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Groups Which Cannot be Realized as Fundamental Groups of the Complements to Hypersurfaces in C^N

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12.1 Introduction

Finding restrictions imposed on a group by the fact that it can appear as a fundamental group of a smooth algebraic variety is an important problem particularly attributed to J.P. Serre. It has rather different aspects in characteristic p and zero and here we will address exclusively the latter case. Most restrictions described in the literature seem to rely on Hodge theory or some clever use of it (cf. [2]). A prototype of such restrictions is evenness of $rk(\pi_1 \,/\, \pi_1' \bigotimes \mathbf{Q})$ where π_1' is the commutator subgroup of the fundamental group π_1. In the case of open non-singular varieties one can apply mixed Hodge theory. This was done by J. Morgan who obtained restrictions on the nilpotent quotients of the fundamental groups [13]. Here I shall describe a different (but also by no means complete) type of restriction on the fundamental groups of open varieties which are complements to hypersurfaces in $\mathbf{C}^n$. It is implicitly contained in previous work on Alexander polynomial of plane curves [4]. For example many knot groups cannot occur as fundamental groups of the complement to an algebraic curve. This gives automatically the same restrictions on the fundamental groups of complements to arbitrary hypersurfaces in $\mathbf{C}^n$ as follows from the well-known argument using Zariski Lefschetz type theorem: For a generic plane H relative to given hypersurface V in $\mathbf{C}^n$ the natural map $\pi_1(H - H \cap V,\ p_0) \to \pi_1(\mathbf{C}^n - V, p_0)(p_0 \in H)$ is an isomorphism, i.e., possible fundamental groups of the complement to hypersurfaces in $\mathbf{C}^n$ are precisely the fundamental groups of the complements to plane curves. Therefore from now on I shall work with plane curves only.

12.2 Alexander Polynomials of Plane Curves

So let C be such a curve having, say, μ components. Let μ denote the number of irreducible components of such a curve C and d be its degree. Then according to van Kampen's theorem [10] $\pi_1(\mathbf{C}^2 - C)$ is generated by d generators belonging to a generic line L which are images of the standard generators of $\pi_1(L - L \cap C)$ (the latter is a free group on d generators). Here the standard generator is a loop which is formed by a path leading from a base point to a point near $C \cap L$, then going once around the nearby poaint of $C \cap L$ and then returning back to the base point traversing the old path in the opposite direction. Generators corresponding to two points of $C \cap L$ belonging to the same irreducible component of C are conjugate in $\pi_1(\mathbf{C}^2 - C)$. The conjugating loop is composed of the path leading from the base point to the vicinity of the first point, a path leading from the vacinity of the first point to the vicinity of the second point which is a pushout into $\mathbf{C}^2 - C$ of a path in the locus of non-singular points of C connecting the first and second points and then returning to the base point along the path comprising the standard generator corresponding to the second point.

In particular this implies that $H_1(\mathbf{C}^2 - C, \mathbf{Z})$ is generated by at most μ generators. On the other hand, linking coefficients with each irreducible component of C provide surjection of this group on $\mathbf{Z}^\mu$. Hence $H_1(\mathbf{C}^2 - C, \mathbf{Z}) = \mathbf{Z}^\mu$ and consequently $\pi_1(\mathbf{C}^2 - \mathbf{C})$ is a normal closure of its μ elements where $\mu = rk\pi_1 / \pi_1'$. This simple (topological) condition prohibits some groups (like free products of $\mathbf{Z}$ with perfect groups) from being fundamental groups of the complement to plane curves.

To state an algebro-geometric restriction, let us consider the homomorphism $\phi : H_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ given by the total linking number with C and the infinite cyclic cover determined by ϕ. Denote it $(\widetilde{\mathbf{C}^2 - C})_\phi$. Then $H_1(\widetilde{\mathbf{C}^2 - \mathbf{C}})_\phi, \mathbf{Q}$ is a module over the group ring of $\mathbf{Z}$ over $\mathbf{Q}$, i.e., over $![t, t^{-1}]$, which as it turns out is a torsion module over this ring (cf. [4]). Hence it is isomorphic to $\bigoplus \mathbf{Q}[t, t^{-1}] / (\lambda_i)$ for some Laurent polynomials λ_i determined up to a unit of $\mathbf{Q}[t, t^{-1}]$. The order $\Delta = \Pi \lambda_i$ of $H_1(\widetilde{\mathbf{C}^2 - \mathbf{C}})_\phi, \mathbf{Q}$ is called the Alexander polynomial of curve C. The special property which is satisfied by the fundamental groups of the complement to plane curves is contained in the following theorem:

Theorem. *The Alexander polynomial Δ of a curve is cyclotomic. In fact Δ divides the product of the characteristic polynomials of monodromies of all singular points of the union of C and the line in infinity.*

In the case when curve C is transversal to the line in infinity one can give a very simple proof of the cyclotomic property of Δ (in this restricted case Kohno's algebraic description of the Alexander polynomial [10] or Randell's interpretation [8] of it as the characteristic polynomial of the monodromy operator of the 2-dimensional singularity defined by the equation of the curve can be used). In fact one has the following:

Proposition

If curve C is transversal to the line in infinity then the automorphism of $H_1(\widetilde{(\mathbf{C}^2 - C)}_\phi, \mathbf{Q})$ induced by the deck transformation of the infinite cyclic cover has a finite order dividing the order of the curve C.

