Vol. 70, No. 1 DUKE MATHEMATICAL JOURNAL © April 1993

TOPOLOGICAL INVARIANTS OF AFFINE HYPERSURFACES: CONNECTIVITY, ENDS, AND SIGNATURE

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1. Introduction. Let V_0 be a hypersurface in $\mathbf{C}^{n+1}$ given by a polynomial equation $P(z_1, \dots, z_{n+1}) = 0$. Suppose that V_0 has an isolated singularity at a point. In this situation the topology of the part $V_{t,\varepsilon}$ of a perturbed hypersurface $P = t$ (for small t) inside of a ball B_ε of radius ε about the singular point is well understood. In particular, $V_{t,\varepsilon}$ is an $(n-1)$-connected parallelizable $2n$-manifold which comes with a monodromy action on the middle-dimensional homology arising when t varies around a small circle about zero in a t-plane. Along these lines, one obtains a beautiful construction of exotic spheres [Br] and rational homology spheres which appear as the boundaries $\partial V_{t,\varepsilon}$ of $V_{t,\varepsilon}$. The invariants determining the type of $\partial V_{t,\varepsilon}$, such as the above mentioned monodromy or the signature of the intersection form on the middle-dimensional homology of $V_{t,\varepsilon}$, can be obtained in some cases directly from the equation P. In particular for the signature, in the case when P is weighted homogeneous, one obtains a combinatorial formula [Br] related to the Dedekind sums [HZ].

The purpose of this paper is to study another situation associated with a polynomial of several variables which exhibits a similar behavior. To be more precise, under certain conditions the affine hypersurface $P(z_1, \dots, z_{n+1}) = t$ for $|t| > N$ is an $(n-1)$-connected $2n$-manifold for which the essential part of its topology is encoded in the intersection form and in the monodromy action on the n-dimensional homology which is induced by going around a circle of a large radius in the t-plane (monodromy at infinity). In particular, one obtains a construction of homology and homotopy spheres as the ends of affine polynomial hypersurfaces.

The topology of affine hypersurfaces was studied recently by Broughton [Brt] who discovered a class of polynomials, which he called the tame polynomials and for which the affine hypersurface $P = t$ is an $(n-1)$-connected $2n$-manifold. Previously, an estimate for the connectivity of affine hypersurfaces was obtained by M. Kato [Ka]. Our class of polynomials is defined in terms of a resolution of the base points of the pencil of hypersurfaces which are closures in $\mathbf{P}^n$ of the hypersurfaces $P(z_1, \dots, z_{n+1}) = t$. Moreover, we describe a wider class of polynomials for which the homology groups of corresponding hypersurfaces vanish in all dimensions except the middle one (Section 2). The examples of polynomials for which the end of $P = t$ is a homology and homotopy sphere are given in Section 3. The analysis

Received 28 March 1992. Revision received 19 September 1992.
Author supported by the National Science Foundation.

of these examples uses calculation of the characteristic polynomial of the monodromy at infinity acting on a middle-dimensional homology of the hypersurfaces defined by commode polynomials nondegenerate for their Newton polyhedron [LS]. Other ingredients are the fact that an end is $\mathbf{Z}/n$-homology sphere if and only if gcd $(\Delta(1), n) = 1$ where $\Delta(t)$ is the characteristic polynomial of this monodromy and the parallelizability of affine hypersurfaces. The Newton polyhedron here means the convex hull in $\mathbf{R}^{n+1}$ of the origin and the points $(d_1, \ldots, d_{n+1})$ such that the monomial $z_1^{d_1} \cdots z_{n+1}^{d_{n+1}}$ has a nonzero coefficient in the polynomial P. Nondegeneracy has the usual meaning due to Koushnirenko [K]: for any face σ of the Newton polyhedron of P the system $Z_1(\partial P_\sigma/\partial Z_1) = \cdots = Z_{n+1}(\partial P_\sigma/\partial Z_{n+1}) = 0$ has no nonzero solutions where P_σ is the sum of monomials $a_{d_1 \cdots d_{n+1}} z_1^{d_1} \cdots z_{n+1}^{d_{n+1}}$ from P for which $(d_1 \cdots d_{n+1}) \in \sigma$. Note that the connectivity issue of $P = t$ for P which is nondegenerate for its Newton polyhedron has been resolved in [Brt] (see also [Brn], [Kh]).

It should be noted that the situation considered here has an overlap with the local situation mentioned at the beginning. More precisely, if P is a weighted homogeneous polynomial having an isolated singularity at the origin, then its Milnor fibre is diffeomorphic to the affine hypersurface $P = t$ for $t \neq 0$. For the case when $P = z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}}$, E. Brieskorn calculated the signature of the Milnor fibre as the difference of the number of integer $(n + 1)$-tuples $(x_1, \ldots, x_{n+1})$, $0 < x_k < a_k$, such that $0 < \sum_{k=1}^{n+1} (x_k/a_k) \bmod 2 < 1$ and such that $1 < \sum_{k=1}^{n+1} (x_k/a_k) \bmod 2 < 2$ (see [Br], [HZ]). In Section 4 we generalize this calculation by finding the signature of the affine hypersurface $P = t$ (for large $|t|$) when P is nondegenerate for its Newton polyhedron. To describe this formula, recall that for a polyhedron Σ in $\mathbf{R}^n$ with vertices having coordinates belonging to the lattice $\mathbf{Z}^n$, one can define the following sequence of integers. Let $a_n(\Sigma)$ be the number of points belonging to the polyhedron and having coordinates in $(1/n)\mathbf{Z}$. Then the generating function for this sequence, i.e. $\sum_{n=1}^{\infty} a_n(\Sigma) t^n$, is a rational function of t. It has form $N_\Sigma(t)/(1 - t)^{\dim \Sigma + 1}$ (see [St]) where $N_\Sigma(t)$ is a polynomial (of degree $\leqslant \dim \Sigma$). Let $f_j(\Sigma)$ be the number of faces of Σ having codimension j and

$$b_p(\Sigma) = \sum_{j=0}^{n} (-1)^{p-j} \binom{\dim \Sigma - j}{p - j} f_j(\Sigma).$$

Suppose that the Newton polyhedron Σ of P is simple; i.e., nonzero vectors on edges merging into one vertex form a basis of the space spanned by Σ for any vertex of Σ. Then the signature of $P = t$ for $|t|$ large is

$$\sum_{p=0}^{n} 2(-1)^i b_{n-2i}(\Sigma) + b_n(\Sigma) + \sum_{\sigma} (-1)^{\dim \sigma} N_\sigma(-1) \tag{1.1}$$

where the second summation is over all faces of the Newton polyhedron Σ. The case considered by E. Brieskorn corresponds to a simplex and, as we will see, is a specialization of (1.1). The proof is obtained by considering the compactification of

$P = t$ in the toric variety corresponding to the Newton polyhedron of P. As part of the proof of (1.1), we also obtain a formula for the χ_y-characteristic of certain hypersurfaces in toric varieties [KKMS] which generalizes a formula of Hirzebruch and Zagier [HZ] for equivariant χ_y-characteristic of the product of cyclic groups acting diagonally on Fermat hypersurfaces. (Their formula also has Brieskorn's formula mentioned above as an immediate consequence.) This quotient is naturally a hypersurface in a weighted projective space which is the toric variety corresponding to the simplex, and our formula in this context coincides with Hirzebruch-Zagier's (see (4.2)).

We would like to point out that another approach to the calculation of χ_y-characteristic was developed in [DH] and also other recent works [N], [NR], [HL] treating other aspects of the behavior of polynomials at infinity.

Finally, my thanks goes to S. Sperber for bringing my attention to the study of behavior of polynomials at infinity in connection with his and A. Adolphson's work on the estimates of exponential sums. My appreciation also goes to the Institute for Advanced Study, where this work began, for its hospitality and support.

2. Vanishing of homology and homotopy groups of affine hypersurfaces below middle dimension.

2.1. Let $P_d(Z_1, \ldots, z_{n+1})$ be a polynomial of degree d. The projective closures of the affine hypersurfaces $P_d(z_1, \ldots, z_{n+1}) = t$ in $\mathbf{C}^{n+1} \subset \mathbf{P}^{n+1}$ form a linear pencil which in turn defines the rational map $\Phi_P: \mathbf{P}^{n+1} \to \mathbf{P}^1$. The undeterminancy points of Φ_P are the base points of the pencil or, what is the same, the intersection of the projective closure of $P = 0$ with the hyperplane at infinity.

