \Lambda / I_a$ (for any $C[H_a]$ module M one has $M \otimes_{C[H_a]} C = M / I_a M$ whereby abuse of notation I_a denotes the augmentation ideal of $C[H_a]$), we obtain:

$$(\Lambda / I_a)^s \to (\Lambda / I_a)^t \to H_1(\tilde{X}, C) \otimes_{C[H_a]} C \to 0. \tag{1.9}$$

The module Λ / I_a is isomorphic to $\oplus_P \Lambda / m_P$ where P runs through all points as in (1.7) and m_P is the maximal ideal of P. This direct sum decomposition can be obtained by selecting polynomials g_P such that $g_P(P)=1$ and $g_P(Q)=0$ for $Q \neq P$, where P and Q are the points given by (1.7), and by mapping $p \in \Lambda / m_P$ to $\tilde{p} g_P$ mod I_a where $\tilde{p}$ is any representative of p. Moreover it is clear from this description of the direct sum decomposition of Λ / I_a that the left map in (1.9) is the direct sum of the maps $(\Lambda / m_P)^s \to (\Lambda / m_P)^t$. On the other hand $\dim \operatorname{Coker}((\Lambda / m_P)^s \to (\Lambda / m_P)^t)$ is equal to the largest order i of the Fitting ideals F_i such that all polynomials from F_i vanish at P. Indeed the corank of a linear map between vector spaces of dimensions s and t is the maximum of i's such that all minors of order $(t-i+1)$ of the matrix of this linear map are zeros. Therefore the claim of Step 3 follows.

Proposition 1.1. *The dimension of the kernel* $\operatorname{Ker}(\Lambda^2 H^1(X, C) \to H^2(X, C))$ *of the map defined by the cup product is one larger than the maximal order of the Fitting ideals vanishing at* $P_0 = (1, \ldots, 1)$.

Proof. Let us apply the results of Steps 2 and 3 to the trivial covering of degree 1 of X. We obtain that

$$k = \dim H_1(X, C) = k + f(P_0, H_1(\tilde{X}, C)) - \dim \operatorname{Ker}(\Lambda^2 H^1(X, C) \to H^2(X, C))$$

and the result follows. □

Theorem 0.1 now follows from Steps 1-3 and Proposition 1.1.

Remark. Each term in the sum of Theorem 0.1 can be interpreted as the homology group of X with coefficients in a 1-dimensional twisted system corresponding to a character of the fundamental group of X. The latter is determined by a choice of a root of unity for each of the generators of $H_1(X, Z) = Z^k$. The term k corresponds to the untwisted system of coefficients.

2. Proof of Proposition 0.2

Recall first the basic properties of support $S(M)$ of finitely generated modules over a commutative ring R (cf. [11]) which is defined as a subvariety of Spec R of prime ideals p such that $M \otimes_R R_p \neq 0$. If

$$0 \to M_1 \to M \to M_2 \to 0 \tag{2.1}$$

is an exact sequence, then $S(M) = S(M_1) \cup S(M_2)$ and for any two finitely generated modules M_1 and M_2 one has $S(M_1 \otimes_R M_2) = S(M_1) \cap S(M_2)$.

Now let us note that in the portion of the exact sequence of the pair $(\tilde{X}, \tilde{pt})$ (where $\tilde{pt}$ is the inverse image of a base point pt under the covering map)

$$0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, pt) \to H_0(\tilde{pt}) \to H_0(\tilde{X}) \tag{2.2}$$

the kernel of the right homomorphism is the augmentation ideal of $\mathbb{C}[Z^n]$. This implies the following "link module sequence" for a 2-complex X:

$$0 \to H_1(\tilde{X}, C) \to A(X) \to I \to 0. \tag{2.3}$$

The localization of this exact sequence at a maximal ideal of a point $P \neq (1, \ldots, 1)$ gives the following sequence

$$0 \to H_1(\tilde{X}, C)_m \to A(X)_m \to I_m \to 0 \tag{2.4}$$

in which I_m is free over the Λ_m module of rank 1. (A direct way to see this is to use exactness of localization applied to $0 \to I \to \Lambda \to C \to 0$ and surjectivity of $I_m \to \Lambda_m$ which follows by writing a/b from Λ as pa/pb where $p(1, \ldots, 1) = 0$ and $b(P) \neq 0$.) Therefore $\Lambda^k A(X)_m$ admits a filtration

$$\Lambda^k A(X)_m = \Lambda_0^k A(X)_m \supset \cdots \supset \Lambda_{k+1}^k A(X)_m = 0$$

such that

$$\Lambda_i^k / \Lambda_{i+1}^k = \Lambda^i (H_1(\tilde{X}) \otimes \Lambda^{k-i} I_m).$$

