Lines on Calabi Yau complete intersections, mirror symmetry, and Picard Fuchs equations

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Introduction and statement of the result. It was suggested (cf. [COGP], [GP]) that (in some circumstances) if V is a Calabi–Yau threefold then one can relate to V a family $W(V)_t,\ t \in \mathbf{C}$ of Calabi–Yau manifolds which are "mirrors" of V, such that one has the following relation between their Euler characteristics: $\chi(V) = -\chi(W(V)_t)$. One of the properties of this correspondence should be the following: the coefficients of the expansion of certain integrals attached to $W(V)_t$ (so called Yukawa couplings) relative to an appropriately chosen parameter are integers from which one may calculate the numbers r_d of rational curves of degree d on a generic Calabi–Yau manifold which is a deformation of V. This was verified in [COGP] and [M1],[M2] in the case when V is the quintic hypersurface in $\mathbf{CP}^4$ for rational curves of a small degree. Other authors ([CL],[S]) have suggested a large list of mirrors of hypersurfaces in weighted projective spaces. The purpose of this note is to verify the above predictions for the remaining types of Calabi–Yau *complete intersections* in complex projective space when $d = 1$ i.e. the case of lines.

A Calabi–Yau threefold W is a Kahler manifold such that $\dim W = 3$, the canonical bundle of W is trivial, and the Hodge numbers satisfy $h^{1,0} = h^{2,0} = 0$. Let W_t be a family of such manifolds and let ω_t be a family of holomorphic 3-forms on W_t (unique up to constant for each t because $h^{3,0}(W_t) = 1$). According to Griffiths transversality ([G]):

$$\kappa_t(k) = \int_W \omega \wedge \frac{d^k \omega_t}{dt^k} \tag{1}$$

is equal to zero for $k \leq 2$. Let $\kappa_{ttt} = \kappa_t(3)$. We assume that the monodromy T about $t = \infty$ acting on $H_3(W_t, \mathbf{Z})$ is maximally unipotent i.e. that $(T-I)^3 \neq 0$ and $(T-I)^4 = 0$. If this is the case then ([M1],[M2]) for $N = logT$ one has $dim(ImN^3) \otimes \mathbf{C} = 1$ (as a consequence of $h^{3,0} = 1$). Let $\gamma_1, \gamma_0 \in H_3(W_t, \mathbf{Z})$ be a basis of $(ImN^2) \otimes \mathbf{C}$ such that $\gamma_0 \in (ImN^3)$ is an indivisible element and $\gamma_1 = 1/\lambda N^2 \tilde{\gamma}_1$ where γ_1 is indivisible and the intersection index of $\tilde{\gamma}_1$ and γ_0 is 1. Let m be defined from the relation $N\gamma_1 = m \cdot \gamma_0$ and let

$$s = \frac{\frac{1}{m}\int_{\gamma_1} \omega}{\int_{\gamma_0} \omega},\ q = e^{2\pi i s}. \tag{2}$$

Then q is indepenedent of a choice of the basis γ_0, γ_1 and the form ω up to root of unity of degree $|m|$ (cf. [M1]).

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In this paper, we use an *a priori* different normalization for the parameter s determined by specifying the asymptotic behavior of s as $t \to \infty$. This normalization is described in (18); it is analogous to that exploited in [COGP] and [M2].

Let V_λ be given in $\mathbf{CP}^5$ by

$$Q_1 = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_4 x_5 x_6 = 0$$

$$Q_2 = x_4^3 + x_5^3 + x_6^3 - 3\lambda x_1 x_2 x_3 = 0 \tag{3}$$

This is a complete intersection which is a Calabi–Yau threefold for generic λ . Let $G_{81} \subset PGL(5, \mathbf{C})$ be the subgroup (of order 81) of transformations $g_{\alpha,\beta,\delta,\epsilon,\mu}$ where $\alpha, \beta, \delta, \epsilon \in \mathbf{Z} \mod 3$ $\mu \in \mathbf{Z}_9$ and $3 \cdot \mu = \alpha + \beta = \delta + \epsilon \mod 3$. These transformations act as:

$$g_{\alpha,\beta,\delta,\epsilon,\mu} : (x_1, x_2, x_3, x_4, x_5, x_6) \to$$

$$\to (\zeta_3^\alpha \cdot \zeta_9^\mu \cdot x_1, \zeta_3^\beta \cdot \zeta_9^\mu \cdot x_2, \zeta_9^\mu \cdot x_3, \zeta_3^{-\delta} \cdot \zeta_9^{-\mu} \cdot x_4, \zeta_3^{-\epsilon} \cdot \zeta_9^{-\mu} \cdot x_5, \zeta_9^{-\mu} \cdot x_6) \tag{4}$$

and preserve both hypersurfaces $\tilde{Q}_i$ given by the equations $Q_i = 0$ $(i = 1, 2)$.

Theorem. *The resolution of singularities $W(V_\lambda)$ of the quotient of V_λ by the action of G_{81} which is a Calabi–Yau manifold satisfies: $\chi(V_\lambda) = -\chi(W(V_\lambda))$. The monodromy of $W(V_\lambda)$ about infinity is maximally unipotent. For q defined for the family $W(V_\lambda)$ by the asymptotic normalization (18), the coefficient of q in the q-expansion of κ_{sss} is equal to the number of lines on a generic non singular complete intersection of two cubic hypersurfaces in $\mathbf{CP}^5$.*

A calculation of the Euler characteristic. The statement on Euler characteristic (as well as the statement on the number of lines) are verified by direct calculation of the quantities involved. The total Chern class $c = 1 + c_1 + c_2 + c_3 \in H^*(V_\lambda, \mathbf{Z})$, of the tangent bundle of V_λ satisfies $c \cdot (1 + 3 \cdot h)^2 = (1 + h)^6$ where h is the generator $H^2(V_\lambda, \mathbf{Z})$ (here $(1 + 3 \cdot h)^2$ and $(1 + h)^6$ respectively the total Chern class of the normal bundle to V_λ in $\mathbf{CP}^5$ and the pulback on V_λ of the total Chern class of $\mathbf{CP}^5$) . The Euler charcateristic of V_λ is c_3 evaluated on its fundamental class which (using the fact that h^3 evaluated on the fundamental class is 9) gives $\chi(V_\lambda) = -144$.

On the other hand according to the "physicist's formula" (cf. [DHVW]) or rather to its reformulation due to Hirzebruch and Hofer (cf. [HH]) the Euler characteristic of a Calabi–Yau resolution of the quotient V_λ/G_{81} can be found as

$$\Sigma_{[g]} \chi(V_\lambda^g / C(g)) \tag{5}$$

where the summation is over all conjugacy classes $[g]$ of elements of G, $C(g)$ denotes the centralizer of g and X^g is the fixed point set of an element g. Because G_{81} is abelian the formula reduces to $\Sigma_g \chi(V_\lambda^g / G_{81})$ where the summation is over all elements of the group. There are 6 curves $C_{i,j}$ having non–trivial stabilizer corresponding to the vanishing of two variables in either of the two sets (x_1, x_2, x_3) or (x_4, x_5, x_6). The Euler characteristic of such a curve, which is a complete intersection of two cubic surfaces in $\mathbf{P}^3$, is -18. The stabilizer of each curve contains 3 elements since for each curve there are 2 elements which have this curve as the fixed point set. Hence the number of elements which have one dimensional fixed point set is 12. The Euler characteristic of the quotient of the one dimensional fixed point set is 2. The zero dimensional fixed point sets $D_{i,j,k}$ on a curve $C_{i,j}$ (i, j are in the same group of variables, and k in another) are obtained by equating to zero a variable in another group. The stabilizer of such zero dimensional fixed point set has order 27. Each zero dimensional fixed point set $D_{i,j,k}$ belongs to 3 curves $C_{i,j}$. Hence the number of elements stabilizing $D_{i,j,k}$ is $27 - 3 \times 2 - 1 = 20$. The number of zero dimensional fixed point sets $D_{i,j,k}$ is 6 and each element with zero dimensional fixed point set stabilizes 2 sets $D_{i,j,k}$. Hence the number of elements with zero dimensional stabilizer is $20 \times 6/2 = 60$ and each such element stablizes 6 points. The quotient of a zero dimensional fixed point set of an element by the group has the Euler characteristic equal to 2. The contribution in (5) from the identity element is