According to Hironaka ([H], p. 141 (*) and p. 142 Main Theorem II), there exists a triple (X, Φ, Ψ) consisting of a nonsingular projective variety X and the morphisms $\Phi: X \to \mathbf{P}^1$ and $\Psi: X \to \mathbf{P}^{n+1}$ such that Ψ is birational and such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & \mathbf{P}^{n+1} \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\Phi_P} & \\ & \mathbf{P}^1 & \end{array}$$

commutes. Moreover, the Ψ-preimage of the hyperplane at infinity is a divisor with normal crossings. In addition, this preimage has the natural stratification. In this stratification the codimension-i strata are connected components of i-fold intersections. Recall that those are the points near which the Ψ-preimage can be given in local coordinates $(x_1, \ldots, x_{n+1})$ by the local equation $x_1 \cdots x_i = 0$. Such a map Φ we shall call a resolution of the base points of P.

2.2. Definition. We say that a polynomial $P(z_1, \ldots, z_{n+1})$ has no singularities at infinity (resp. only isolated singularities at infinity, resp. isolated singularities at infinity which have **Q**-spheres as their links) if there is a resolution of the base points of P such that

(a) the dimension of the singular locus of the part of each fibre of Φ in $\Psi^{-1}(\mathbf{C}^{n+1}) \subset X$ has the dimension of a singular locus at most zero;

(b) each fibre of Φ except the fibre over the point of $\mathbf{P}^1$ at infinity intersects each stratum of the Ψ-preimage of the hyperplane at infinity transversely (resp. intersections with n-dimensional strata have isolated singularities and the intersections with strata of lower dimension are transversal, resp. intersection with n-dimensional strata have only isolated singularities which are homology **Q**-spheres).

Remark. Condition (b) implies that the only fibre of Φ having a common component with the Ψ^{-1}-preimage of the hyperplane at infinity in $\mathbf{P}^{n+1}$ is the fibre over the point of $\mathbf{P}^1$ at infinity.

The values of t for which the fibre of Φ is either singular or is not transversal to a stratum of the stratification of the Ψ-preimage of the hyperplane at infinity is called atypical (see [Brt]). We shall call Φ a good resolution of the base points of the pencil defined by P.

2.3. Examples.

2.3.1. Condition (b) excludes the resolution of the base points of the polynomial $P(x, y) = x^2y + x$, which produces the situation mentioned in the remark above. The projective closure of an element of the corresponding pencil is given by $x^2y + xu^2 = tu^3$. The base locus consists of two points $M(u = 0, x = 0, y = 1)$ and $N(u = 0, x = 1, y = 0)$. At N each curve of the pencil is nonsingular and the multiplicity of intersection of any two curves is equal to 3. After one blowup at M, the pullback of the pencil has the exceptional curve as a fixed component with multiplicity 2. Removal of this component gives the pencil with the base point over M such that all curves, except the one corresponding to $t = 0$, are nonsingular with the mutual multiplicity of intersection equal to 2. The curve corresponding to $t = 0$ has a node at the base point. The resolution of the base points can be achieved by three blowups at N (which produce exceptional curves e_0, e_0', e_0'') and three blowups over M (which produce exceptional curves e_1, e_2, e_3). This leads to the resolution of Figure 1.

Here the fibre corresponding to $t = 0$ is the union of curves $\ell_1\ell_2$ (projecting into components $xy + 1 = 0$ and $x = 0$ of $P = 0$) and e_2. The fibre over $t = \infty$ is the union of the proper preimage of $u = 0$ and e_1, e_0, e_0'.

2.3.2. An example of singularity at infinity. Let $P(X, Y, Z) = XYZ - X - Y$. The corresponding pencil $XYZ - XU^2 - YU^2 = tU^3$ has as the base locus the union of three lines ℓ_1: $[U = 0, X = 0]$, ℓ_2: $[U = 0, Y = 0]$; ℓ_3: $[U = 0, Z = 0]$. All surfaces of the pencil have the singularity of type A_1 at the points P_1: $(1, 0, 0, 0)$, P_2: $(0, 1, 0, 0)$ for any t (the hessian is nonzero at these points), the singularity of type A_2 at P_3: $(0, 0, 1, 0)$ for $t \neq 0$, and the singularity of type A_3 at P_3 for $t = 0$. (The proper preimages of the surfaces of the pencil after the blowup at P_3 for $t \neq 0$

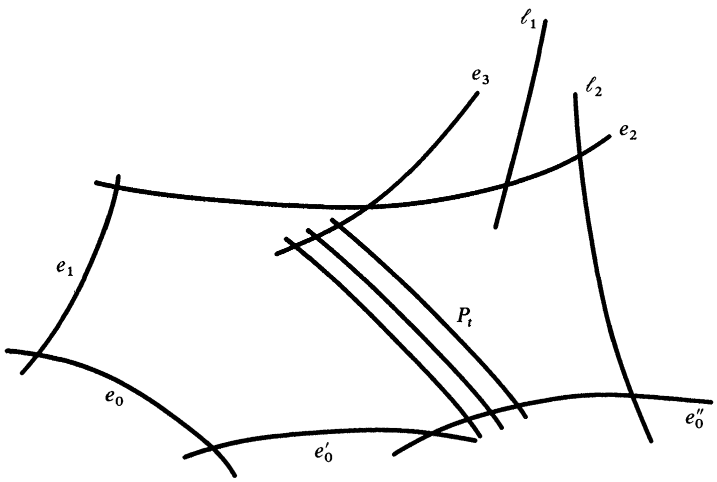

FIGURE 1

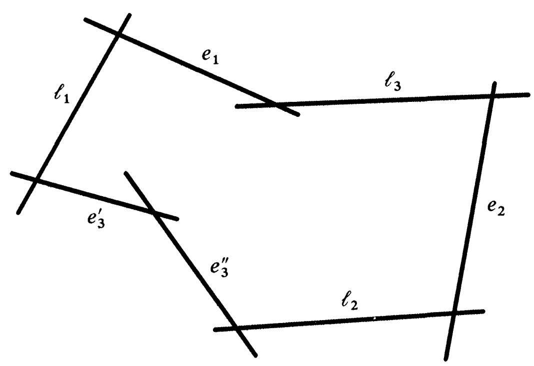

FIGURE 2

are nonsingular and have intersection with the exceptional plane consisting of two lines, and for $t = 0$ these preimages have A_1-singularity.) Blowing up at P_1, P_2, and P_3 adds to the base locus one line, e_1 and e_2, for each of the two points where the surfaces of the pencil have A_1-singularity (corresponding to the tangent cone to the surface which is independent of t) and the pair of lines e_3', e_3'' corresponding to A_2 singularities at P_3 (see Figure 2). All surfaces of the pencil are nonsingular except for the one corresponding to $t = 0$ which has A_1-singularity at the intersection of e_3' and e_3''. Blowing up along one of these lines, say e_3', produces the pencil of nonsingular surfaces transversal for any $t \neq 0$ to the preimage of the hyperplane at infinity $U = 0$, the exceptional planes obtained from blowing up P_1, P_2, P_3, and the exceptional set over e_3'. For $t = 0$ the transversality to the exceptional set over e_3'

fails. Blowups along other components of the base locus decrease the order of tangency of the surfaces of the pencil to each other and eventually give the pencil which is base-point–free and where transversality to the stratum (of codimension 1) of the preimage of $U = 0$ fails for $t = 0$ at one point. (The intersection of the surface with this stratum has a node at the point where transversality fails.) This point gives the isolated singularity at infinity.

2.3.3. Example of polynomial having no singularities at infinity. Let $P = X^2 + Y^5$. The compactification of $\mathbf{P}^2$ of the elements of the corresponding pencil gives the family of curves $X^2 \cdot Z^3 + Y^5 = tZ^5$ which has one base point at $Z = 0$, $X = 1$, $Y = 0$. All the curves of the pencil have at this point the singularity locally equivalent to $u^3 + v^5 = 0$. The multiplicity of the intersection of any two curves of the pencil at this point is 25. The blowup at this point gives the pencil with one base point such that all curves of the pencil have at this point an ordinary cusp (i.e. locally are given by $u^2 + v^3 = 0$) and have the multiplicity of intersection of any pair of curves at this point equal to 12. An additional 12 blowups over the base point will produce the pencil of curves such that any curve from it will intersect only the last of 14 exceptional curves, and these intersections will be transversal.

2.3.4. More generally, if P is any polynomial nondegenerate for its Newton polyhedron, then P does not have singularities at infinity in the sense of 2.2. Indeed, it is shown in [LS] that a toric desingularization of the toric variety canonically corresponding to the Newton polyhedron of P provides the resolution of the base locus with all added strata transversal to the hypersurfaces of the pencil (also see Section 4 below).

2.4. THEOREM. *Let $P(z_1, \ldots, z_{n+1})$ be a polynomial which has no singularities at infinity (resp. only isolated singularities whose links are $\mathbf{Q}$-spheres). Then for any t the hypersurface $P = t$ is $(n-1)$-connected (resp. has vanishing homology with $\mathbf{Q}$-coefficients in all dimensions except n).*

Proof. First, we shall prove this for typical values of t. Let t_0 be such a value and $t_1 \cdots t_k$ be all atypical values of t. The t-plane can be retracted on a sufficiently small regular neighborhood of a system of nonintersecting paths connecting and $t_1 \cdots t_k$ as in Figure 3.