Because $\Lambda^{k-i} I_m = 0$ for $i \neq k$, $k-1$ and $\Lambda^1 I_m = \Lambda^0 I_m = \Lambda_m$ we obtain

$$0 \to \Lambda^k (H_1(\tilde{X})_m) \to A(X)_m \to \Lambda^{k-1}(H_1(\tilde{X})_m) \to 0. \tag{2.5}$$

Using the relationship for supports implied by (2.1) and $V(\Lambda^k H_1(X)) \subset V(\Lambda^{k-1} H_1(\tilde{X}))$ one derives that $V(\Lambda^k A(X)_m) = V(\Lambda^{k-1} H_1(\hat{X}))$ and Proposition 0.2 follows.

Remark. As pointed out in the Introduction this proposition is not new. Hillman [6, p. 49] showed that determinantal ideals of sequence (2.3) satisfy $E_k(A) \supseteq E_{k-1}(H_1(\tilde{X})) I^{k-1}$ and $E_{k-1}(H_1(\tilde{X})) \supseteq E_k(A(X)) I^{\binom{k-1}{2}}$. These inclusions are stronger than Proposition 0.2.

Note, however, that for general X there is no equality of Crowell–Strauss type, i.e., $E_0(H_1(\tilde{X})) = (\Delta_1(A)) I^{\binom{k-2}{2}}$ (cf. [4]): the proof of these relations relies on the existence of presentation of $\pi_1(X)$ with $k+1$ generators and k relators.

3. Characteristic varieties of plane algebraic curves

Let $C = C_1 \cup \cdots \cup C_\mu$ be a reduced algebraic curve in $\mathbb{C}^2$ having irreducible components $C_1, \ldots, C_\mu$. Let d_i be the degree of the component C_i. Then $H_1(\mathbb{C}^2 - C_1 \cup \cdots \cup C_\mu, Z) = Z^\mu$ where the isomorphism is given by assigning to a 1-cycle the collection of its linking numbers with the components of C. $\mathbb{C}^2 - C$ has the homotopy type of a 2-dimensional complex (cf. [8]). Let $V_k(C) \subset (\mathbb{C}^*)^\mu$ (respectively $V_k(C_{\hat\mu}) \subset (\mathbb{C}^*)^{\mu-1}$) be the kth characteristic variety of $\mathbb{C}^2 - \bigcup C_i$ where $i = 1, \ldots, \mu$ (respectively $i = 1, \ldots, \mu - 1$). Let $Z_{d_1,\ldots,d_\mu} \subset (\mathbb{C}^*)^{\mu-1}$ be the set of the zeros of the polynomial $P = (t_1^{d_1} \cdots t_{\mu-1}^{d_{\mu-1}})^{d_\mu} - 1$, and let H be the hyperplane $t_\mu = 1$ in $(\mathbb{C}^*)^\mu$ identified with $(\mathbb{C}^*)^{\mu-1}$ using remaining coordinates. Let $O = S(\Lambda / I_{\mu-1})$.

Theorem 3.1. *The following inclusions take place*:

$$V_{k-1}(C_{\hat\mu}) \cap (Z \cup V_k(C_{\hat\mu})) \supseteq V_k(C) \cap H \supseteq V_{k-1}(C_{\hat\mu}) \cap (O \cup V_k(C_{\hat\mu})). \tag{3.1}$$

This theorem is actually proven below in a stronger form which looks identically with the Traldi relations [13] for determinantal ideals of a link.

Theorem 3.1′. *The following inclusions take place*:

$$E_{k-1}(C_{\hat\mu}) + (P) E_k(C_{\hat\mu}) \subseteq \Phi(E_k(C)) \subseteq E_k(C_\mu) + I_{\mu-1} E_k(C_\mu) \tag{3.2}$$

where (P) is the principal ideal generated by P, I is the augmentation ideal and Φ is the homomorphism of specialization of t_μ into 1.