$$\chi(V_\lambda/G_{81}) = \frac{1}{|G_{81}|}\Sigma_g \chi(V_\lambda^g)$$

which is equal to $1/81(-144 + 60 \times 6 + 12 \times (-18)) = 0$. Hence using (5) the Euler characteristic of a Calabi–Yau resolution is $0 + 2 \times 60 + 2 \times 12 = +144$.

A method for constructing the Picard Fuchs equations. To find the Picard Fuchs equations for the periods of $W(V_\lambda)$ we shall extend to complete intersections the Griffiths description of cohomology classes of hypersurfaces using meromorphic forms on the ambient space. Let $T(\tilde{Q}_1 \cap \tilde{Q}_2)$ be a small tubular neighbourhood of $\tilde{Q}_1 \cap \tilde{Q}_2$ in $\mathbf{CP}^5$ and $\partial(T(\tilde{Q}_1 \cap \tilde{Q}_2))$ be the boundary of $T(\tilde{Q}_1 \cap \tilde{Q}_2)$. Then

$$H^3(\tilde{Q}_1 \cap \tilde{Q}_2) = H^3(\tilde{Q}_1 \cap \tilde{Q}_2)^* = H^7(T(\tilde{Q}_1 \cap \tilde{Q}_2, \partial T(\tilde{Q}_1 \tilde{Q}_2)) = H^7(\mathbf{CP}^5, \mathbf{CP}^5 - T(\tilde{Q}_1 \cap \tilde{Q}_2))$$

(use Poincare duality, retraction combined with Lefschetz duality, and excision). The latter group is isomorphic to $H^6(\mathbf{CP}^5 - \tilde{Q}_1 \cap \tilde{Q}_2)$ as follows from the exact sequence of the pair. The Mayer Vietoris sequence combined with these isomorphisms gives the identification:

$$H^5(\mathbf{CP}^5 - (\tilde{Q}_1 \cup \tilde{Q}_2)/Im\ (H^5(\mathbf{CP}^5 - \tilde{Q}_1) \oplus H^5(\mathbf{CP}^5 - \tilde{Q}_2)) = H^3(\tilde{Q}_1 \cap \tilde{Q}_2) \quad (7)$$

An alternative description of this isomorphism can be obtained by interpreting a meromorphic 5-form on $\mathbf{CP}^5$ having poles along $\tilde{Q}_1 \cup \tilde{Q}_2$ as a functional on $H_3(\tilde{Q}_1 \cap \tilde{Q}_2)$ which is given by assigning to a 3-cycle γ representing a homology class in the latter group the integral over a 5-cycle in $\mathbf{CP}^5 - (\tilde{Q}_1 \cup \tilde{Q}_2)$; This 5-cycle is the restriction to γ of a torus fibration on which $T(\tilde{Q}_1 \cap \tilde{Q}_2) - (\tilde{Q}_1 \cup \tilde{Q}_2) \cap T(\tilde{Q}_1 \cap \tilde{Q}_2)$ retracts as a consequence of the non–singularity of $\tilde{Q}_1 \cap \tilde{Q}_2$. Moreover in the isomorphism (7) the filtration by the total order of the pole corresponds to the Hodge filtration on $H^3(\tilde{Q}_1 \cap \tilde{Q}_2)$ (details of this will appear elsewhere). The residues of the meromorphic 5-forms which are G_{81}-invariant give the forms on V_λ which descend to V_λ/G_{81}; The pull–back of these forms, which give a basis of $H^3(W(V_\lambda)$, are

$$\frac{(x_1x_2x_3)^{i-1}(x_4x_5x_6)^{n-i-1}\Omega}{Q_1^iQ_2^{n-i}} \tag{8}$$

where $n = 2, 3, 4, 5$ and Ω is the Euler form:

$$\Omega = \sum(-1)^i x_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_6.$$

