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Homotopy groups of the complements to singular hypersurfaces, II

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Introduction

The fundamental groups of complements to algebraic curves in $\mathbb{CP}^2$ were studied by O. Zariski almost 60 years ago (cf. [Z]). He showed that these groups are affected by the type and position of singularities. Zariski and Van Kampen (see [K]) described a general procedure for calculating these groups in terms of the behavior of the intersection of the curve with a generic line as one varies this line in a pencil. For the curves with mild and few singularities, these fundamental groups are abelian. For example, if C is an irreducible curve, having only singular points near which C can be given in some coordinate system by the equation $x^2 = y^2$, then its complement has an abelian fundamental group (cf. [F], [D]). On the other hand, one knows that there is an abundance of curves with nonabelian fundamental groups of complements (for example, branching curves of generic projections on a plane of surfaces embedded in some projective space). Some explicit calculations were made by Zariski. For example, for the curve given by the equation $f_2^3 + f_3^2 = 0$, where $f_k(z_0, z_1, z_2)$ is a generic form of degree k, the corresponding fundamental group is $\mathrm{PSL}_2(\mathbb{Z})$ (cf. [Z]).

If one thinks of the high-dimensional analog of these results, one can immediately notice that the class of fundamental groups of the complements to hypersurfaces in a projective space coincides with the class of fundamental groups of complements to the curves. Indeed by a Zariski–Lefschetz-type theorem (cf. [H]) the fundamental group of the complement to a hypersurface V is the same as the fundamental group of the complement to the intersection $V \cap H$ inside H for a generic plane H. In this article we will show, however, that the homotopy group $\pi_{n-k}(\mathbb{C}^{n+1} - V)$, where k is the dimension of the singular locus of V, exhibits properties rather similar to those properties of the fundamental group discovered by Zariski. (By abuse of notation we

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often will omit specifying the base point in homotopy groups, except when doing so causes confusion.) Actually, as in the case of curves (cf. [Z], [K]), we study a somewhat more general case of hypersurfaces in affine space (which is motivated by a desire to apply these results to the covers of $\mathbb{CP}^{n+1}$ of arbitrary degree, the branching locus of which contains V). It appears that the group $\pi_{n-k}(\mathbb{CP}^{n+1} - (V \cup H))$, where H is "a hyperplane at infinity", at least after taking a tensor product with $\mathbb{Q}$, has an algebro-geometric (rather than a homotopy-theoretic) meaning: it depends on the "local type and position" of singularities of a section of V by a generic linear subspace of $\mathbb{C}^{n+1}$ of codimension k (see Examples 5.4). This group has a description in the spirit of geometric topology similar to the one given by the van Kampen theorem in which the Artin braid group is replaced by a certain generalization (see Section 2). For the most part we allow V to have a certain type of singular behavior at infinity (see Definition 1.4). However the information on the homotopy groups we obtain in the affine situation when V is nonsingular at infinity, as we show below, is equivalent to the information on the homotopy groups of complements to projective hypersurfaces (see Lemma 1.13). An a priori construction of hypersurfaces with isolated singularities and nontrivial $\pi_n(\mathbb{C}^{n+1} - V)$, like the one with generic projections mentioned above in the case of fundamental groups, seems absent in our context (though the case of weighted homogeneous hypersurfaces discussed in Section 1 provides a class of hypersurfaces with one singular point in $\mathbb{C}^{n+1}$ and nontrivial π_n). Nevertheless one can, starting from the equation of curves with nontrivial fundamental groups of the complements, construct equations of hypersurfaces having a nontrivial higher homotopy of their complements (see Examples 5.4). In particular one obtains hypersurfaces with the same local data, but with distinct higher homotopy groups.

A more detailed summary of this article is the following: In Section 1 we start with a study of the complements to nonsingular hypersurfaces in $\mathbb{C}^{n+1}$. This, in particular, implies that the dimension $n-k$, where k as above is the dimension of the singular locus of V, is the lowest in which nontrivial homotopy groups π_i $(i \geq 2)$ can appear. It also allows one to reduce the study of π_{n-k} to the case when V has only isolated singularities. Moreover we also derive a relation between $\pi_n(\mathbb{CP}^{n+1} - V)$ and $\pi_n(\mathbb{C}^{n+1} - V)$ where V is a hypersurface with isolated singularities and no singularities at infinity.

In Section 2 we outline a procedure for finding $\pi_n(\mathbb{C}^{n+1} - V)$ using a generic pencil of hyperplane sections of V. Such a pencil defines the geometric monodromy homomorphism of the fundamental group of the space of parameters of the pencil corresponding to nonsingular members of this pencil. The target of this homomorphism is the fundamental group of the space of certain embeddings of $V \cap H$ into $H = \mathbb{C}^n$, where H is a generic element of the pencil. The latter group has a natural linear representation over $\mathbb{Z}[t, t^{-1}]$. Then the homotopy group in question is expressed in terms of the geometric monodromy and this linear representation (in some cases, information on certain degeneration operators will be needed). In the case of curves, the whole procedure coincides with van Kampen's (cf. [K]); the group of embeddings is the Artin braid group and the geometric monodromy is the braid monodromy (cf. [Mo]). Theorem 2.4 reduces then to van Kampen's theorem. The linear representation mentioned above is the classical Burau representation, and the object which Theorem 2.4 calculates is the Alexander module of the curve (cf. [L2], [L4]).

Section 3 describes a necessary condition for the vanishing of $\pi_n(\mathbb{C}^{n+1} - V)$. Use of this result shows that in some cases the contribution from the degeneration operators, appearing in the description of $\pi_n(\mathbb{C}^{n+1} - V)$ via pencils in Section 2, can be omitted. The vanishing results here are parallel to those on the commutativity of the fundamental groups of complements in the case of curves (cf. [Ab], [N]). The issue of explicit numerical conditions on singularities, which will assure the vanishing of the homotopy groups, is more algebro-geometric in nature than most issues treated here and will be discussed elsewhere. Note also that another vanishing result for π_2 of the complement to an image of a generic projection was obtained in [L3].

The next section, Section 4, gives restrictions on $\pi_n(\mathbb{C}^{n+1} - V)$ imposed by the local type of singularities and the behavior of V at infinity. These results are generalizations of divisibility theorems for Alexander polynomials in [L2]. As a corollary, we show that in the case of the absence of singularities at infinity the order of $\pi_n(\mathbb{C}^{n+1} - V) \otimes \mathbb{Q}$ coincides with the characteristic polynomial of the monodromy operator acting on H_n of the Milnor fiber of a (nonisolated) singularity of the defining equation of V at the origin in $\mathbb{C}^{n+2}$. These characteristic polynomials were also considered in [Di1], [Di2].