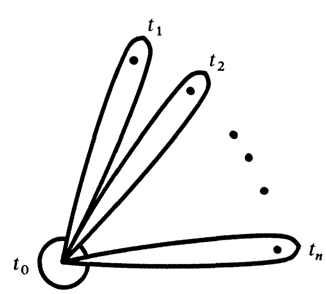

FIGURE 3

It follows from condition (b) in 2.2 that this retraction induces retraction of $\mathbf{C}^{n+1}$ on the space obtained by taking a disjoint union of the neighborhoods $U(P_{t_1}), \ldots, U(P_{t_k})$ of special fibres and identifying them along the chosen generic fibre P_{t_0}. In the case when one does not have singularities at infinity, P_{t_k} is obtained from P_{t_0} by collapsing to a point a complex of dimension n (because the singularity is isolated), i.e. by attaching cells of dimension $(n+1)$. This does not affect the homotopy of dimensions up to $n-1$. Hence $\pi_i(P_{t_0}) = \pi_i(\mathbf{C}^{n+1}) = 0$ for $i \leqslant n-1$.

In the case when links at infinity are $\mathbf{Q}$-spheres, the theorem follows by induction from the following lemma.

LEMMA. *If, as above, $U(P_{t_s})$ is a neighborhood of the atypical fibre corresponding to t_s and V is a union along P_{t_0} of several neighborhoods of atypical fibres corresponding to $t_1, \ldots, t_{s-1}$, then*

$$H_i\Big(V \bigcup_{P_{t_0}} U(P_{t_s}), \mathbf{Q}\Big) = H_i(V, \mathbf{Q}) \qquad \text{for } i \leqslant n-1. \tag{2.4.1}$$

The theorem follows immediately from this result, because this lemma implies that $H_i(P_{t_0}, \mathbf{Q}) = H_i(\mathbf{C}^{n+1}, \mathbf{Q})$ for $i \leqslant n-1$.

On the other hand, the lemma follows from the exact sequence

$$\to H_i(V) \to H_i\Big(V \bigcup_{P_{t_0}} U(P_{t_s})\Big) \to H_i\Big(V \bigcup_{P_{t_0}} U(P_{t_s}, V)\Big) \to H_{i-1}(V) \to , \tag{2.4.2}$$

the excision, and the isomorphism $H_i(U(P_{t_k}), P_{t_0}) = 0$ for $i \leqslant n$. The latter isomorphism is equivalent to

(a) $H_i(P_{t_0}) \to H_i(U(P_{t_k}))$ is an isomorphism for $i \leqslant n-1$;

(b) the map in (a) is surjective for $i = n$.

(a) and (b) are clearly true if t_k is an atypical value corresponding to the fibre $P = t_k$ having isolated singularity in the $\mathbf{C}^{n+1} \subset X$. ($X$, as above, is a good resolution of the base locus of the pencil.) If t_k corresponds to the fibre having a singular point at infinity (i.e., either is singular itself or is nontransversal to a codimension-1 stratum H), let B_∞ be the intersection of a small ball about this singular point with $\mathbf{C}^{n+1} \subset X$ (i.e. B_∞ is equivalent to $(z_1, \ldots, z_{n+1})$ such that $|z_1^2| + \cdots + |z_{n+1}^2| \leqslant 0$, $z_1 \neq 0$). We have the homotopy equivalences

$$\begin{aligned} P_{t_0} - P_{t_0} \cap B_\infty &\simeq U(P_{t_k}) - U(P_{t_k}) \cap B_\infty \\ P_{t_0} \cap \partial B_\infty &\simeq U(P_{t_k}) \cap \partial B_\infty . \end{aligned} \tag{2.4.3}$$

Because P_{t_0} can be assumed to be transversal to ∂B_∞, the maps $H_i(P_{t_0} \cap \partial B_\infty) \to$

$H_i(U(P_{t_k}) \cap \partial B_\infty)$ are isomorphisms for any i. The Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\longrightarrow & H_i(P_{t_0} \cap \partial B_\infty) & \longrightarrow & H_i(P_{t_0} \cap B_\infty) & \oplus & H_i(P_{t_0} - B_\infty \cap P_{t_0}) & \longrightarrow & H_i(P_{t_0}) & \longrightarrow \\
\longrightarrow & H_i(U(P_{t_k}) \cap \partial B_\infty) & \longrightarrow & H_i(U(P_{t_k}) \cap B_\infty) & \oplus & H_i(U(P_{t_k}) - B_\infty \cap U(P_{t_k})) & \longrightarrow & H_i(U(P_{t_k})) & \longrightarrow \\
\longrightarrow & H_{i-1}(P_{t_0} \cap \partial B_\infty) & \longrightarrow & H_{i-1}(P_{t_0}) \cap B_\infty & \oplus & H_{i-1}(P_{t_0} - B_\infty \cap P_{t_0}) & \longrightarrow & & \\
 & \downarrow\wr & & \downarrow & & \downarrow\wr & & & \\
\longrightarrow & H_{i-1}(U(P_{t_k}) \cap \partial B_\infty) & \longrightarrow & H_{i-1}(U(P_{t_k}) \cap B_\infty) & \oplus & H_{i-1}(U(P_{t_k}) - U(P_{t_k}) \cap B_\infty) & \longrightarrow, & &
\end{array} \tag{2.4.4}$$

and the five lemma show that it is enough to verify that $H_i(P_{t_0} \cap B_\infty, \mathbf{Q}) \to H_i(U(P_{t_k}) \cap B_\infty, \mathbf{Q})$ are isomorphisms for $i \leqslant n-1$ and are surjective for $i = n$. In fact, we show that

(c) $H_i(U(P_{t_k}) \cap B_\infty, \mathbf{Q}) = 0$ unless $i = 0, 1$ and $H_1(U(P_{t_k}) \cap B_\infty, \mathbf{Q}) = \mathbf{Q}$; (2.4.5)

(d) $H_i(P_{t_0} \cap B_\infty, \mathbf{Q}) = 0$ for $i \neq n$, $i \neq 0, 1$ and $H_1(P_{t_0} \cap B_\infty, \mathbf{Q}) = \mathbf{Q}$;

and generators of H_1's correspond to each other.

Note that $U(P_{t_k}) \cap B_\infty = \overline{B_\infty} \cap \overline{P_{t_k}} - H \cap \overline{P_{t_k}}$ where $\overline{P_k}$ is the closure of P_{t_k} in X, and here by abuse of notation we use H for $H \cap \overline{B_\infty}$. Indeed, the retraction of a regular neighborhood of $\overline{P_{t_k}}$ inside B_∞ on $\overline{P_{t_k}} \cap B_\infty$ preserves H. In the exact sequence

$$\begin{aligned}
&\to H_{i+1}(\overline{B_\infty} \cap \overline{P_{t_k}}, \overline{B_\infty} \cap \overline{P_{t_k}} - H \cap \overline{P_{t_k}}) \\
&\to H_i(\overline{B_\infty} \cap \overline{P_{t_k}} - H \cap \overline{P_{t_k}}) \\
&\to H_i(\overline{B_\infty} \cap \overline{P_{t_k}}) \to ,
\end{aligned} \tag{2.4.6}$$

one has $H_i(\overline{B_\infty} \cap \overline{P_{t_k}}) = 0$ for $i > 0$ because $\overline{P_{t_k}} \cap \overline{B_\infty}$ is a cone. On the other hand, $H_i(\overline{B_\infty} \cap \overline{P_{t_k}}, \overline{B_\infty} \cap \overline{P_{t_k}} - H \cap \overline{P_{t_k}}) \simeq H_i(T(H \cap \overline{P_{t_k}}), \partial_1)$ where $T(H \cap \overline{P_{t_k}})$ is a regular neighborhood of $H \cap \overline{P_{t_k}}$ inside $\overline{P_{t_k}}$ and ∂_1 is the relevant portion of this boundary (see Figure 4) (i.e. the complement in $\partial T(H \cap \overline{P_{t_k}})$ to a neighborhood of $H \cap \partial T(H \cap \overline{P_{t_k}})$).

The latter group by Poincaré duality with $\mathbf{Q}$-coefficients is isomorphic to $H^{2n-i}(T(H \cap \overline{P_{t_k}}), \partial_2)$ (where $\partial_2 = \partial B_\infty \cap \partial T(H \cap \overline{P_{t_k}})$). Using retraction of $(T(H \cap \overline{P_{t_k}}), \partial_2)$ on $(H \cap \overline{P_{t_k}}, \partial)$ and by applying Poincaré duality again with $\mathbf{Q}$-coefficients, we obtain that the last group is isomorphic to $H_{i-2}(H \cap \overline{P_{t_k}}, \mathbf{Q})$ which is zero unless $i = 2$ and is $\mathbf{Q}$ if $i = 2$. This implies (c).