Proof. Clearly Theorem 3.1′ implies Theorem 3.1. The proof of Theorem 3.1′ follows closely the pattern of the Hillman proof of Traldi relations for the links. Let $\tilde X$ be the universal Abelian covering of $\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i$ and $\tilde Y$ be the universal Abelian cover of $\mathbb{C}^2 - \bigcup_{i=1}^{\mu-1} C_i$. Let p_X and p_Y be the corresponding projections. First notice that one has

$$A\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right) \to A\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right) \to H_1\left(p_Y^{-1}\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right), p_Y^{-1}(pt)\right) \to 0 \tag{3.3}$$

where the left arrow is the multiplication by $t_\mu - 1$. This follows from the exact homology sequence corresponding to the exact sequence of the chain complexes

$$0 \to C_*\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i, \tilde{pt}\right) \to C_*\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i, \tilde{pt}\right) \to C_*\left(p_Y^{-1}\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right), \tilde{pt}\right) \to 0 \tag{3.4}$$

where the left homomorphism is the multiplication by $t_\mu - 1$. (3.4) is induced by the deck transformations of the infinite cyclic covering

$$\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i \to p_Y^{-1}\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right).$$

Hence $H_1(p_Y^{-1}(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i), p_Y^{-1}(pt))$ is the specialization of $A(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i)$. Next let us consider the exact sequence of the triple $(\tilde{Y}, p_Y^{-1}(\mathbb{C}^2 - C_i), p_Y^{-1}(pt))$. Using excision one obtains:

$$\begin{aligned} H_2\left(\tilde{Y}, p_Y^{-1}(\mathbb{C}^2) - \bigcup_{i=1}^{\mu} C_i\right) &\to H_1\left(p_Y^{-1}\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right), p_Y^{-1}(pt)\right) \\ &\to H_1(\tilde{Y}, p_Y^{-1}(pt)) \\ &\to H_1\left(\tilde{Y}, p_Y^{-1}\left(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i\right)\right). \end{aligned} \tag{3.5}$$

One has $H_1(\tilde{Y}, p_Y^{-1}(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i)) = 0$. On the other hand $H_2(\tilde{Y}, p_Y^{-1}(\mathbb{C}^2 - \bigcup_{i=1}^{\mu} C_i))$ can be identified with $\Lambda/(P)$. Therefore for an ideal J containing the principal ideal (P) one has the following sequence:

$$0 \to \Lambda/J \to \Phi(A(C)) \to A(C_\mu) \to 0. \tag{3.6}$$

Then the argument proceeds as in Hillman's proof [6, p. 87] of the corresponding result in the case of links in 3-sphere which gives the relation of Theorem 3.1. □

The following is a generalization of the divisibility theorem from [8].

Let $C = \bigcup_{i=1}^{\mu} C_i$ as above be a reduced algebraic curve in $\mathbb{C}^2$ having μ components. Let S^3 be a sphere in $\mathbb{C}^2$ of a sufficiently large radius (alternatively this S^3 is the boundary of a sufficiently small tubular neighborhood in $\mathbb{CP}^2$ of the line $\mathbb{CP}^2 - \mathbb{C}^2$ in infinity). Let $C_\infty = S^3 \cap C$ be the link of C in infinity. Let

$$K = \operatorname{Ker}(\pi_1(S^3 - C_\infty) \to H_1(\mathbb{C}^2 - C) = Z^\mu).$$

Then K/K' carries the module structure over the ring of Laurent polynomials. The corresponding ith characteristic variety we shall denote as $V_i(S^3 - C_\infty)$.

Theorem 3.2. *The following inclusion takes place* $V_i(\mathbb{C}^2 - C) \subset V_i(S^3 - C_\infty)$.

Proof. As was shown in [8] one has the following surjection of the fundamental groups $\pi_1(S^3 - C_\infty) \to \pi_1(\mathbb{C}^2 - C)$. Therefore $\pi_1'(\mathbb{C}^2 - C)/\pi_1''(\mathbb{C}^2 - C)$ surjects onto K/K'. Hence $\Lambda^i(\pi_1'(\mathbb{C}^2 - C)/\pi_1''(\mathbb{C}^2 - C))$ surjects onto $\Lambda^i(K/K')$ and our claim follows. □

Corollary 3.3. *Let* $C = \bigcup_{i=1}^{\mu} C_i$ *be a curve having* μ *components which are of degrees* $d_1, \ldots, d_\mu$ *such that all components are transversal to the line in infinity. Then* $V_1(\mathbb{C}^2 - C)$ *belongs to the set of zeros of the polynomial* $t_1^{d_1} \cdots t_\mu^{d_\mu} - 1$.