In the last section, Section 5, we give two methods for constructing hypersurfaces for which $\pi_n(\mathbb{C}^{n+1} - V) \neq 0$, and we calculate the homotopy groups in these cases. The first method is based on a generalization of Zariski's example, mentioned in the first paragraph of this Introduction. We show that if f_k denotes a generic form of degree k of $n+2$ variables, p_i $(i = 1, \ldots, n+1)$ are positive integers and $q_i = (\prod_{i=1}^{n+1} p_i)/p_i$, then the order $\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ as a module over $\mathbb{Q}[t, t^{-1}]$ (see Section 1), where V is given by equation

$$(0.1) \qquad f_{q_1}^{p_1} + \cdots + f_{q_{n+1}}^{p_{n+1}} = 0,$$

is the characteristic polynomial of the monodromy of the singularity $x_1^{p_1} + \cdots + x_{n+1}^{p_{n+1}} = 0$. The hypersurface (0.1) and the nonvanishing of π_n were described in [L1] as a consequence of the fact that singularities of (0.1) form a finite set in $\mathbb{CP}^{n+1}$, which is a *complete intersection* of $n+1$ hypersurfaces

$f_{q_1} = \cdots = f_{q_{n+1}} = 0$ in $\mathbb{CP}^{n+1}$ (see also [L5]). The second method, based on the use of the Thom–Sebastiani theorem, results in examples of hypersurfaces with the same collection of singularities, but with distinct groups π_n of the complement.

The results in this article were partly outlined in the announcement [L1]. On the other hand, the latter contained a number of results on the relationship of the homotopy groups in question with the Hodge theory of the cyclic covers of $\mathbb{CP}^{n+1}$ branched over V; this is described in [L5]. Note that since [L1] appeared, [Di1], [Di2] and [Deg], all related to the case of hypersurfaces in $\mathbb{CP}^{n+1}$, came out.

1. Preliminaries

This section describes the topology of complements to nonsingular hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^{n+1}$ (Lemmas 1.1 and 1.5 and Corollary 1.2), the case of weighted homogeneous hypersurfaces (Lemma 1.11), the homology of the complements (Lemma 1.6) and the relationship between complements in affine and projective spaces (Lemma 1.13).

LEMMA 1.1. *Let V be a nonsingular hypersurface in $\mathbb{C}^{n+1}$ that is transversal to the hyperplane at infinity (resp. a nonsingular hypersurface in $\mathbb{CP}^{n+1}$). Then $\pi_i(\mathbb{C}^{n+1} - V) = 0$ (resp. $\pi_i(\mathbb{CP}^{n+1} - V) = 0$) for $1 < i \leq n$ and $\pi_1(\mathbb{C}^{n+1} - V) = \mathbb{Z}$ (resp. $\mathbb{Z}/d\mathbb{Z}$, d is the degree of V).*

Proof. The statement about the fundamental groups follows immediately from the Zariski theorem by applying it to a section of the hypersurface by a generic plane. A hypersurface satisfying the conditions of the lemma is isotopic to the hypersurface given by the equation: $z_1^d + \cdots + z_{n+1}^d + 1 = 0$ (resp. the projective closure of this). The d-fold cover of the complement to the hypersurface $z_0^d + z_1^d + \cdots + z_{n+1}^d = 0$ in $\mathbb{CP}^{n+1}$ can be identified with the hypersurface $\mathcal{V}$ in $\mathbb{C}^{n+2}$ given by $z_0^d + z_1^d + \cdots + z_{n+1}^d = 1$. This is n-connected because, for example, it is diffeomorphic to the Milnor fiber of the isolated singularity $z_0^d + \cdots + z_{n+1}^d = 0$. Hence our claim follows in the projective case. The d-fold cover of the complement in $\mathbb{C}^{n+1}$ is obtained by removing from $\mathcal{V}$ the hyperplane section H_0 given by $z_0 = 0$. In other words, this d-fold cover is the hypersurface in $\mathbb{CP}^{n+2} - (H_0 \cup H_\infty) = \mathbb{C}^* \times \mathbb{C}^{n+1}$ that is transversal to these hyperplanes (H_∞ is the hyperplane at infinity). Hence the Lefschetz-type theorem applied to the closure of $\mathcal{V}$ in $\mathbb{CP}^{n+2}$, which is an ample divisor transversal to $H_0 \cup H_\infty$, implies that $\pi_i(\mathcal{V} - H_0) = \pi_i(\mathbb{C}^*)$. □

COROLLARY 1.2. *If V is a nonsingular hypersurface transversal to the hyperplane at infinity, then $\mathbb{C}^{n+1} - V$ is homotopy equivalent to the wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^{n+1} \vee \cdots \vee \mathbb{S}^{n+1}$.*

Proof. Indeed the lemma implies that the CW-complex $\mathbb{C}^{n+1} - V$ is a $(\mathbb{Z}, n+1)$-complex in the sense of Dyer [Dy] and hence is homotopy equivalent to the wedge as above (i.e., as a consequence of the stable triviality of $\pi_{n+1}(\mathbb{C}^{n+1} - V)$ as a $\mathbb{Z}[t, t^{-1}]$-module (cf. [Wh], thm. 14) and the fact that stably trivial modules over $\mathbb{Z}[t, t^{-1}]$ are free). □

Remark 1.3. It is interesting to see how such a wedge comes up geometrically. Let us compare the complements in $\mathbb{C}^{n+1}$ to the quadric hypersurfaces Q_1 and Q_0, where Q_ϵ is given by $z_1^2 + \cdots + z_{n+1}^2 = \epsilon$ $(0 \le \epsilon \le 1)$. The complement to Q_0 fibers over $\mathbb{C}^*$ using the map $(z_1, \ldots, z_{n+1}) \to z_1^2 + \cdots + z_{n+1}^2$. The fiber is homotopy equivalent to $\mathbb{S}^n$. Hence the complement to the singular quadric hypersurface can be identified with $\mathbb{S}^1 \times \mathbb{S}^n$. On the other hand, the degeneration of the nonsingular quadric hypersurface Q_ϵ $(\epsilon \ne 0)$ into the singular one when $\epsilon \to 0$ results in the collapse of the vanishing cycle $\mathbb{S}^n$ which is the boundary of a relative vanishing cycle (this relative vanishing cycle can be given explicitly as the set Δ_ϵ of points $(z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ such that $|z_1|^2 + \cdots + |z_{n+1}|^2 \le \epsilon$.) In particular $\mathbb{C}^{n+1} - Q_1 - \Delta_1 = \mathbb{C}^{n+1} - Q_0$. The complement to Q_1 can hence be obtained from the complement to Q_0 by attaching an $(n+1)$-cell. Therefore the complement to the nonsingular quadric hypersurface can be identified with $\mathbb{S}^1 \times \mathbb{S}^n \cup_{*\times\mathbb{S}^n} e_{n+1} = \mathbb{S}^1 \vee \mathbb{S}^{n+1}$.

Definition 1.4. Let V be a hypersurface in $\mathbb{CP}^{n+1}$ and H be a hyperplane. A point of V will be called a singular point at infinity if it is a singular point of $V \cap H$. The subvariety of singular points of V at infinity will be denoted as $\operatorname{Sing}_\infty(V)$.