To derive (d) one similarly considers the exact sequence

$$\begin{aligned}
&H_{i+1}(\overline{B_\infty} \cap \overline{P_{t_0}}, \overline{B_\infty} \cap \overline{P_{t_0}} - H \cap \overline{P_{t_0}}) \\
&\quad \to H_i(\overline{P_{t_0}} \cap \overline{B_\infty} - H \cap \overline{P_{t_0}}) \\
&\quad \to H_i(\overline{B_\infty} \cap \overline{P_{t_0}}) \to .
\end{aligned} \tag{2.4.7}$$

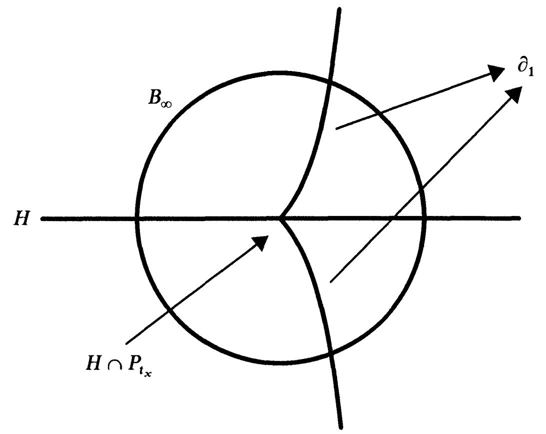

FIGURE 4

We have

$$H_{i+1}(\overline{B_\infty} \cap \overline{P_{t_0}}, \overline{B_\infty} \cap \overline{P_{t_0}} - H \cap \overline{P_{t_0}}) = H_{i+1}(T(H \cap \overline{P_{t_0}}), \partial_1).$$

(Again, $T(H \cap P_{t_0})$ is the regular neighborhood of $H \cap P_{t_0}$ inside $\overline{P_{t_0}}$ and ∂_1 is the portion of the boundary outside ∂B_∞.) The latter group is isomorphic to

$$H^{2n-i-1}(T(H \cap P_{t_0}), \partial_2)(\partial_2 = T(H \cap P_{t_0}) \cap \partial B_\infty)$$
$$= H^{2n-i-1}(H \cap P_{t_0}, \partial(H \cap P_{t_0})) = H_{i-1}(H \cap P_{t_0}).$$

This group is trivial for $0 < i - 1 \leqslant n - 2$ because $H \cap P_{t_0}$ is equivalent to a bouquet of $(n - 1)$-spheres, which implies (d).

The vanishing of homology in dimensions below n for atypical fibres is standard for fibres corresponding to singularities in $\mathbf{C}^{n+1}$ (degeneration corresponds to the collapsing of n-spheres) and because singularities at infinity which we allow do not affect homology in these dimensions (as follows from (c) and (d)).

3. Homology and homotopy spheres as ends of affine hypersurfaces.

3.1. The purpose of this section is to give a condition when the intersection of the sphere of a sufficiently large radius with a hypersurface V_{t_0}: $P(z_1, \ldots, z_{n+1}) = t_0$, where t_0 is a typical value of P, is a homology sphere for homologies with coefficients in $\mathbf{Z}$, $\mathbf{Z}/n$, $\mathbf{Z}$ or $\mathbf{Q}$. The answer is given in terms of the monodromy at infinity, i.e. the operator T on $H_n(V_{t_0}, \mathbf{Z})$ corresponding to the diffeomorphism of V_{t_0} induced by moving t around a loop γ in a t-plane starting at t_0 and containing inside all atypical values $t_1, \ldots, t_k$ (see Figure 3). For $t \in \gamma$, $\Phi^{-1}(t)$ (where Φ: $X \to \mathbf{P}^1$ is a good resolution of $P = t$) is transversal to all strata of the preimage X_∞ of the hyperplane at infinity. So one also has transversality to ∂B_R for a large R because ∂B_R is also the boundary of the regular neighborhood of X_∞. In particular, $\Phi|_{\Phi^{-1}(\gamma)\cap B_R}$ is a locally trivial fibration and the diffeomorphism of monodromy is well defined up to homotopy. The characteristic polynomial of T on $H_n(V_{t_0}, \mathbf{Z})$ will be denoted $\Delta(s)$.

3.2. THEOREM. *Let V_t be the part of an affine hypersurface $P = t$ inside the ball B_k of a sufficiently large radius R. Assume that D has no singularities at infinity. Then the boundary ∂V_t is a $\mathbf{Z}/n$-sphere if and only if* $\gcd(\Delta(1), n) = 1$. *In particular, it is a rational homology sphere if $\Delta(1) \neq 0$ and is a $\mathbf{Z}$-sphere if $\Delta(1) = \pm 1$.*

Proof. First, notice that the geometric monodromy at infinity can be chosen to act trivially on the boundary of V_t. Indeed, this monodromy is a product of the local monodromies about singular points, and it is well known that those can be chosen to act trivially on the boundary. Hence the same is true for the monodromy at infinity. Now using the fact that V_t is homotopy equivalent to a bouquet of spheres (V_t is $(n-1)$-connected by 2.4 and has the homotopy type of an n-dimensional complex), one obtains the exact sequence

$$0 \to H_n(\partial V_t, K) \to H_n(V_t, K) \to H_n(V_t, \partial V_t, K) \to H_{n-1}(\partial V_t, K) \to 0 \qquad (3.2.1)$$

where K is a ring of coefficients. The invariance of $H_n(\partial V_t, K)$ under monodromy implies that it is a subgroup of $\operatorname{Ker}(T - Id)$ where T, as above, is the monodromy operator on $H_n(V_t, K)$. Hence if $\Delta(1)$ is a unit in K then $H_n(\partial V_t, K) = 0$. Hence, by Poincaré duality, $H_{n-1}(\partial V_t, K) = 0$ and the result follows.

3.3. Note that V_t is a parallelizable manifold. Indeed, it is clearly stably parallelizable, has homotopy type of a complex of dimension n, and hence is parallelizable (see [Krv]).

3.4. To construct examples one should have a reasonable class of polynomials for which one can readily calculate the monodromy at infinity. As explained in the introduction, to get something different from the local case one needs a class wider than weighted homogeneous polynomials. We will use the following result from [LS] calculating the monodromy for commode polynomials nondegenerate for its Newton polyhedron. (See Introduction. Recall that commode means that Newton polyhedron contains points on all coordinate axes.)

THEOREM 3.4.1. *Let P be a commode, nondegenerate for its Newton polyhedron. Let $\sigma_1, \ldots, \sigma_N$ be a collection of faces of the Newton polyhedron such that the minimal dimension of coordinate planes (i.e. given by the vanishing of several coordinates) containing σ equals* $\dim \sigma + 1$ *and such that none of $\sigma - 1, \ldots, \sigma_N$ is in the origin. Let $\ell_i = m_i$ be the equations with integer, relatively prime coefficients containing this face. (ℓ_i is the linear form of* $\dim \sigma_i + 1$ *variables.) Let* $\operatorname{vol}(\sigma_i)$ *be the volume of the face σ_i relative to the lattice induced on the plane carrying σ_i by the integral lattice of the space in which the Newton polyhedron of P is constructed. Then the charac-*

teristic polynomial of the monodromy at infinity is equal to

$$\frac{1}{1-z}\prod_i (1 - z^{m_i})^{(\dim \sigma_i)!\,\mathrm{vol}(\sigma_i)(-1)^{\dim \sigma_i}}. \tag{3.4.1.1}$$

We also will need the following corollary from the global analog of the Thom-Sebastiani theorem [Nm].

THEOREM 3.4.2. *Let $P(z_1, \dots, z_{n+1})$ and $Q(w_1, \dots, w_{k+1})$ be two polynomials with disjoint variables which do not have singularities at infinity. Then the monodromy at infinity of $P(z_1, \dots, z_{n+1}) + Q(w_1, \dots, w_{n+1})$ is the tensor product of the monodromies of P and Q.*

In particular, the value at 1 of the characteristic polynomial of $P + Q$ is a product up to the sign of the values of one of the characteristic polynomial in the roots of another.