Calculating the Picard–Fuchs Equation. A cohomology class in $H^3(V_\lambda)$ by (7) is represented by a differential form

$$\eta = \sum_{i=1}^{n} \frac{P_i}{Q_1^iQ_2^{n-i}}\Omega$$

where $\deg(P_i) = 3(n-2)$ and $n \geq 2$. Relations among forms of this type arise from consideration of forms $d\phi$, where

$$\phi = \frac{\sum(x_iA_j - A_ix_j)dx_1 \cdots \hat{dx_i} \cdots \hat{dx_j} \cdots dx_6}{Q_1^iQ_2^j}.$$

The relations take the form

$$\frac{i\sum A_i\frac{\partial Q_1}{\partial x_i}}{Q_1^{i+1}Q_2^j} + \frac{j\sum A_i\frac{\partial Q_2}{\partial x_j}}{Q_1^iQ_2^{j+1}} \equiv \frac{\sum\frac{\partial A_i}{\partial x_i}}{Q_1^iQ_2^j} \quad (\text{mod exact}) \tag{9}$$

In addition, a form with poles along only one of the forms Q_i is equivalent to zero:

$$\frac{P}{Q_i^j}\Omega \equiv 0 \quad (\text{mod exact}) \tag{10}$$

We will now describe a procedure for finding canonical representations for meromorphic forms modulo the relations (1) and (2), by constructing an explicit representation of these relations.

Let J_1 and J_2 represent the rows of the jacobian matrix of (Q_1, Q_2):

$$J_i = \begin{pmatrix} \frac{\partial Q_i}{\partial x_1} & \ldots & \frac{\partial Q_i}{\partial x_6} \end{pmatrix}$$

If $n > 2$ is an integer, we construct an $(n-1) \times 6(n-2)$ matrix B_n as follows:

$$B_n = \begin{pmatrix} (n-2)J_1 & 0 & 0 & \ldots & 0 & 0 \\ J_2 & (n-3)J_1 & 0 & \ldots & 0 & 0 \\ 0 & 2J_2 & (n-4)J_1 & \ldots & 0 & 0 \\ 0 & 0 & 3J_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 2J_1 & 0 \\ 0 & 0 & 0 & 0 & (n-3)J_2 & J_1 \\ 0 & 0 & 0 & 0 & 0 & (n-2)J_2 \end{pmatrix}$$

Let I_{n-1} denote the $(n-1)\times(n-1)$ identity matrix. We consider the module presented by the $(n-1)\times(8n-14)$ matrix $K_n = (B_n\ Q_1 I_{n-1}\ Q_2 I_{n-1})$:

$$S^{8n-14} \to S^{n-1} \to M_n^* \to 0,$$

where S is the graded polynomial ring in the variables $x_1, \dots, x_6$.

When $n > 2$ Let M_n denote the part of M_n^* which is homogeneous of degree $3(n-2)$, and let $M_2 = \mathbf{C}$.