LEMMA 1.5. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ having the dimension of $\operatorname{Sing}(V) \cup \operatorname{Sing}_\infty(V)$ (resp. $\operatorname{Sing}(V)$) equal to k. If $\mathbb{CP}^{n-k+1}$ is a generic linear subspace of $\mathbb{CP}^{n+1}$ of codimension k, then $\mathbb{CP}^{n-k+1} \cap V$ has isolated singularities including infinity (resp. isolated singularities) and $\pi_{n-k}(\mathbb{CP}^{n+1} - V \cup H) = \pi_{n-k}(\mathbb{CP}^{n-k+1} - (V \cup H) \cap \mathbb{CP}^{n-k+1})$ (resp. $\pi_{n-k}(\mathbb{CP}^{n+1} - V) = \pi_{n-k}(\mathbb{CP}^{n-k+1} - V \cap \mathbb{CP}^{n-k+1})$). Moreover $\pi_1(\mathbb{CP}^{n+1} - V \cup H) = \mathbb{Z}$ (resp. $\pi_1(\mathbb{CP}^{n+1} - V) = \mathbb{Z}/d\mathbb{Z}$) and $\pi_i(\mathbb{CP}^{n+1} - V \cup H) = \pi_i(\mathbb{CP}^{n+1} - V) = 0$ for $2 \le i < n-k$.*

Proof. The first part is a consequence of the Lefschetz theorem (cf. [H]). If L is a generic subspace of codimension $k+1$ in $\mathbb{CP}^{n+1}$, then $V \cap L$ is a nonsingular hypersurface in L that is transversal to $L \cap H$, and the claim follows from Lemma 1.1. □

LEMMA 1.6. *Let V be a hypersurface with isolated singularities including singularities at infinity. Then $\mathrm{H}_i(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z}) = 0$ for $2 \le i \le n-1$. If V is nonsingular, then the vanishing also takes place for $i = n$. If*

$n \geq 2$, then for $i = n$ this group is isomorphic to $\mathrm{H}^{n+1}(V, H \cap V, \mathbb{Z})$. *Moreover* $\mathrm{H}_1(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}$ *unless* $\dim V = 1$, *in which case this group is the free abelian group of the rank equal to the number of irreducible components of* V.

Proof. Calculating the group in the lemma for $i \leq n - 1$ using the Lefschetz theorem can be reduced to the case where $i = n$ (as was done above for homotopy groups). For $i = n$, the group in the lemma is isomorphic to $\mathrm{H}_{n+1}(\mathbb{CP}^{n+1} - H, \mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z})$ as follows from the exact sequence of the pair $(\mathbb{CP}^{n+1} - H, \mathbb{CP}^{n+1} - (V \cup H))$. Using duality one can identify this with $\mathrm{H}^{n+1}(V \cup H, H, \mathbb{Z})$. Using excision, one sees that this group is isomorphic to $\mathrm{H}^{n+1}(V, H \cap V, \mathbb{Z})$, which proves the lemma. The remaining case of curves is well known (cf. [L]). □

LEMMA 1.7. *If V and $V \cap H$ are $\mathbb{Q}$-manifolds, then* $\mathrm{H}^{n+1}(V, H \cap V, \mathbb{Q}) = 0$.

Proof. The claim is equivalent to the injectivity of $\mathrm{H}^i(V, \mathbb{Q}) \to \mathrm{H}^i(H \cap V, \mathbb{Q})$ for $i = n + 1$ and to the surjectivity for $i = n$. This follows from the Poincaré duality with $\mathbb{Q}$-coefficients and the Lefschetz theorem, which implies that the homology groups dual to the cohomology groups involved are isomorphic to either $\mathbb{Q}$ or 0 depending on the parity of n. □

Remark 1.8. An easily verifiable condition for a hypersurface V to be a $\mathbb{Q}$-manifold is the following: Let V have only isolated singularities and for each of the singularities the characteristic polynomial of the monodromy operator does not vanish at 1. Then V is a $\mathbb{Q}$-manifold. This is an immediate consequence of (1) the fact that the condition on the monodromy is equivalent to the condition that the link of each singularity is a $\mathbb{Q}$-sphere, and (2) the Zeeman spectral sequence (cf. [Mc]).

Let V be a hypersurface in $\mathbb{C}^{n+1}$ having only isolated singularities including infinity. According to Lemma 1.5, one can identify $\pi_n(\mathbb{C}^{n+1} - V)$ with $\mathrm{H}_n(\widetilde{\mathbb{C}^{n+1} - V}, \mathbb{Z})$, where $\widetilde{\mathbb{C}^{n+1} - V}$ is the universal cover of the space in question. The group $\mathbb{Z}$ of deck transformations acts on $\mathrm{H}_n(\widetilde{\mathbb{C}^{n+1} - V}, \mathbb{Z})$. This action on $\pi_n(\mathbb{C}^{n+1} - V)$ can be described as $\beta \to [\alpha, \beta] - \beta$, where $\beta \in \pi_n(\mathbb{C}^{n+1} - V)$, $\alpha \in \pi_1(\mathbb{C}^{n+1} - V)$ and $[,]$ is the Whitehead product (alternatively this action of the fundamental group on homotopy groups is the one given by the change of the base point). The structure of $\pi_n(\mathbb{C}^{n+1} - V)$ as the module over the group ring of the fundamental group, i.e., over $\mathbb{Z}[t, t^{-1}]$, becomes particularly simple after we take a tensor product with $\mathbb{Q}$. Namely we have the following isomorphism of $\mathbb{Q}[t,t^{-1}]$-modules:

$$\pi_n(\mathbb{C}^{n+1}-V)\otimes\mathbb{Q}=\bigoplus_i \mathbb{Q}[t,t^{-1}]/(\lambda_i)\oplus\mathbb{Q}[t,t^{-1}]^{\kappa} \tag{1.1}$$

for some polynomials λ_i defined up to a unit of $\mathbb{Q}[t,t^{-1}]$.

Definition 1.9. If $\kappa=0$ in equation (1.1), then the product $\prod_i \lambda_i$ is called the order of $\pi_n(\mathbb{C}^{n+1}-V)\otimes\mathbb{Q}$ (as a module over $\mathbb{Q}[t,t^{-1}]$). If $\kappa\neq 0$, then the order is defined to be 0.

Remark 1.10. We will see below that, for V with only isolated singularities, $\pi_n(\mathbb{C}^{n+1}-V)$ is always a torsion module (see Remark 4.4 and Lemma 1.12). In the low-dimensional cases, if one works with the homology of infinite cyclic covers, one obtains results similar to those in this article. The order of the corresponding $\mathbb{Q}[t,t^{-1}]$-module in the case where $n=1$ is the Alexander polynomial of the curves studied in [L2]. Note that if $n=0$, then the 1-dimensional homology over $\mathbb{Z}$ of the infinite cyclic cover of $\mathbb{C}-V$ (i.e., of the complement to, say, d points) is a free module over $\mathbb{Z}[t,t^{-1}]$ of rank $d-1$ (cf. [A]).