3.5. Let $A_k(x, y, z) = ax^3 + bx^2y^2 + cy^3 + dz^k((k, 6) = 1)$ be a nondegenerate polynomial. The characteristic polynomial of the monodromy at infinity is

$$\frac{(s^{6k}-1)^2(s^3-1)^2(s^k-1)}{(s^{3k}-1)^2(s^6-1)^2(s-1)}. \tag{3.5.1}$$

Hence the end of $A = t$ is a $\mathbf{Z}/n$ sphere if $(n, k) = 1$. This Newton polygon is not simple, however (see Figure 5). (Recall that a polyhedron is called simple if the

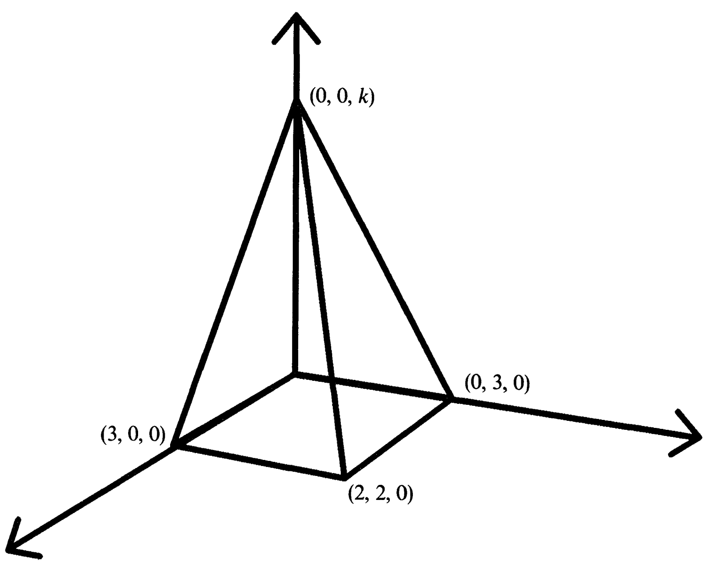

FIGURE 5

minimal vectors with the integral ends on all edges containing each vertex form a **Q**-basis of the ambient space.) Next is an example of a simple Newton polyhedron.

3.6. Let b, ℓ, c be integers such that $\gcd(b, e) = 1$, ℓ-odd, and $\gcd(c - 2b, \ell) = 1$. Let $B_{b,\ell,c}(x, y, z) = \alpha x^2 + \beta y^{2\ell} + \gamma z^c + \delta x z^{2b} + \varepsilon y^\ell z^{2b}$ with generic α, β, γ, δ, ε. Theorem 3.4.1 gives the following for the characteristic polynomial of the monodromy:

$$\Delta_{b,\ell,c}(s) = \frac{(1 - s^{4b\ell})^3(1 - s^{c\ell})(1 - s^{2\ell})(1 - s^2)(1 - s^c)}{(1 - s^{4b\ell})(1 - s^{c\ell})(1 - s^c)(1 - s^{4b})(1 - s^{2\ell})^2(1 - s)}$$

$$= \frac{(1 - s^{4b\ell})^2(1 + s)}{(1 - s^{4b})(1 - s^{2\ell})}. \tag{3.6.1}$$

In particular, $\Delta(1) = 4b\ell$. (Note however that the case $\ell = 1$ can be analyzed alternatively by ad hoc methods. For example, $x' = x + z^2$, $y' = y + z^b$ reduces this to the weighted homogeneous case.) (See Figure 6.)

3.7. The examples in the last two paragraphs give rise to polynomials of larger number variables which have the homotopy spheres as their links at infinity. Let us assume, for example, that $4b \equiv 1 (\text{mod } 3)$ and $\ell \equiv 2 (\text{mod } 3)$.

The characteristic polynomial of the monodromy for u^3 is $s^2 + s + 1$ and $\Delta_{b,\ell,c}(w) = \Delta_{b,\ell,c}(w^2) = 1$. Hence the remark after Theorem 3.4.2 shows that the end

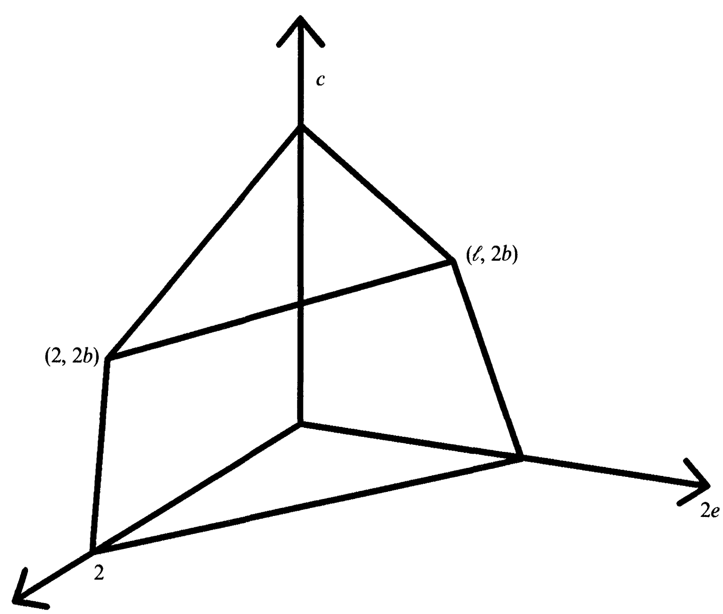

FIGURE 6

of affine hypersurface given by

$$x^2 + y^{2\ell} + z^c + xz^{2b} + y^\ell z^{2b} + u^3 + v_1^2 + \cdots + v_m^2 = t \tag{3.7.1}$$

is a homotopy sphere (m-even) of dimension congruent to $+1 \bmod 4$.

One obtains another example of homotopy spheres at infinity in the case $4b\ell \equiv 1(15), \ell \equiv 1(15)$ by taking the sum of $P_{b,\ell,c}(x, y, z) + u^3 + v^5$. The characteristic polynomial of the monodromy of $u^3 + v^5$ is $(s^{15} - 1)(s - 1)/(s^5 - 1)(s^3 - 1)$, i.e. a polynomial having as its roots the primitive roots of unity of degree 15. If ζ is such a root, then our assumptions imply $\Delta_{b,\ell,c}(\zeta) = 1$. Hence

$$x^2 + y^{2\ell} + z^c + xz^{2b} + y^\ell z^{2b} + u^3 + v^5 + u_1^2 + \cdots + u_m^2 = t \tag{3.7.2}$$

(m-even) is a homotopy sphere of dimension congruent to $-1 \bmod 4$. One can apply the formulas of the next section to determine the differentiable types of these spheres.

3.8. Finally, notice that the homology spheres one obtains are the boundaries of the plumbing of disk bundles over spheres according to a contractible graph. The vertices of this graph correspond to those 2-dimensional orbits of a toric desingularization of the canonical toric variety corresponding to the Newton polyhedron Σ of the polynomial P which do not belong to $\mathbf{C}^3$ containing $\mathbf{P} = t$ (see [LS] and 4.1). Recall that the canonical toric variety is the toric variety corresponding to the fan composed of cones on which the support function of polyhedron (i.e. $h(x) = \min_{y \in \Sigma}(x, y)$ is linear). The corresponding curves are intersections of the closure $P = t$ with each of these orbits $T_1, \ldots, T_N$. The Euler classes of the bundles which one needs to plumb and intersection indices can be found in terms of triple intersections $T_i T_j T_k$ using the fact that the closure of $P = t$ is linearly equivalent to $-\Sigma h(T_i) T_i$. (Here T_i, by abuse of notation, denotes the vector of the fan corresponding to the orbit T_i.) For example, for $B_{1,1,3} = x^2 + y^2 + xz^2 + yz^2 + z^3$ the canonical toric variety can be described as follows. The equations of faces are $x = 0$, $y = 0$, $z = 0$, $-x - y - z = -3$, $-2x - 2y - z = -4$ (where signs in the equations are chosen in such a way that the value of the left-hand side on the face will be less than the value inside the polyhedron). The minimal vectors on the rays of the fan are $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, $e_4 = (-1, -1, -1)$, $e_5 = (-2, -2, -1)$. A triple of vectors forms a cone from the fan if and only if the corresponding faces define a vertex of the polyhedron. Hence the cones of the fan are $C_1:(e_1, e_2, e_3)$, $C_2:(e_1, e_2, e_4)$, $C_3:(e_1, e_3, e_5)$, $C_4:(e_2, e_4, e_5)$, $C_5:(e_1, e_4, e_5)$, $C_6:(e_2, e_3, e_5)$. The only cone which requires a subdivision to obtain a desingularization is C_6. If $e_6 = (-1, -1, 0)$, then $C_6 = C_6' \cup C_6''$ where $C_6' = (e_2, e_3, e_6)$ and $C_6'' = (e_2, e_5, e_6)$. Let h_i be the minimum of e_i on the Newton polyhedron of $B_{1,1,3}$. Then $h_i = 0$ for $1 \leqslant i \leqslant 3$, $h_4 = -3$, $h_5 = -4$, $h_6 = -2$. Hence the closure of $B_{1,1,3}$ in the above desingularization of the canonical toric variety of $B_{1,1,3}$ is linearly equivalent to $3T_4 + 4T_5 + 3T_6$ where T_i is the closure of the orbit corresponding to e_i. Using the fact that $T_i T_j T_k$ $(i \neq j \neq k)$ is equal to one if corresponding faces of the polyhedron

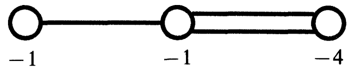

FIGURE 7

have a vertex in common and equal zero otherwise and replacing closures of orbits by linear combinations of different orbits, one obtains the plumbing graph in Figure 7 for the link at infinity.