Proof. The transversality condition implies that the link in infinity is the Hopf link L of $l = d_1 + \cdots + d_\mu$ components (i.e., the union of fibers of the Hopf fibration). The fundamental group G_l of this link fits in the following central extension

$$0 \to Z \to G_l \to F(l-1) \to 0 \tag{3.7}$$

where $F(l-1)$ is the free group on $l-1$ generators. Because the corresponding sequence of H_1's is exact (as follows for example from the five term sequence of lower degree terms of the spectral sequence of extension (3.7)) one obtains the isomorphism of G'_l/G''_l and $F'(l-1)/F''(l-1)$ as an Abelian group. Moreover the module structures over $Z[t_1, t_1^{-1}, \ldots, t_l, t_l^{-1}]$ and $Z[t_1, t_1^{-1}, \ldots, t_{l-1}^{-1}]$ agree if one puts $t_1 \cdots t_l = 1$. On the other hand the latter module appears as Coker $\Lambda^{\binom{l}{3}} \to \Lambda^{\binom{l}{2}}$ of terms of Koszul resolution corresponding to the sequence $(t_1 - 1, \ldots, t_l - 1)$. Therefore one has the exact sequence of $\Lambda = Z[Z^{l-1}]$ modules

$$0 \to F'(l-1)/F''(l-1) \to \Lambda^l \to I \to 0. \tag{3.8}$$

Hence

$$\begin{aligned} 0 &\to G'_l/G''_l \to (Z[t_1, \ldots, t_l]/(t_1 \ldots t_l - 1))^l \\ &\to I \otimes Z[t_1, \ldots, t_l]/(t_1 \ldots t_l - 1) \to 0. \end{aligned} \tag{3.9}$$

The spectral sequence of the extension $0 \to G'_l \to K \to Z^{l-\mu}$ in which K defined above fits degenerates in the term E_2 because G_l is free. Hence one has

$$0 \to G'_l/G''_l \to K/K' \to Z^{l-\mu} \to 0 \tag{3.10}$$

which implies that $V_1(S^3 - C_\infty)$ is the set of zeros of $t_1^{d_1} \cdots t_\mu^{d_\mu} - 1$. $\square$

Remark. This corollary provides information on the asymptotic behavior of the Betti numbers when the Galois group of the covering grows. For example if $G_a = \bigoplus_{i=1}^{\mu} Z/nZ$, then the first Betti number is bounded by a polynomial in n of degree $\mu - 1$. This is a generalization of the boundedness of the first Betti number for cyclic covers (cf. [14, 8]).

4. Examples

Example 4.1. Let us show that Theorem 0.1 implies the classical formula for the first Betti number of a finite cyclic covering of a complex with $H_1(X, Z) = Z$ (cf. [12, 14, 8]). Recall that this Betti number for the a-fold cyclic covering of X in terms of the cyclic decomposition of

$$\pi'_1/\pi''_1 \otimes Q = \oplus\, Q[t, t^{-1}]/(\lambda_i) \tag{4.1}$$

(the left-hand side has the module structure described in the Introduction) is increased by one the sum over i of the number of common roots of λ_i and $t^a - 1$. For a module given by (4.1) the matrix of presentation is diagonal with λ_i's on the diagonal and hence the ith Fitting ideal is $(\lambda_i, \dots, \lambda_n)$. From $\lambda_{i+1} | \lambda_i$ it follows that the ith characteristic variety is the set of zeros of i and the classical formula mentioned above follows from Theorem 0.1 (using also the fact that the value of the Alexander polynomial $\Delta(t) = \prod \lambda_i(t)$ at 1 is not zero).

Example 4.2. Let us consider the coverings of the complement in $\mathbb{C}^2$ to the union of two lines. Clearly π_1 of this space is Z^2 and the universal (Abelian) covering is equivalent to a point. Hence the characteristic varieties are empty. Therefore the first Betti number of a finite Abelian covering as in the Introduction is 2.

Example 4.3. Let us consider a covering of $\mathbb{C}^2 - (L_1 \cup L_2 \cup L_3)$ where the L_i's are the lines passing through a fixed point. Suppose that it corresponds to the mod n reduction $Z^3 \to (Z/n)^3$. The argument used in the proof of the corollary to Theorem 3.2 shows that

$$\pi_1'/\pi_1'' \otimes Q = Q[t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}]/(t_1 t_2 t_3 - 1). \tag{4.2}$$

Hence the first characteristic variety is given by $t_1 t_2 t_3 - 1 = 0$ and the next ones are empty. Hence the first Betti number of the covering in question is $n^2 + 2$. (This cover is homeomorphic to the subset of $\mathbb{C}^3$ given by $x^n + y^n + z^n = 0$, $x \neq 0$, $y \neq 0$, $z \neq 0$ and one can verify this answer directly.)

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