To see this, let us put $\pi_{1,a} = \pi_1(\mathbf{C}^2 - C)$ and $\pi_1 = \pi_1(\mathbf{CP}^2 - \overline{C})$ where $\overline{C}$ is the projective completion of C in $\mathbf{CP}^2$. Then one has the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \pi_1^\phi & \to & \pi_1 & \to & \mathbf{Z}/d\mathbf{Z} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \pi_{1,a}^\phi & \to & \pi_{1,a} & \to & \mathbf{Z} & \to & 0 \end{array}$$

The surjections on the right ends of each row are homomorphisms of the fundamental group on H_1 for $\mathbf{CP}^2 - \overline{C}$ and $\mathbf{C}^2$ respectively. The groups on the left side are the kernels of these surjections. The middle vertical arrow is induced by inclusion and is a surjection because any loop in $\mathbf{CP}^2 - \overline{C}$ can be moved to miss the line in infinity. π_1^ϕ and $\pi_{1,a}^\phi$ are the fundamental groups of the cyclic coverings having as the Galois groups $\mathbf{Z}/d\mathbf{Z}$ and $\mathbf{Z}$ respectively. The action of the Galois group on the homology of each of these cyclic coverings coincides with the action induced by each of these two sequences on the abelianization of the left terms. We will need the following (cf. [11], p. 509):

Lemma *If C is transversal to the line in infinity then* $\mathrm{Ker}(\pi_{1,a} \to \pi_1) = \mathbf{Z}$ *and belongs to the center of* $\pi_{1,a}$.

We shall postpone the proof for a moment. This lemma implies that the left arrow in the diagram above is an isomorphism because the kernel of the middle map injects by the map ϕ into $\mathbf{Z}$. Hence $H_1(\widetilde{(\mathbf{C}^2 - C)}_\phi, \mathbf{Z} \simeq \pi_{1,a}^\phi)/(\pi)1, a^\phi)' \simeq \pi_1^\phi/(\pi_1^\phi)' \simeq H_1(\widetilde{\mathbf{CP}^2 - C}_\phi, \mathbf{Z})$. Moreover the action of the generator of $\mathbf{Z}$ in the diagram above on abelianization $\pi_{1,a}^\phi$ coincides with the action of $\mathbf{Z}/d\mathbf{Z}$ on the abelianization of π_1/π_1' under the above isomorphism and our claim about the order of the action of the automorphism induced by deck transformation follows.

As far as the proof of the lemma is concerned notice that the fact that the kernel of $\pi_1(\mathbf{C}^2 - C) \to \pi_1(\mathbf{CP}^2 - C)$ is a normal closure of a loop γ in the vicinity of the line in infinity L_∞ which surrounds this line follows from the comparison of van Kampen's presentations of the fundamental groups of the complement in $\mathbf{CP}^2$ to $C \cup L_\infty$ and C which use as generators the standard generators of $\pi_1(D - D \cap C \cap L_\infty)$ for a generic line D. To see that γ is in the center of $\pi_1(\mathbf{C}^2 - C)$, select a line L_t which is small perturbation of L_∞ and write the van Kampen presentation for $\pi_1(\mathbf{C}^2 - C, L_t \cap D)$ usiong generators in $\pi_1(L_t - \cap L_t(C \cup L_\infty))$. Let D_∞ be a small disk in D about $D \cap L_\infty$, parameterizing lines in the pencil of lines containing L_∞ and L_t.

Then the union of affine parts (i.e., in $\mathbf{C}^2$) of these lines minus $C \cup L_\infty$ is the trivial fibration over $D_\infty - D_\infty \cap L_\infty$ with fibre $\mathbf{C}$ minus d points because this fibration extends over D_∞, (as follows from our assumption that C is transversal to L_∞). The generator of π_1 $(D_\infty$ which can be identified with v hence commutes with all generators of $(\mathbf{C}^2 - C)$ and the lemma follows.

Note that if the line in infinity is not transversal to the curve then the relationship between $\pi_1(\mathbf{C}^2 - C)$ and $\pi_1(\mathbf{CP}^2 - C)$ is more subtle. For example if C is given in $\mathbf{C}^2$ by equation $x^2 = y^3$ then $\pi_1(\mathbf{C}^2 - c) = \{a, b, /aba = bab\}$ and $\pi_1(\mathbf{CP}^2 - C) = \mathbf{Z}/3\mathbf{Z}$ as follows for example from Abhyankar's calculations [1].

A series of calculations of the fundamental groups of the complements is discussed in [5]. In particular the class of possible fundamental groups includes braid groups (Zariski, Moishezon) and groups of torus knots of type (p, q) (M. Oka). The cyclotomic property of Alexander polynomials of algebraic curves discussed above prohibits, however, many knot or link groups to be fundamental groups of an algebraic curve. For example, the fundamental group of figure eight knot (Alexander polynomial is $t^2 - 3t + 1$) cannot occur as a fundamental group of a complement to a plane algebraic curve. The lower central series of this group at the same time stabilizes in the second term. Indeed if Γ_i is the lower central series of π_1 (i.e., $\Gamma_1 = \pi_1$ and $\Gamma_{i+1} = [\pi_1, \Gamma_i]$ then $Hom(\Gamma_2/\Gamma_3, \mathbf{Z})$ can be identified with the kernel of the cup product map: $\Lambda^2 H^1 \to H^2$ (cf. [8]) which is trivial for a space with $H^1 = \mathbf{Z}$. Hence the methods of [13] does not outrule these groups as the fundamental groups of an open algebraic variety.

Finally note that the theory of Alexander polynomials of plane curves has a high dimensional generalization in which the first homology of the infinite cyclic cover are replaced by the first non vanishing homotopy group of the infinite cyclic cover which imposes a non-trivial restriction on the possible homotopy types of the complements to hypersurfaces in $\mathbf{C}^n$ (cf. [6]).

12.3 References

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