LEMMA 1.11. *Let $f(z_1,\dots,z_n)$ be a weighted homogeneous polynomial having an isolated singularity at the origin and let V be given by $f=0$. Then $\pi_n(\mathbb{C}^{n+1}-V)=\mathrm{H}_n(M_f,\mathbb{Z})$, where M_f is the Milnor fiber of the singularity of f. If one introduces the structure of $\mathbb{Z}[t,t^{-1}]$-modules on $\mathrm{H}_n(M_f,\mathbb{Z})$ by defining the action of t as the action of the monodromy operator, then this is an isomorphism of $\mathbb{Z}[t,t^{-1}]$-modules.*

Proof. Clearly $\mathbb{C}^{n+1}-V$ can be retracted on the complement of the link of the singularity of f. Hence this follows from the exact sequence of fibration and the $n-1$ connectedness of the Milnor fiber. □

LEMMA 1.12. *Let $P_V(t)$ be the order of $\pi_n(\mathbb{CP}^{n+1}-(V\cup H))\otimes\mathbb{Q}$ as the $\mathbb{Q}[t,t^{-1}]$-module. If $\mathrm{H}^{n+1}(V,H\cap V,\mathbb{Q})=0$, then $P_V(1)\neq 0$. In particular this homotopy group is a torsion module.*

Proof. Let us consider the exact sequence corresponding to the following exact sequence of the chain complexes of the universal cyclic cover $(\mathbb{CP}^{n+1}\widetilde{-(V\cup H)})_\infty$:

$$\begin{aligned}0\to C_*\big((\mathbb{CP}^{n+1}\widetilde{-(V\cup H)})_\infty\big)&\to C_*\big((\mathbb{CP}^{n+1}\widetilde{-(V\cup H)})_\infty\big)\\ &\to C_*\big(\mathbb{CP}^{n+1}-(V\cup H)\big)\to 0,\end{aligned} \tag{1.2}$$

where the left homomorphism is the map of free $\mathbb{Q}[t,t^{-1}]$-modules induced by the multiplication by $t-1$ ($C_*(X)$ denotes the chain complex of a cell complex

X corresponding to a cell structure, which we assume is equivariant whenever the group action is involved). We obtain

$$\begin{aligned} \to \mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty, \mathbb{Q}\big) &\to \mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty, \mathbb{Q}\big) \\ &\to \mathrm{H}_n(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Q}) \to \end{aligned} \tag{1.3}$$

(the left homomorphism is the multiplication by $t-1$). The right group in the exact sequence (1.3) is trivial by the assumption of Lemma 1.12 and by the isomorphism $\mathrm{H}_n(\mathbb{CP}^{n+1} - (V \cup H)_1\mathbb{Z}) = \mathrm{H}^{n+1}(V, V \cap H_1\mathbb{Z})$ of Lemma 1.6. Hence the multiplication by $t-1$ in $\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q} = \mathrm{H}_n(\mathbb{CP}^{n+1} \widetilde{- (V \cup H)}_\infty, \mathbb{Q})$ is surjective. Therefore its cyclic decomposition has neither free summands nor summands of the form $\mathbb{Q}[t, t^{-1}]/(t-1)^\kappa \mathbb{Q}[t, t^{-1}]$, with $\kappa \in \mathbb{N}$. □

LEMMA 1.13. *Let H be a generic hyperplane and V a hypersurface of degree d with isolated singularities in $\mathbb{CP}^{n+1}$. Let $d\mathbb{Z}$ be the subgroup of $\mathbb{Z}$ of index d. Then $\pi_n(\mathbb{CP}^{n+1} - V)$ is isomorphic to the module of covariants of $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))^{d\mathbb{Z}}$ (i.e., the quotient by the submodule of images by the action of the augmentation ideal of the subgroup $\pi_n/(t^d - 1)\pi_n$) with the standard action of $\mathbb{Z}/d\mathbb{Z}$.*

Proof. First let us show that the module of covariants in the lemma is isomorphic to

$$\mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_d, \mathbb{Z}\big), \tag{1.4}$$

where $(\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_d$ is the d-fold cyclic cover of the corresponding space. Indeed the sequence of the chain complexes similar to that of (1.2),

$$\begin{aligned} 0 \to C_*\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty\big) &\to C_*\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty\big) \\ &\to C_*\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_d\big) \to 0, \end{aligned} \tag{1.5}$$

where the left homomorphism is the multiplication by $t^d - 1$, gives rise to the homology sequence

$$\begin{aligned} \to \mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty, \mathbb{Z}\big) &\to \mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_\infty, \mathbb{Z}\big) \\ &\to \mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_d, \mathbb{Z}\big), \end{aligned} \tag{1.6}$$

in which the right homomorphism is surjective because of the vanishing of $\pi_{n-1}(\mathbb{CP}^{n+1} - (V \cup H))$, which proves our claim.

To conclude the proof of the lemma we need to show that

$$\mathrm{H}_n\big((\mathbb{CP}^{n+1} \widetilde{- (V \cup H)})_d, \mathbb{Z}\big) = \mathrm{H}_n\big((\widetilde{\mathbb{CP}^{n+1} - V})_d, \mathbb{Z}\big), \tag{1.7}$$

where $\widetilde{(\mathbb{CP}^{n+1} - V)}_d$ is the universal (d-fold) cyclic cover of $\mathbb{CP}^{n+1} - V$. This will follow from the vanishing of the relative group $\mathrm{H}_i(\widetilde{(\mathbb{CP}^{n+1} - V)}_d, \widetilde{(\mathbb{CP}^{n+1} - (V \cup H))}_d, \mathbb{Z})$ for $i = n, n+1$. Let W_d be the cyclic d-fold cover of $\mathbb{CP}^{n+1}$ branched over V and let $Z \subset W_d$ be the ramification locus (isomorphic to V). Let H_d be the submanifold of W_d that maps onto $H \cap V$. Let us consider the regular neighborhoods $T(H_d)$ and $T(Z)$ in W_d of H_d and Z, respectively. The boundary of $T(H_d) - T(H_d) \cap T(Z)$ contains the part ∂_1, which is the part of the boundary of $T(H_d)$. We denote ∂_2 the part of the boundary of $T(H_d) - T(H_d) \cap T(Z)$ that is complementary to ∂_1. By excision

(1.8)

$$\mathrm{H}_i\big(\widetilde{(\mathbb{CP}^{n+1} - V)}_d, \widetilde{(\mathbb{CP}^{n+1} - (V \cup H)}_d, \mathbb{Z}\big) = \mathrm{H}_i(T(H_d) - T(H_d) \cap T(Z), \partial_1, \mathbb{Z}).$$

Moreover

$$\begin{aligned}\mathrm{H}_i(T(H_d) - T(H_d) \cap T(Z), \partial_1, \mathbb{Z}) &= \mathrm{H}^{2n+2-i}(T(H_d) - T(H_d) \cap T(Z), \partial_2, \mathbb{Z}) \\ &= \mathrm{H}^{2n+2-i}(H_d, H_d \cap Z, \mathbb{Z})\end{aligned}$$

by duality, excision and retraction. The assumption that H is generic implies that H_d, which is a cyclic branched cover of H of degree d with branching locus $H \cap Z$, is nonsingular and, hence, $\mathrm{H}^{2n+2-i}(H_d, H_d \cap Z, \mathbb{Z}) = \mathrm{H}_{i-2}(H_d - H_d \cap Z, \mathbb{Z})$. The latter group, which is the homology group of the affine hypersurface transversal to the hyperplane at infinity, is trivial except for $i = 2$ and $i = n+2$. This implies the lemma. □

2. Calculation of homotopy groups using generic pencils

In this section we shall describe a method for calculating $\pi_n(\mathbb{C}^{n+1} - V)$ using the monodromy action on homotopy groups. A typical homotopy group supporting such an action is π_{n-1} of the complement to a nonsingular hypersurface in $\mathbb{C}^n$, which appears as the intersections of V with an element of a generic pencil of hyperplanes. The action on such homotopy groups is obtained by moving a hyperplane from the pencil along a loop in the subset of the parameter space of the pencil corresponding to hyperplanes transversal to V. Monodromy action is the composition of "geometric monodromy" with values in the fundamental group of the space of certain embeddings and a linear representation of this group over $\mathbb{Z}[t, t^{-1}]$. In the case where $n = 1$, this construction, with the monodromy taking values in the fundamental group of the space of embeddings (which in this case is Artin's braid group), reduces to the van Kampen theorem (cf. [K]). The composition of this monodromy with the Burau representation leads to a calculation of the Alexander module of the curve (cf. [L4]).