To see the relationship between M_n and $H^3(V_\lambda)$, suppose that ω belongs to $\mathrm{Fil}^i H^3(V_\lambda)$ i.e. the Hodge filtration of ω is i. We may represent ω in the form

$$\omega = (\sum_{k=1}^{i+1} \frac{p_k}{Q_1^k Q_2^{i+1-k}})\Omega.$$

Define a homogeneous map ϕ_i from $\mathrm{Fil}^i H^3(V_\lambda)$ to S^{i+1} by setting $\phi_i(\omega) = (p_1, \dots, p_{i+1})$, and we let $\overline{\phi}_i$ denote the composition of ϕ_i with the projection map to M_{i+2}^*. It follows from our description of the relations (9) and (10) that

$$0 \to Fil^{i-1}H^3(V_\lambda) \to Fil^i H^3(V_\lambda) \to M_{i+2}^* \to 0 \tag{11}$$

is exact. We may now briefly describe an algorithm for putting a cohomology class ω, presented as above, into a standard form. This standard form will consist of elements $m_k(\omega) \in M_k$ for $k = 2, \dots, n+2$ with the property that two forms ω and ω' represent the same cohomology class if and only if $m_k(\omega) = m_k(\omega')$ for all k in this range.

Step 1. Compute Grobner bases for the modules M_n for $n = 0, \dots, i+1$. Such a calculation provides a canonical form for elements of M_{i+2} represented as vectors in S^{i+1}.

Step 2. Reduce the vector $(p_1, \dots, p_{i+1})$ to canonical form modulo the image of I_{i+2} using Step 1. Suppose that $m_{i+2}(\omega)$ is this canonical form. In the reduction process, compute a vector A so that

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{i+1} \end{pmatrix} = m_{i+2}(\omega) + K_{i+2}A.$$

Step 3. Let $A_{i,j}$ denote the subvector of A consisting of the entries $A_i, \dots, A_j$. We denote by $\nabla \cdot A_{i,i+5}$ the usual "divergence" of the 6–vector $A_{i,i+5}$ relative to the x_i: $\nabla \cdot A_{i,i+5} = \Sigma_k \frac{\partial A_k}{\partial x_k}$. Construct a new vector p' in S^i representing $\omega - m_{i+2}(\omega)$ (which, by the lemma, belongs to Fil^{i-1}) by defining:

$$\begin{aligned} p_1 &= \nabla \cdot A_{1,6} + A_{6i+1} \\ p_2 &= \nabla \cdot A_{7,12} + A_{6i+2} + A_{7i+3} \\ &\vdots \\ p_{i-1} &= \nabla \cdot A_{6i-11,6i-6} + A_{7i-1} + A_{8i+1} \\ p_i &= \nabla \cdot A_{6i-5,6i} + A_{8i+2} \end{aligned}$$

Repeat Steps 2 and 3 for p', and continue decreasing i by one each time, until $i = 2$.

We must apply this algorithm in one concrete situation, which we now describe. Define 3-forms ω_i, for $i = 2, 3, \ldots$ by the formula

$$\omega_n = (-1)^n (n-2)! \sum_{i=1}^{n-1} \frac{\lambda^n (x_1 x_2 x_3)^{i-1} (x_4 x_5 x_6)^{n-i-1}}{Q_1^i Q_2^{n-i}} \Omega.$$

These define forms on the complement of $\tilde{Q}_1 \cup \tilde{Q}_2$ which, by the residue construction, define cohomology classes on V_λ invariant under the automorphism group G_{81}. In fact, these forms span the space of G_{81}–invariant three forms on V_λ, and therefore span $H^3(W_\lambda)$ for the mirror manifold.

Let $z = \lambda^{-6}$, so that z is a uniformizing parameter at ∞ for the parameter space of V_λ. In terms of the derivation

$$\Theta = z\frac{d}{dz} = -\frac{1}{6}\lambda\frac{d}{d\lambda} \tag{12}$$

we have the following fundamental relation:

$$\Theta\omega_i = -\frac{i}{6}\omega_i + \omega_{i+1}. \tag{13}$$

It follows from this relation, $rkH^3(W(V_\lambda) = 4$, and the G_{81} invariance of the forms ω_i that ω_6 is dependent on the forms $\omega_2, \ldots, \omega_5$. By analogy with Morrison ([M2]), we postulate a relationship of the following form:

$$\omega_6(z) = \sum_{i=2}^{5} \frac{a_i z + b_i}{z-1} \omega_i(z) \tag{14}$$

where the a_i and b_i are small rational numbers. Once the a_i and b_i are known, it is straightforward to compute the Picard–Fuchs equation as in [M2]. The most powerful tool available for carrying out the calculations described in the reduction algorithm and computing the relation (14) is the Macaulay program of Bayer and Stillman ([Mac]). It has one sizeable limitation which limits its direct application to our problem – it computes Grobner bases over a finite field, whereas at first glance our problem requires computing over the rational function field $\mathbf{C}(\lambda)$. However, if we assume the form of the relation we seek is as in (14), we may avoid this problem by exploiting the Chinese Remainder Theorem:

Step 1. Set the parameter value λ to various *constant* values λ_0 in the finite field $\mathbf{F}_p$. Now use Macaulay to apply the reduction algorithm in the corresponding fiber of the family and find the relations:

$$\omega_6(\lambda_0^{-6}) = \sum h_i(\lambda_0^{-6})\omega_i(\lambda_0^{-6}).$$

Here the h_i are constants in $\mathbf{F}_p$, and these relations are the specializations of the relation (14).

Step 2. Knowledge of the values of the h_i for, say, three distinct λ_0 determines the a_i and b_i mod p. Now repeat the calculation in Step 1 for various different choices of p (again using Macaulay), then apply the Chinese remainder theorem. (This is not totally straightforward, since the a_i and b_i are rational numbers, not integers, and we have no proved *a priori* estimate on their denominators; we guessed that the denominators involved powers of two and three, found some reasonable a_i and b_i, then verified that those coefficients worked for many choices of prime p.)

Using this method, we found the following relation:

$$\omega_6 = \frac{z-7}{3(z-1)}\omega_5 + \frac{z+55}{36(z-1)}\omega_4 + \frac{z-65}{216(z-1)}\omega_3 + \frac{1}{81(z-1)}\omega_2. \tag{15}$$

The associated Picard–Fuchs equation, calculated using this relation and (13), is the generalized hypergeometric equation:

$$(\Theta^4 - z(\Theta+1/3)^2(\Theta+2/3)^2)F = 0 \tag{16}$$

In particular, this implies that the monodromy at $\lambda = \infty$ is maximally unipotent.

Computing the Yukawa Coupling. To determine the expansion of the Yukawa coupling from the equation, we again follow [M2]. The holomorphic solution F_0 to (5) is

$$F_0(z) = \sum_{n=0}^{\infty} \left(\frac{(3n)!}{(n!)^3}\right)^2 \left(\frac{z}{3^6}\right)^n .$$

We let F_1 denote the unique solution to (16) which involves $\log(z)$ (but no higher powers of $\log(z)$) and such that

$$s(z) = F_1(z)/F_0(z)$$

has the property

$$s(z) \sim \log(3^{-6}z) = -6\log(3\lambda) \quad \text{as } z \to 0. \tag{17}$$

(This is the asymptotic normalization mentioned at the beginning of the paper, in this special case.) If we let

$$W = F_0\Theta F_1 - F_1\Theta F_0$$

then the Yukawa potential κ_{sss}, expressed in the canonical parameter $q(z) = \exp(s(z))$, and normalized so that its leading term is 9 (=the degree of our Calabi–Yau family V_λ) is

$$\kappa_{sss} = -9\frac{F_0^4}{W^3(s(q)-1)}.$$

To determine the predicted number of rational curves of given degree, we write κ_{sss} in the form

$$\kappa_{sss} = 9 + \sum \frac{n_d d^3 q^d}{1-q^d}.$$

With our choices of normalization, we obtain integral values for the n_d, and record them in Table 1.