4. Invariants of affine hypersurfaces nondegenerate for their Newton polyhedron. In this section, after a brief review of toric varieties, we shall calculate the signature of an affine hypersurface given by a polynomial nondegenerate for its Newton polyhedron as well as the χ_y-characteristic of its natural compactification.

4.1. First, we shall recall that a toric variety is an algebraic variety X, say of dimension $(n + 1)$, which contains a torus $T = (\mathbf{C}^*)^{n+1}$ as an open set and such that the action of T on itself extends to the action on X. Let $(\mathbf{Z}^{n+1})_{\mathrm{chr}} = \mathrm{Hom}(T, \mathbf{C}^*)$ be the lattice of characters, $(\mathbf{Z}^{n+1})_{\mathrm{sgp}} = \mathrm{Hom}(\mathbf{C}^*, T)$ is dual to the lattice of 1-parameter subgroups, and $(\mathbf{R}^{n+1})_{\mathrm{chr}} = (\mathbf{Z}^{n+1})_{\mathrm{chr}} \otimes \mathbf{R}$ (resp. $(\mathbf{R}^{n+1})_{\mathrm{sgp}} = (\mathbf{Z}^{n+1})_{\mathrm{sgp}} \otimes \mathbf{R}$). The category of compact toric varieties corresponding to a fixed torus T is equivalent to the category of fans in $(\mathbf{R}^n)_{\mathrm{sgp}}$.

Recall that a fan is a union of a collection of strongly convex rational polyhedral cones (i.e. having form $\Sigma a_i v_i$, $v_i \in (\mathbf{Z}^{n+1})_{\mathrm{sgp}}$, $a_i \in \mathbf{R}$, $a_i \geqslant 0$) such that every face of each cone belongs to the fan and the intersection of any two faces belongs to the fan. (In this correspondence, each cone σ of the fan defines an affine variety which is Spec $k[S]$ where S is the semigroup of characters belonging to the σ. Adjacency of the cones in the fan defines gluing of these affine sets.) To each cone of dimension i corresponds the orbit of T of codimension i. Each divisor is linearly equivalent to the one supported on the closures of codimensional-1 orbits. Assignment of multiplicities along orbits of codimension one defines a function (a support function of the divisor) on $(\mathbf{R}^n)_{\mathrm{sgp}}$ linear on each cone of the fan. Vice versa, a function of $(\mathbf{R}^{n+1})_{\mathrm{sgp}}$ linear on each cone of the fan which takes integral values on the minimal vectors on the rays of the fan, defines the divisor. Classes of linear equivalences of divisors are in 1-to-1 correspondence with the classes of functions as above up to addition of a linear function on $(\mathbf{R}^{n+1})_{\mathrm{sgp}}$. If Σ is a polyhedron in $(\mathbf{R}^{n+1})_{\mathrm{chr}}$ with integral vertices, one associates to it the fan consisting of the cones on which the support function corresponding to Σ: $x \to \min_{v \in \Sigma}(x, v)$ is linear. If Σ is a simple polyhedron (see 3.5, i.e. minimal integral vectors on the edges containing each vertex form that is a basis of $(\mathbf{Z}^{n+1})_{\mathrm{chr}} \otimes \mathbf{Q}$), then the corresponding toric variety is a $\mathbf{Q}$-manifold having only quotient singularities. The cones of the corresponding fan are simplicial ones, and there is 1-to-1 correspondence in which faces of dimension i correspond to cones of codimension i in the fan. The divisor corresponding to the support function of Σ defines a base-point–free linear system (see [O], Theorem 2.22). If $\Sigma(P)$ is the Newton polyhedron of a commode nondegenerate relative to the $\Sigma(P)$

polynomial P, then the projective closures of the elements of the pencil $P = t$ are nonsingular hypersurfaces transversal to all orbits of $(\mathbf{C}^*)^{n+1}$. (I.e., all strata and the single blowup along each component of the base locus of the pencil gives a base-point-free pencil ([LS]).

4.2. Let us describe the numerical data of a polyhedron which we shall use to express the topological information about hypersurfaces. Let $\#\Sigma(j)$ be the number of faces of dimension j of a simple polyhedron. We let $P_\Sigma(t) = \sum_{j=0}^{n+1} \#\Sigma(j)(t-1)^j$. Note that $P_\Sigma(t) = \sum_{p=0}^{n} h^{p,p} t^p$ where $h^{p,p}$ are the Hodge numbers of corresponding toric varieties (i.e. $h^{p,p} = H^p(X, \tilde{\Omega}^p)$ where $\tilde{\Omega}^p = j_* \Omega_U^p$, U the nonsingular locus of X, $j\colon U \hookrightarrow X$ the natural embedding of U) (see [O] (Theorem 3.11)).

The generating function $\sum_{s=0}^{\infty} I_s t^s$ for the number I_s of the points in the lattice $(1/s)(\mathbf{Z}^{n+1})$ which belong to a polyhedron Δ has form $N_\Delta(t)/(1-t)^{\dim \Delta + 1}$ where $N_\Delta(t)$ is a polynomial of degree not exceeding the dimension of Δ.

4.3. Finally, recall that on a manifold (or on a $\mathbf{Q}$-manifold X) with a line bundle $\mathscr{L}$ one defines $\chi^p(X, \mathscr{L}) = \sum_{i=0}^{\dim x} (-1)^i \dim H^i(X, \Omega_X^p \otimes \mathscr{L})$ and $\chi_y(X, \mathscr{L}) = \sum_{p=0}^{\dim X} \chi^p(X, \mathscr{L}) y^p$.

THEOREM. *Let S_Δ be a generic element in the linear system $L(\Delta)$ on a toric variety T_Δ which corresponds to a simple polyhedron Δ. Then*

$$\chi_y(S_\Delta) = \chi_y(T_\Delta) - (-y)^{\dim \Delta} \sum_{\delta} (-1)^{\dim \delta} N_\delta\left(-\frac{1}{y}\right) \tag{4.3.1}$$

where δ runs through all faces of Δ.

Proof. The proof follows from the following series of steps. As usual, $T(\sigma)$ will denote the closure of the orbit corresponding to the cone σ of the fan.

Step 1.

$$\chi^p(T_\Delta, \mathscr{L}) = \sum_{\sigma} (-1)^{\operatorname{codim} \sigma} \binom{n+1-\operatorname{codim} \sigma}{p - \operatorname{codim} \sigma} \chi(T(\sigma), \mathscr{L}) \tag{4.3.2}$$

where $\mathscr{L}$ is a line bundle on T_Δ and summations over all cones σ of the fan define a toric variety T_Δ.

Indeed, the Ishida complex (see [O], p. 120) provides a resolution

$$0 \to \Omega_{T_\Delta}^p \to \mathscr{K}^0(T_{\Delta,p}) \to \mathscr{K}^1(T_{\Delta,p}) \to \cdots \to \mathscr{K}^p(T_{\Delta,p}) \to 0$$

where

$$\mathscr{K}^j(T_{\Delta,p}) = \bigoplus_{\sigma \in F(j)} O_{T(\sigma)} \otimes \Lambda^{p-j}((\mathbf{Z}^{n+1})_{\mathrm{chr}} \cap \sigma^\perp)$$

with a summation running over the collection $F(j)$ of all cones in the fan having dimension j and with $\sigma^\perp$ being the subspace in $(\mathbf{R}^{n+1})_{\mathrm{chr}}$ annihilating $\sigma \in (\mathbf{R}^{n+1})_{\mathrm{sgp}}$.

The right-hand side in the last equality is a free $O_{T(\sigma)}$-mode of rank $\binom{n+1-\dim\sigma}{p-\dim\sigma}$. Hence

$$\chi(\Omega^p_{T_\Delta}, \mathscr{L}) = \sum_\sigma (-1)^{\dim\sigma} \binom{n+1-\dim\sigma}{p-\dim\sigma} \chi(T(\sigma), \mathscr{L}). \tag{4.3.3}$$

Step 2. If $\mathscr{L}_\Delta$ is the line bundle on the toric variety which corresponds to the polyhedron Δ and $P_\delta(i) = \chi(T_\delta, \mathscr{L}^i_\delta)$ in the Hilbert polynomial of the toric variety corresponding to the face δ of Δ, then

$$\chi_y(T_\Delta, S^i_\Delta) = \sum_\sigma (-y)^{\dim\sigma}(1+y)^{\operatorname{codim}\sigma} P_\sigma(i) \tag{4.3.4}$$

where σ runs over all cones of the fan.