We will start by specifying the loops such that, by moving along these loops, we will get the information needed for the calcultions of $\pi_n(\mathbb{C}^{n+1} - V)$.

Definition 2.1. Let $t_1, \dots, t_N$ be a finite set of points in $\mathbb{C}$. A system of generators of $\gamma_i \in \pi_1(\mathbb{C} - \{t_1, \dots, t_N\}, t_0)$ is called *good* if each of the loops $\gamma_i : \mathbb{S}^1 \to \mathbb{C} - \{t_1, \dots, t_N\}$ extends to a map of the disk $D^2 \to \mathbb{C}$ with nonintersecting images for distinct subindices i.

A standard method for constructing a good system of generators is to select a system of small disks Δ_i about each of t_i, $i = 1, \dots, N$, to choose a system of N nonintersecting paths δ_i connecting the base point t_0 with a point of $\partial\Delta_i$ and to take $\gamma_i = \delta^{-1} \circ \partial\Delta_i \circ \delta_i$ (with, say, the counterclockwise orientation of $\partial\Delta_i$).

Let V be a nonsingular hypersurface in $\mathbb{C}^n$ that is transversal to the hyperplane at infinity. Let us consider a sphere $\mathbb{S}^{2n-1}$ in $\mathbb{C}^n$ of a sufficiently large radius. Let $\partial_\infty V = V \cap \mathbb{S}^{2n-1}$. Let us consider the space $\mathrm{Emb}(V, \mathbb{C}^n)$ of *submanifolds* of $\mathbb{C}^n$, which are diffeomorphic to V and isotopic to the chosen embedding of V, such that for any $V' \in \mathrm{Emb}(V, \mathbb{C}^n)$ one has $V' \cap \mathbb{S}^{2n-1} = \partial_\infty V$. We assume the compact open topology on this space of submanifolds. Let us describe a certain linear representation of $\pi_1(\mathrm{Emb}(V, \mathbb{C}^n))$ (over $\mathbb{Z}[t, t^{-1}]$), which after the choice of a basis gives a homomorphism into $\mathrm{GL}_r(\mathbb{Z}[t, t^{-1}])$, where r is the rank of $\tilde{\mathrm{H}}_n(\mathbb{C}^n - V, \mathbb{Z})$ (the reduced homology of the complement).

Let $\mathrm{Diff}(\mathbb{C}^n, \mathbb{S}^{2n-1})$ be the group of diffeomorphisms of $\mathbb{C}^n$ acting as the identity outside $\mathbb{S}^{2n-1}$. This group can be identified with $\mathrm{Diff}(\mathbb{S}^{2n}, D_{2n})$ of the diffeomorphisms of the sphere fixing a disk (cf. [AnBuKa]). Let $\mathrm{Diff}(\mathbb{C}^n, V)$ be the subgroup of $\mathrm{Diff}(\mathbb{C}^n, \mathbb{S}^{2n-1})$ of those diffeomorphisms which take the hypersurface V into itself. The group $\mathrm{Diff}(\mathbb{C}^n, \mathbb{S}^{2n-1})$ acts transitively on $\mathrm{Emb}(V, \mathbb{C}^n)$ (cf. [C], p. 116) with the stabilizer $\mathrm{Diff}(\mathbb{C}^n, V)$. This implies the following exact sequence:

$$\begin{aligned} \pi_1(\mathrm{Diff}(\mathbb{S}^{2n}, D_{2n})) \to \pi_1(\mathrm{Emb}(V, \mathbb{C}^n)) \to \pi_0(\mathrm{Diff}(\mathbb{C}^n, V)) \to \\ \to \pi_0(\mathrm{Diff}(\mathbb{S}^{2n}, D_{2n})) \to . \end{aligned} \tag{2.1}$$

Any element in $\mathrm{Diff}(\mathbb{C}^n, V)$ induces the self-map of $\mathbb{C}^n - V$ and the self-map of the universal (resp. cyclic in case $n = 1$) cover of this space. Hence it induces an automorphism of $\mathrm{H}_n(\widetilde{\mathbb{C}^n - V}, \mathbb{Z}) = \pi_n(\mathbb{C}^n - V)$, $(n > 1)$. The composition of the boundary homomorphism in (2.1) with the map of $\pi_0(\mathrm{Diff}(\mathbb{C}^n, V))$ just described results in

$$\lambda : \pi_1(\mathrm{Emb}(V, \mathbb{C}^n)) \to \mathrm{Aut}(\pi_n(\mathbb{C}^n - V)). \tag{2.2}$$

In case $n = 1$, V is just a collection of points in $\mathbb{C}$,

$$\pi_1(\mathrm{Emb}(V, \mathbb{C})) = \pi_0(\mathrm{Diff}(\mathbb{C}, V))$$

is Artin's braid group, and this construction gives the homomorphism of the braid group into $\mathrm{Aut}(\mathrm{H}_1(\widetilde{\mathbb{C} - V}, \mathbb{Z}))$, which after the choice of the basis in $\mathrm{H}_1(\widetilde{\mathbb{C} - V})$ corresponding to the choice of the generators of the braid group, gives the reduced Burau representation. This construction coincides with the one described in [A].

Now we can define the relevant monodromy operator corresponding to a loop in the parameter space of a linear pencil of hyperplane sections. Let V be a hypersurface in $\mathbb{CP}^{n+1}$, which has only isolated singularities, and H be the hyperplane at infinity (which we shall assume is transversal to V). Let L_t, $t \in \mathbb{C}$, be a pencil of hyperplanes, the projective closure of which has the base locus $B \subset H$ such that B is transversal to V. Let $t_1, \dots, t_N$ denote those t for which $V \cap L_t$ has a singularity. We assume that for any i the singularity of $V \cap L_{t_i}$ is outside H. Over $\mathbb{C} - \{t_1, \dots, t_N\}$ the pencil L_t defines a locally trivial fibration τ of $\mathbb{C}^{n+1} - V$ with a nonsingular hypersurface in $\mathbb{C}^n$ as a fiber transversal to the hyperplane at infinity. The restriction of this fibration on the complement to a sufficiently large ball is trivial, as follows from the assumptions on the singularities at infinity. Let $\gamma : [0, 1] \to \mathbb{C} - \{t_1, \dots, t_N\}$ be a loop with the base point t_0. The choice of a trivialization of the pullback of the fibration τ on $[0, 1]$, using γ, defines a loop e_γ in $\mathrm{Emb}(V \cap L_{t_0}, L_{t_0})$. Different trivializations define homotopic loops in this space. By abuse of language we shall assume that $e_\lambda \in \pi_1(\mathrm{Emb}(V \cap L_{t_0}, L_{t_0}))$.