Extrapolations. We know that the Picard–Fuchs equation associated to the quintic hypersurface is the generalized hypergeometric equation with parameters $\{1/5, \ldots, 4/5\}$, while that for the complete intersection of two cubics is the hypergeometric equation with parameters $\{1/3, 1/3, 2/3, 2/3\}$. It seems reasonable us to extrapolate from this that the equations for the remaining types of Calabi–Yau complete intersections are hypergeometric as well; with parameters as given in the following table:

Description	Parameters
4 quadrics in $\mathbf{P}^7$	$\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$
2 quadrics and cubic in $\mathbf{P}^6$	$\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$
2 cubics in $\mathbf{P}^5$	$\{\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\}$
Quartic and quadric in $\mathbf{P}^5$	$\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}\}$
Quintic in $\mathbf{P}^4$	$\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$

Based on this hypothesis we calculated the Yukawa potential in each of these cases. There are two constants which must be chosen for each such calculation; one of these forces κ_{sss} to have initial term the degree of the variety, while the other determines the asymptotic behavior of the coordinate s in terms of the "hypergeometric" variable z as in equation (16). In each case, we made the choice

$$s(z) \sim \log(z) - \sum d_i \log(d_i) \tag{18}$$

where the d_i are the degrees of the hypersurfaces defining the complete intersection. With these choices, we obtained the correct values for the number of straight lines in each case, and integral values for the predicted number of rational curves. The results of our calculations are summarized in the Table 1.

Table 1.
Numerical Results

Predicted Number of Rational Curves of Given Degree
For Various Types of Complete Intersection Calabi–Yau Manifolds

Degree	$V_{3,3} \subset \mathbf{P}^5$	$V_{2,4} \subset \mathbf{P}^5$
1	1053*	1280*
2	52812	92288
3	6424326	15655168
4	1139448384	3883902528
5	249787892583	1190923282176
6	62660964509532	417874605342336
7	17256453900822009	160964588281789696
8	5088842568426162960	66392895625625639488
9	1581250717976557887945	28855060316616488359936
10	512045241907209106828608	13069047760169269024822656
Degree	$V_{2,2,2,2} \subset \mathbf{P}^7$	$V_{2,2,3} \subset \mathbf{P}^6$
1	512*	720*
2	9728	22428
3	416256	1611504
4	25703936	168199200
5	1957983744	21676931712
6	170535923200	3195557904564
7	16300354777600	517064870788848
8	1668063096387072	89580965599606752
9	179845756064329728	16352303769375910848
10	20206497983891554816	3110686153486233022944

(*) These numbers coincide with those given in [L] p. 52. The number of lines on $V_{2,2,2,2}$ (resp. $V_{2,2,3}$) is not given explicitly there (only as part of theorem 3). It is easy to check that the lines belonging to a quadric in $\mathbf{P}^7$ form a cycle on the Grassmanian $Gr(1,7)$ of lines in $\mathbf{P}^7$ which is homologous to $4\Omega_{4,6}$ ($\Omega_{p,q}$ denotes the Schubert cycle consisting of lines in a generic $\mathbf{P}^q$ intersecting generic $\mathbf{P}^p \subset \mathbf{P}^q$). Its 4-fold self–intersection equals 512, which gives the number of lines on $V_{2,2,2,2}$. On the other hand, the lines in $\mathbf{P}^6$ which belong to a generic hypersurface of degree 3 (resp. 2) form the cycle in $Gr(1,6)$ homologous to $18\Omega_{2,5} + 27\Omega_{3,4}$ (resp. $4\Omega_{3,5}$). The intersection index: $(18\Omega_{2,5} + 27\Omega_{3,4})(4\Omega_{3,5})^2$ equals 720 which gives the number of lines on $V_{3,2,2}$.

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P.S. Besides calculation in [L] of the number of lines on generic complete intersection of arbitrary dimension on the case when it is finite the case of lines on three dimensional complete intersections with $K = 0$ also treated in S.Katz paper in Math. Zeit. 191 (1986) .293-296. We are thanking S.Katz for pointing this out.