Indeed, it follows from Step 1 that

$$\chi^p(T_\Delta, S^i_\Delta) = \sum_\sigma (-1)^{\dim\sigma} \binom{n+1-\dim\sigma}{p-\dim\sigma} P_\sigma(i). \tag{4.3.5}$$

Hence

$$\chi_y(T_\Delta, S^i_\Delta) = \sum_{p=0}^{n+1} y^p \chi^p(T_\Delta, S^i_\Delta) = \sum_\sigma (-1)^{\dim\sigma} P_\sigma(i) \sum_{p=0}^{n+1} y^p \binom{n+1-\dim\sigma}{p-\dim\sigma}$$

$$= \sum_\sigma (-1)^{\dim\sigma} P_\sigma(i) y^{\dim\sigma} \sum_{p=\dim\sigma}^{n+1} \binom{n+1-\dim\sigma}{p-\dim\sigma} y^{p-\dim\sigma}$$

$$= \sum_\sigma P_\sigma(i)(-y)^{\dim\sigma}(1+y)^{n+1-\dim\sigma},$$

as claimed in Step 2.

Step 3. According to Hirzebruch's formula ([Hir], §16, (11*)),

$$\chi_y(S_\Delta) = \sum_{i=0}^{\infty} (-y)^i (\chi_y(T_\Delta, S^i_\Delta) - \chi_y(T, S^{-i-1}))$$

$$= \chi_y(T_\Delta) - (1+y) \sum_{i=0}^{\infty} (-y)^i \chi_y(T_\Delta, S^{-i-1}_\Delta).$$

Using Step 2, we obtain

$$\chi_y(S_\Delta) = \chi_y(T_\Delta) - (1+y) \sum_\sigma (-y)^{\dim\sigma}(1+y)^{\operatorname{codim}\sigma} \sum_{i=0}^{\infty} (-y)^i P_\sigma(-i-1). \tag{4.3.6}$$

Step 4. Now let us consider the generating function $\sum_{n=0}^{\infty} t^n P_\sigma(n)$ for the values of the Hilbert polynomial of the closure of orbit corresponding to the cone σ. This is a rational function having the form $N_\sigma(t)/(1-t)^{\operatorname{codim}\sigma+1}$. From the combinatorial point of view this rational function is the generating function for the number of points of the lattice $(1/n)\mathbf{Z}^{\dim\delta}$ which belong to the face δ of the polyhedron Δ, corresponding to the cone σ of the fan.

We claim that

$$\sum_{i=0}^{\infty}(-y)^i P_\delta(-i-1) = \frac{y^{\dim\delta} N_\delta(-1/y)}{(1+y)^{\dim\delta+1}}. \tag{4.3.7}$$

Indeed, if $P(n)$ is an integer-valued polynomial function vanishing at zero and $Q(t)=\sum_{h=0}^{\infty}P(n)t^n$, then $\sum_{n=0}^{\infty}P(-n)t^n = -Q(1/t)$ as follows from identities $\sum_{h=0}^{\infty}\binom{h}{k}t^n = t^k/(1-t)^{k+1}$, $\sum_{n=0}^{\infty}\binom{-n}{k} = (-1)^k t/(1-t)^{k+1}$, and the fact that an integer-valued function of integer variable is a combination of $\binom{n}{k}$ with integer coefficients. Therefore, if $P(n)$ is a polynomial as above,

$$\sum_{i=0}^{\infty}(-y)^i P(-i-1) = -\frac{1}{y}\sum_{i=0}^{\infty}(-y)^i P(-i) = \frac{1}{y}Q\left(-\frac{1}{y}\right) = \frac{y^{\deg P}N(-1/y)}{(1+y)^{\deg P+1}}.$$

The case of P with $P(0)\neq 0$ follows from the case just considered using the last identity for $\tilde{P}(n)=P(n)-P(0)$.

Step 5. Combining the formula of Step 4 with calculations of Step 3, we obtain

$$\begin{aligned}\chi_y(S_\Delta) &= \chi_y(T_\Delta) - (1+y)\sum_\sigma(-y)^{\dim\sigma}(1+y)^{\operatorname{codim}\sigma}\frac{y^{\operatorname{codim}\sigma}N_\sigma(-1/y)}{(1+y)^{\operatorname{codim}\sigma+1}}\\ &= \chi_y(T_\Delta) - (-y)^{n+1}\sum_\sigma(-1)^{\operatorname{codim}\sigma}N_\sigma\left(-\frac{1}{y}\right)\\ &= \chi_y(T_\Delta) - (-y)^{\dim\Delta}\sum_\sigma(-1)^{\dim\delta}N_\delta\left(-\frac{1}{y}\right).\end{aligned}$$

Q.E.D.

4.4. Corollary (Hirzebruch-Zagier [HZ]). *Let S be the quotient of the hypersurface V_N: $Z_0^N+\cdots+Z_n^N=0$ in $\mathbf{P}^n$ by the action of $G=G_{b_0}\times\cdots\times G_{b_n}$ where G_{b_i} is a cyclic group of order b_i, (assume $\gcd(b_0,\ldots,b_n)=1$, $b_i|N$), acting by multiplication of the ith coordinate by a primitive root of degree b_i of unity. Then $\chi_y(V_N)^G = \sum_{k=0}^{n}(-1)^{k-1}(1-(-1)^n\tilde{N}_k)$ where $\chi_y(V_N)^G = \sum_p \chi^p(V_N)^G y^p$ with $\chi^p(V_N)^G=\sum_i(-1)^i H^i(V_N,\Omega^p_{V_N})^G$ being the alternating sum of G-invariant parts of cohomology and $\tilde{N}_k$ is the number of integer solutions of $j_0b_0+\cdots+j_nb_n=k$ such that $0<j_i<N/b_i$.*

Proof. Note first that $\chi_y(V_N)^G$ is the χ_y-characteristic of the quotient of V_N by the action of G induced from the action of $\mathbf{P}^n$. The action outside of coordinate hyperplanes is free. Hence $\Omega^p_{V_N/G}$, i.e. the direct image of the sheaf of p-forms on the quotient of this complement, is the sheaf of G-invariant forms i.e. $(\Omega^p_{V_N})^G$. The standard spectral sequence gives $\chi_y(V_N)^G = \chi_y(V_N/G)$.

On the other hand, V_N/G is the hypersurface in the weighted projective space (i.e. the quotient of $\mathbf{P}^n$ by the mentioned action of G). This hypersurface can be described as the hypersurface in toric variety corresponding to the polyhedron in the hyperplane in $\mathbf{R}^{n+1}$ given by $b_0x_0 + \cdots + b_ny_n = N$ consisting of points in this hyperplane satisfying $x_i \geqslant 0$ $(i = 0, \ldots, n)$. Recall how it comes about. The group $G = G_{b_0} \times \cdots \times G_{b_n}$ is the subgroup of the torus $\mathbf{T}^n$ and the action of G on $\mathbf{P}^n$ is the restriction of the action of $\mathbf{T}^n$. The quotient $\mathbf{P}^n/G$ is the toric variety for the torus $\mathbf{T}_G = \mathbf{T}^n/G$. Hence for the lattices of characters, one has

$$0 \to \operatorname{Chr} \mathbf{T}_G \to \operatorname{Chr} \mathbf{T} \to G \to 0. \tag{4.4.1}$$

Vice versa, this exact sequence associates with each toric variety corresponding to the polyhedron Δ_G in $\operatorname{Chr} \mathbf{T}_G$, the polyhedron Δ in $\operatorname{Chr} \mathbf{T}$, and the toric variety corresponding to Δ_G in the G-quotient of the toric variety corresponding to Δ. Now $\mathbf{P}^n$ with a diagonal hypersurface of degree N in it corresponds to the polyhedron given by $\tilde{x}_i \geqslant 0$ $(i = 0, \ldots, n)$ inside hyperplane $\sum_{i=0}^n \tilde{x}_i = N$. The map $\tilde{x}_i = b_ix_i$ gives the sequence

$$0 \to L_{b_0, \ldots, b_n} \to L_{1, \ldots, 1} \to G \to 0 \tag{4.4.2}$$

where $L_{b_0, \ldots, b_n}$ is the intersection of integer lattice in $\mathbf{R}^{n+1}$ with the hyperplane $\sum b_ix_i = 0$. The description of weighted projective space as a toric variety now follows.