Definition 2.2. The monodromy operator corresponding to γ is the element $\lambda(e_\gamma)$ in

$$\mathrm{Aut}(\pi_n(L_{t_0} - L_{t_0} \cap V)),$$

where λ is the map in (2.2).

Next we will need to associate with each pair (L_{t_i}, γ) consisting of (1) a singular fiber L_{t_i} and (2) a loop γ with the base point t_0 in the parameter space of the pencil (where it bounds a disk Δ_{t_i} not containing other singular points of the pencil) certain homomorphisms

$$\pi_{n-1}(L_{t_i} - L_{t_i} \cap V) \to \pi_n(L_{t_0} - L_{t_0} \cap V)/\,\mathrm{im}(\Gamma - I), \tag{2.3}$$

where Γ is the monodromy operator corresponding to γ.

First let us note that the module on the right in the map (2.3) is isomorphic to the homology $\mathrm{H}_n(\widetilde{\tau^{-1}(\partial \Delta_{t_i})}, \mathbb{Z})$ of the infinite cyclic cover of the restriction of the fibration τ on the boundary of Δ_{t_i}. This follows immediately from the Wang sequence of a fibration over a circle and the vanishing of the homotopy of $L_{t_0} - L_{t_0} \cap V$ in dimensions below n. Let B_i be a polydisk in $\mathbb{C}^{n+1}$ about the singular point of $L_{t_i} \cap V$ such that $B_i = \Delta_{t_i} \times B$ for a certain polydisk B in L_{t_i}. Then $\widetilde{\tau^{-1}(\Delta_{t_i})} - B_i$ is a trivial fibration over Δ_{t_i} with the infinite cyclic cover $L_{t_i} \widetilde{- L_{t_i}} \cap V$ as a fiber. In particular one obtains the map

(2.4)

$$\begin{aligned}\pi_{n-1}(L_{t_i} - L_{t_i} \cap V) &= \mathrm{H}_{n-1}(L_{t_i} \widetilde{- L_{t_i}} \cap V, \mathbb{Z}) \to \mathrm{H}_n(\tau^{-1}(\partial\Delta_{t_i}) \widetilde{-} \partial\Delta_{t_i} \times B, \mathbb{Z}) \\ &= \mathrm{H}_{n-1}(L_{t_i} \widetilde{- L_{t_i}} \cap V, \mathbb{Z}) \oplus \mathrm{H}_n(L_{t_i} \widetilde{- L_{t_i}} \cap V, \mathbb{Z}).\end{aligned}$$

(The latter isomorphism is the Künneth decomposition of

$$\tau^{-1}(\partial\Delta_{t_i}) \widetilde{-} \partial\Delta_{t_i} \times B = \mathbb{S}^1 \times (L_{t_i} - \tilde{L}_{t_i} \cap V).$$

This map associates to a cycle its product with $\mathbb{S}^1$.)

Definition 2.3. The degeneration operator is the map (2.3) given by composition of the map (2.4) with the map

$$\mathrm{H}_n(\tau^{-1}(\widetilde{\partial\Delta_{t_i}}) - \partial\Delta_{t_i} \times B, \mathbb{Z}) \to \mathrm{H}_n(\tau^{-1}(\widetilde{\partial\Delta_{t_i}}), \mathbb{Z}) = \pi_n(L_{t_0} - L_{t_0} \cap V)/(\Gamma - I)$$

induced by inclusion.

THEOREM 2.4. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ having only isolated singularities and transversal to the hyperplane H at infinity. Consider a pencil of hyperplanes in $\mathbb{CP}^{n+1}$, the base locus of which belongs to H and is transversal in H to $V \cap H$. Let $\mathbb{C}^n_t$ $(t \in \Omega)$ be the pencil of hyperplanes in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$ defined by L_t (where $\Omega = \mathbb{C}$ is the set parametrizing all elements of the pencil L_t different from H). Denote by $t_1, \dots, t_N$ the collection of those t for which $V \cap L_t$ has a singularity. Assume that the pencil was chosen so that $L_t \cap H$ has at most one singular point outside H. Let t_0 be different from either of t_i $(i = 1, \dots, N)$. Let γ_i $(i = 1, \dots, N)$ be a good collection, in the sense described above (Definition 2.1), of paths in Ω based in t_0 and forming a basis of $\pi_1(\Omega - \{t_1, \dots t_N\}, t_0)$. Let $\Gamma_i \in \mathrm{Aut}(\pi_n(\mathbb{C}^n_{t_0} - V \cap \mathbb{C}^n_{t_0}))$ be the monodromy automorphism corresponding to γ_i. Let $\sigma_i : \pi_{n-1}(\mathbb{C}^n_{t_i} - V \cap \mathbb{C}^n_{t_i}) \to \pi_n(\mathbb{C}^n_{t_0} - V \cap \mathbb{C}^n_{t_0})^{\Gamma_i}$ be the degeneration operator of the homotopy group of a special element of the pencil into the corresponding quotient of covariants constructed above. Then*

$$\begin{aligned}&\pi_n(\mathbb{C}^{n+1} - V \cap \mathbb{C}^{n+1}) \\ &\quad = \pi_n(\mathbb{C}^n_{t_0} - V \cap \mathbb{C}^n_{t_0})/(\mathrm{im}(\Gamma_1 - I), \mathrm{im}\,\sigma_1, \dots, \mathrm{im}(\Gamma_N - I), \mathrm{im}\,\sigma_N).\end{aligned} \tag{2.5}$$

Proof. Let $P : \mathbb{C}^{n+1} \to \Omega$ be the projection defined by the pencil $\mathbb{C}^n_t$. Let $T(\mathbb{C}^n_{t_i})$ be the intersection of the tubular neighborhood of L_{t_i} in $\mathbb{CP}^{n+1}$, with the finite part $\mathbb{C}^{n+1}$ which can be taken as $P^{-1}(\Delta_i)$, where $\Delta_i \subset \Omega$ is a small disk about t_i $(i = 1, \dots, N)$. Each loop γ_i is isotopic to the loop having the standard form $\delta_i^{-1} \circ \partial\Delta_i \circ \delta_i$ (δ_i, as above, is a system of paths in Ω connecting t_0 to $\partial\Delta_i$ and nonintersecting outside t_0). We shall assume from now on that the γ_i have such a form. The restriction P_V of P on $\mathbb{C}^{n+1} - V \cap \mathbb{C}^{n+1}$ defines over $\Omega - \bigcup_i(\gamma_i \cup \Delta_i)$ a locally trivial fibration and, therefore, $\mathbb{C}^{n+1} - V \cap \mathbb{C}^{n+1}$ is homotopy equivalent to $P_V^{-1}(\bigcup_i(\gamma_i \cup \Delta_i))$. The latter space is homotopy equivalent to

$$\bigcup_{T(\mathbb{C}^n_{t_0}) - V \cap T(\mathbb{C}^n_{t_0})} T(\mathbb{C}^n_{t_i}) - V \cap T(\mathbb{C}^n_{t_i}) \quad (i = 1, \ldots, N), \tag{2.6}$$

with the embedding of the common part of the spaces in the union in each of them depending on the trivialization of $P_V^{-1}(\delta_i)$ over δ_i. The space in formula (2.6) can be decribed as a disjoint union of indicated spaces in which two points are identified if and only if they both are images of the same point in $T(\mathbb{C}^n_{t_0}) - V \cap T(\mathbb{C}^n_{t_0})$. We are going to calculate the homology of the infinite cyclic cover of the space (2.6) by repeated use of the Mayer–Vietoris sequences. First we claim that if $t_{0,i}$ denotes the endpoint of the path δ_i and Γ'_i is the automorphism of $\pi_n(\mathbb{C}^n_{t_{0,i}} - \mathbb{C}^n_{t_{0,i}} \cap V)$ induced by the monodromy corresponding to the loop $\partial\Delta_i$, then

$$\mathrm{H}_n(\widetilde{T(\mathbb{C}^n_{t_i}) - V \cap T(\mathbb{C}^n_{t_i})}, \mathbb{Z}) = \pi_n(\mathbb{C}^n_{t_{0,i}} - V \cap \mathbb{C}^n_{t_{0,i}})/(\mathrm{im}(\Gamma'_i - I), \mathrm{im}\, \sigma'_i) \tag{2.7}$$

for any i ($i = 1, \ldots, N$), where $\widetilde{X}$ denotes the universal cyclic cover of a space X. To verify equation (2.7) let us consider a small polydisk $B_i \subset T(\mathbb{C}^n_{t_i})$ for which the projection P induces a splitting $B_i = \Delta'_i \times D^n_i$ as a product of a 2-disk $\Delta'_i \subset \Delta_i$, $t_i \in \Delta'_i$ and an n-disk D^n_i in $\mathbb{C}^n_{t_i}$ such that D^n_i contains the singular point of $V \cap \mathbb{C}^n_{t_i}$. One has a natural retraction $B_i - B_i \cap V$ onto $\partial B_i - \partial B_i \cap V$, which shows that $T(\mathbb{C}^n_{t_i}) - T(\mathbb{C}^n_{t_i}) \cap V$ is homotopy equivalent to $T(\mathbb{C}^n_{t_i}) - T(\mathbb{C}^n_{t_i}) \cap V - B_i$. Let us decompose the latter as:

$$P_V^{-1}(\Delta_i - \Delta'_i) \cup (P_V^{-1}(\Delta'_i) - B_i). \tag{2.8}$$

The first component in this union, which we shall call Θ_1, is a locally trivial fibration over the homotopy circle $\Delta_i - \Delta'_i$ with the fiber $\mathbb{C}^n_{t_0} - V \cap \mathbb{C}^n_{t_0}$. The second component, which we denote by Θ_2, is fibered over the 2-disk Δ'_i with the fiber $\mathbb{C}^n_{t_i} - \mathbb{C}^n_{t_i} \cap V$ and, hence, is homotopy equivalent to this fiber. The intersection Θ_0 of two pieces in formula (2.8), which is the preimage of the boundary circle $\partial\Delta'_i$, forms part of a fibration over a disk and, hence, is homotopy equivalent to $\mathbb{C}^n_{t_i} - \mathbb{C}^n_{t_i} \cap V$. This defines the decomposition of the infinite cyclic covers:

$$\widetilde{T(\mathbb{C}^n_{t_i}) - T(\mathbb{C}^n_{t_i} \cap V)} = \tilde{\Theta}_1 \cup_{\tilde{\Theta}_0} \tilde{\Theta}_2. \tag{2.9}$$

The split of Θ_0 as a direct product $(\mathbb{C}^n_{t_i} - \mathbb{C}^n_{t_i} \cap V) \times \mathbb{S}^1$ defines the splitting of the infinite cyclic cover of $\tilde{\Theta}_0$ as $(\mathbb{C}^n_{t_i} - \widetilde{\mathbb{C}^n_{t_i} \cap V}) \times \mathbb{S}^1$. Therefore

$$\text{(2.10)} \quad \mathrm{H}_j(\tilde{\Theta}_0, \mathbb{Z}) = H_{j-1}(\mathbb{C}^n_{t_i} - \widetilde{\mathbb{C}^n_{t_i}} \cap V, \mathbb{Z}) \oplus \mathrm{H}_j(\mathbb{C}^n_{t_i} - \widetilde{\mathbb{C}^n_{t_i}} \cap V, \mathbb{Z}) \quad (j \in \mathbb{Z}).$$

Next let us consider the Mayer–Vietoris homology sequence corresponding to decomposition (2.9):

(2.11)

$$\begin{aligned} &\to \mathrm{H}_n(\tilde{\Theta}_0, \mathbb{Z}) \to \mathrm{H}_n(\tilde{\Theta}_1, \mathbb{Z}) \oplus \mathrm{H}_n(\tilde{\Theta}_2, \mathbb{Z}) \to \mathrm{H}_n(T(\mathbb{C}^n_{t_i}) - \widetilde{T(\mathbb{C}^n_{t_i}) \cap V}, \mathbb{Z}) \\ &\qquad \to \mathrm{H}_{n-1}(\tilde{\Theta}_0) \to \mathrm{H}_{n-1}(\tilde{\Theta}_1) \oplus \mathrm{H}_{n-1}(\tilde{\Theta}_2). \end{aligned}$$

The group $\mathrm{H}_n(\tilde{\Theta}_1, \mathbb{Z})$ can be identified as above with $\pi_n(\mathbb{C}^n_{t_{0,i}} - \mathbb{C}^n_{t_{0,i}} \cap V)/(\Gamma'_i - I)$; and the left homomorphism in the sequence (2.11) takes the second summand in equation (2.10), in the case where $j = n$, isomorphically to $\mathrm{H}_n(\tilde{\Theta}_2, \mathbb{Z})$. Hence the Coker of the left homomorphism in the sequence (2.11) coincides with $\pi_n(\mathbb{C}^n_{t_{0,i}} - \mathbb{C}^n_{t_{0,i}} \cap V)/(\Gamma'_i - I, \operatorname{im} \sigma'_i)$. Moreover the same argument shows that the homomorphism of $\mathrm{H}_{n-1}(\tilde{\Theta}_0, \mathbb{Z})$ in sequence (2.11) is an injection, because $\mathrm{H}_{n-2}(\mathbb{C}^n_{t_{0,i}} - \widetilde{\mathbb{C}^n_{t_{0,i}}} \cap V, \mathbb{Z})$ is isomorphic to π_{n-2} of the same space and, therefore, is trivial (see Lemma 1.5), which implies equation (2.7).