The formula of Theorem 4.3 in the present case, i.e. the case when the polyhedron is a simplex can be simplified. Indeed, $N_\delta(t) = \sum_{\tau \leqslant \sigma} N^*_\delta(t)$ where $N^*_\delta(t)$ for a simplex which is a convex hull of points $x_0, \ldots, x_d$ has as a coefficient of t^s the cardinality of integer vectors $\sum_{i=0}^d \lambda_i x_i$, $\Sigma\lambda_i = s$, $0 < \lambda_i < 1$ (see [BM], Lemma 3). Hence $\sum_\delta (-1)^{\dim \delta} N_\delta(-1/y) = \sum_\tau N^*_\tau(-1/y)(\sum_{\delta \geqslant \tau} (-1)^{\dim \tau})$. On the other hand,

$$\sum_{\tau \leqslant \delta} (-1)^{\dim \delta} = \begin{cases} 1 & \tau = \varnothing \\ (-1)^{\dim \Delta} & \tau \text{ is whole simplex } \Delta \\ 0 & \text{otherwise}. \end{cases} \tag{4.4.3}$$

Also $\chi_y(T_\Delta) = \sum_{x=0}^n (-y)^k$ because $h^p(T_\Delta, \Omega^q)$, as follows from computation of these numbers for any toric variety (see [O]), depends only on the numerology of faces, i.e. is the same for T_Δ as for $\mathbf{P}^n$. Hence using $N^*(t) = t^{n+1}N^*(1/t)$ (see [BM]), we obtain

$$\chi_y(V_N)^G = \sum_{k=0}^{n} (-y)^k - (-y)^n\left(1 + N_\Delta^*\left(-\frac{1}{y}\right)\right)$$

$$= \sum_{k=1}^{n} (-y)^{k-1} - \frac{1}{(-y)}(-1)^n N_\Delta^*(-y)$$

$$= \sum_{k=1}^{n} (-y)^{k-1}(1 - (-1)^n \tilde{N}_k)$$

because the coefficient of t^k in $N^*(t)$ has interpretation as the number of the integer lattice points of the form $(\dots, \lambda_i(N/b_i), \dots)$ with $\Sigma\lambda_i = s$, $0 < \lambda_i < 1$.

4.5. THEOREM. *Let V_Δ be a generic hypersurface given by a polynomial commode and nondegenerate for its Newton polyhedron Δ and having odd number of variables (i.e. $n = \dim_{\mathbf{R}} V_\Delta \equiv 0 (\mathrm{mod}\ 4)$). Let $P_\Delta(t) = \sum b_i t^i$ be the polynomial corresponding to Δ (see 4.2). Then the signature of V_Δ is given by*

$$\tau(V_\Delta) = 2\sum_{i=1}^{(n-1)/2} (-1)^i b_{n-2i} + b_n - \sum_{\delta} (-1)^{\dim \delta} N_\delta(-1) \tag{4.5.1}$$

where summation is over all the faces of Δ.

Proof. Let S_Δ be the generic element of the linear system on toric variety T_Δ corresponding to toric variety defined by Δ. We have $V_\Delta \subset S_\Delta$.

Step 1. The group $H^n(S_\Delta, V_\Delta)$ coincides with the subgroups of $H^n(S_\Delta)$ which is the image of the map $H^n(T_\Delta) \to H^n(S_\Delta)$ induced by inclusion.

The fact that $H^n(S_\Delta, V_\Delta)$ is a subgroup of $H^n(S_\Delta)$ follows from the exact sequence of the pair and $H^{n-1}(V_\Delta) = 0$ (the latter is because V_Δ is $(n-1)$-connected). Moreover, the image of $H^n(T_\Delta) \to H^n(S_\Delta)$ belongs to the subgroup of invariant cycles (of the pencil defined by the polynomial in question) and because all cycles in $H^n(V_\Delta)$ are vanishing Im $H^n(T_\Delta) \subset$ Im $H^n(S_\Delta, V_\Delta)$. Our claim will follow from equality of the ranks of these groups.

Let $T_\infty = T_\Delta - \mathbf{C}^{n+1}$, $S_\infty = S_\Delta - V_\Delta$ be the parts at infinity. Then S_∞, T_∞ are unions of transversally intersecting divisors and $H^n(S_\Delta, V_\Delta) \simeq H_n(S_\infty)$ (Lefshetz duality and retraction), $H^n(T_\Delta) = H_{n+z}(T_\infty)$. We deduce the required equality of ranks: rk $H^n(S_\infty) =$ rk $H^{n+2}(T_\infty)$ from the Mayer-Vietoris spectral sequences

$$\begin{aligned} E_1^{p,q}(S) &= H^p(S_\infty^{[q]}) \Rightarrow H^n(S_\infty), \\ E_1^{p,q}(T) &= H^p(T_\infty^{[q]}) \Rightarrow H^n(T_\infty), \end{aligned} \tag{4.5.2}$$

where $S_\infty^{[q]}$ (resp. $T_\infty^{[q]}$) denotes the disjoint union of $(q+1)$-fold intersections of irreducible components of S_∞ (resp. T_∞). Moreover, the E_1-terms of these sequences are related by the Gysin maps $H^p(S_\infty^{[q]}) \to H^{p+2}(T_\infty^{[q]})$ which by Poincaré duality correspond to $H_{2n-p-2q-2}(S_\infty^{[q]}) \to H_{2n-p-2-2q}(T_\infty^{[q]})$ which are isomorphisms for $2n - p - 2q - 2 < n - q - 1$; i.e., $p + q > n - 1$ and surjective for $p + q = n - 1$. The diagram

$$\begin{array}{ccc} d_1\colon H^p(S_\infty^{[q]}) & \longrightarrow & H^p(S_\infty^{[q+1]}) \\ \big\downarrow & & \big\downarrow \\ d_1\colon H^p(T_\infty^{[d]}) & \longrightarrow & H^p(T_\infty^{[q+1]}) \end{array} \qquad (4.5.3)$$

commutes and induces isomorphisms on $E_2^{p,q}$ for $p + q \geqslant n$. The rest of the differentials are trivial (see [GS]) which implies isomorphism $H^s(S_\infty) \to H^s(T_\infty)$ for $s \geqslant n$.

Step 2. The restriction of the intersection form in $H^n(S_\Delta)$ on the image of $H^n(T_\Delta)$ coincides with the form $(x, y) \to (x \cap y \cap [S_\Delta])[T_\Delta]$. ($[S_\Delta] \in H^2(T_\Delta)$ is the cohomology class dual to S_Δ, and $[T_\Delta]$ is the fundamental class of T_Δ.) The primitive decomposition gives $H^n(T_\Delta) = \bigoplus P_i \cup [S]^{n/2-i}$ where P_i's are the primitive cohomology of dimension i. The form $(x, y) \to x \cap y \cap [S]^{n-2i+2}$ is definite having signature $(-1)^{(n-i)/2}$. (All classes in $H^i(T_\Delta)$ are algebraic and hence have Hodge type $(i/2, i/2)$.) One has $\dim P_i = b_i(T_\Delta) - b_{i-2}(T_\Delta)$ where $b_i(T_\Delta)$ is the ith Betti number of T_Δ coinciding with the ith coefficient of the polynomial $P_\Delta(t)$. Hence the signature of the intersection form on $H^n(S_\Delta, V_\Delta)$ is $b_n - 2b_{n-2} + 2b_{n-q} \cdots$.

Step 3. Now $S_\Delta = V_\Delta \cup S_\infty$, and the theorem follows from the fact that the signature of S_Δ is $\chi_1(S_\Delta)$ (see [Hir]) and the additivity of the signature.

4.6. Corollary (Brieskorn [Br]). *The signature of the hypersurface* $z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}} = 1$ *is equal to the difference between the number of integer vectors* $(j_1, \ldots, j_{n+1})$ *such that* $0 < j_i < a_i$ $(i = 1, \ldots, n+1)$ *for which* $[\sum_{i=1}^{n=1} j_i/a_i] \equiv 0 \pmod 2$ *and those for which* $[\sum_{i=1}^{n=1} j_i/a_i] \equiv 1 \pmod 2$. (*Here* [ ] *denotes the integer part.*)

Proof. The Newton polyhedron of the polynomial in the left-hand side is simplex $\sum(x_i/a_i) \leqslant 1$, $x_i \geqslant 0$. It is equivalent to the simplex which is the convex hull Σ of the point in $\mathbf{R}^{n+2}$ with coordinates $(0, \ldots, a_i, \ldots, 0)$ $i = 1, n+1, (0, \ldots, k)$ where k is the least common multiple of a_i's. For simplex $b_{2i} = 1$ and as in 4.4,

$$\sum_\delta (-1)^{\dim \delta} N_\delta(-1) = \sum_\tau \left(\sum_{\delta \geqslant \tau} (-1)^{\dim \delta} \right) N_\tau^*(-1) = 1 + N_\Sigma^*(-1).$$

The coefficient of t^s has an interpretation as # of λ_i's such that $(\ldots, \lambda_i a_i, \ldots, j_{n=1} k) \in \mathbf{Z}^{n+2}$ and $\sum \lambda_i = S$. Letting $j_i = \lambda_i a_i$, we obtain the corollary.

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