Next we shall calculate the homology of the infinite cyclic cover in formula (2.6) inductively, using the Mayer–Vietoris sequence corresponding to this decomposition. Because $\pi_i(\mathbb{C}^n_{t_0} - \mathbb{C}^n_{t_0} \cap V) = 0$ for $2 \le i \le n-1$ (see Lemma 1.5), the terms in the Mayer–Vietoris sequence below dimension n vanish. The cokernel of the map

(2.12)

$$\begin{aligned} &\mathrm{H}_n(T(\mathbb{C}^n_{t_0}) - \widetilde{T(\mathbb{C}^n_{t_0})} \cap V, \mathbb{Z}) \to \mathrm{H}_n(\mathbb{C}^n_{t_{0,i}} - \widetilde{\mathbb{C}^n_{t_{0,i}}} \cap V, \mathbb{Z})/(\operatorname{im}(\Gamma'_i - I), \operatorname{im} \sigma'_i) \\ &\qquad \oplus \mathrm{H}_n(\mathbb{C}^n_{t_{0,j}} - \widetilde{\mathbb{C}^n_{t_{0,j}}} \cap V, \mathbb{Z})/(\operatorname{im}(\Gamma'_j - I), \operatorname{im} \sigma'_j) \end{aligned}$$

is isomorphic to $\mathrm{H}_n(\mathbb{C}^n_{t_0} - \widetilde{\mathbb{C}^n_{t_0}} \cap V, \mathbb{Z})/ \operatorname{im}(\Gamma_i - I), \operatorname{im} \sigma_i, \operatorname{im}(\Gamma_j - I), \operatorname{im} \sigma_j)$ by linear algebra. Now the theorem follows. □

3. A vanishing theorem

In this section we give a necessary condition for the vanishing of $\pi_n(\mathbb{C}^{n+1} - V)$. This is useful in applications of Theorem 2.4, because it allows one to dispose of degeneration operators in some cases. We give a numerical condition for such vanishings in the case of curves. The key step in deriving the vanishing of $\pi_n(\mathbb{C}^{n+1} - V)$ is the following counterpart of the commutativity of the fundamental group in the case of curves.

THEOREM 3.1. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ that has only isolated singularities, including singularities at infinity, and let H be the hyperplane at infinity. Suppose that a nonsingular projective variety W and a birational map $\phi : W \to \mathbb{CP}^{n+1}$ are such that the union of the strict transform V' of V, the strict transform H' of H, and the exceptional locus E of ϕ form a divisor with normal crossings on W. Assume that V' is an ample divisor on W. Then the action of $\pi_1(\mathbb{CP}^{n+1} - (V \cup H)) = \mathbb{Z}$ on $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))$ is trivial.*

Proof. Let $U_{V'}$ be a tubular neighborhood of V' in W. First note that the map $\pi_i((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V')) \to \pi_i(W - (E \cup V' \cup H'))$, induced by inclusion, is an isomorphism for $i \leq n-1$ and is surjective for $i = n$. This follows immediately from the assumption of the ampleness of V' and the Lefschetz theorem for open varieties (cf. [H]). In the setting of this reference, one applies Theorem 2 from [H] to W embedded into $\mathbb{CP}^N$ using a multiple of the line bundle corresponding to V' and taking V' as the hyperplane section at infinity. Because $W - (V' \cup H' \cup E) = \mathbb{CP}^{n+1} - (V \cup H)$, we find that $\psi_i : \pi_i((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V')) \to \pi_i(\mathbb{CP}^{n+1} - (V \cup H))$ is an isomorphism for $i \leq n-1$ and is surjective for $i = n$.

Let α be the boundary of a 2-disk, which is normal to V' in $U_{V'}$ at a point of V' outside $E \cup H'$. The action of α (considered as an element of $\pi_1((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V')))$ on $\pi_n((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V'))$ is trivial. Indeed $(U_{V'} - V') - (U_{V'} - V') \cap (E \cup H')$ is a (trivial) circle bundle over $V' - V' \cap (E \cup H')$, because $V' \cup H' \cup E$ is assumed to be a divisor with normal crossings. Moreover the projection map λ induces the isomorphism $\lambda_* : \pi_i((U_{V'} - V') - (U_{V'} - V') \cap (E \cup H')) \to \pi_i(V' - V' \cap H' \cap E)$ for $i \geq 2$. If $\gamma \in \pi_i((U_{V'} - V') - (U_{V'} - V') \cap (E \cup H'))$ then $\lambda_*(\alpha \cdot \gamma) = \lambda_*(\alpha) \cdot \lambda_*(\gamma) = \lambda_*(\gamma)$, i.e., $\alpha \cdot \gamma = \gamma$, and our claim follows. This also concludes the proof of the theorem, because ψ_n is surjective and because α is taken by ψ_1 onto the generator of $\pi_1(\mathbb{CP}^{n+1} - (V \cup H))$. □

THEOREM 3.2. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ satisfying all conditions of Theorem 3.1. Assume also that $\mathrm{H}^{n+1}(V, V \cap H, \mathbb{Z}) = 0$. Then $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))$ vanishes.*

Proof. Let us consider the Leray spectral sequence

$$(3.1) \quad E^2_{p,q} = \mathrm{H}_p(\mathbb{Z}, \mathrm{H}_q(\widetilde{\mathbb{CP}^{n+1} - (V \cup H)}, \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z})$$

associated with the classifying map of $\mathbb{CP}^{n+1} - (V \cup H)$ into $\mathbb{S}^1 = K(\mathbb{Z}, 1)$ corresponding to the generator of $\mathrm{H}^1(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z}) = \mathbb{Z}$. The homotopy fiber of this map is the universal cyclic cover $(\widetilde{\mathbb{CP}^{n+1} - (V \cup H)})$. The sequence

(3.1) implies that $\mathrm{H}_0(\mathbb{Z}, \mathrm{H}_n((\widetilde{\mathbb{CP}^{n+1} - (V \cup H)}), \mathbb{Z}))$, which is isomorphic to the module of covariants

$$\mathrm{H}_n\big((\widetilde{\mathbb{CP}^{n+1} - (V \cup H)}), \mathbb{Z}\big)^{\pi_1(\mathbb{CP}^{n+1}-(V\cup H))} = \mathrm{H}_n(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z}).$$

The group in the left-hand side is isomorphic to the module of covariants $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))^{\pi_1(\mathbb{CP}^{n+1}-(V\cup H))}$. Hence the result follows from the above theorem, and from the vanishing of $\mathrm{H}_n(\mathbb{CP}^{n+1} - (V \cup H), \mathbb{Z})$ (the latter is a consequence of Lemma 1.6 and the assumption we made on the cohomology of $(V, V \cap H)$). □

COROLLARY 3.3. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ satisfying the condition of Theorem* 3.1. *If V and $V \cap H$ are $\mathbb{Q}$-manifolds, then $\pi_n(\mathbb{CP}^{n+1} - (V \cap H)) = 0$.*

Proof. This follows from Theorem 3.2 and Lemma 1.7. □

Remark 3.4. In the case where $n = 1$, the argument given in the proof of Theorem 3.1 can be strengthened to show that $V' \cdot V' > 0$ implies that the fundamental group of the complement is abelian (see [Ab], [N]). Recall that, for example, for a curve C of degree d, which has δ nodes and κ cusps, this implies the commutativity of $\pi_1(\mathbb{CP}^{n+1} - C)$, provided that $4\delta + 6\kappa < d^2$. For the application of the technique of Section 2 the following result is useful:

LEMMA 3.5. *Let C be a curve in $\mathbb{CP}^2$ that has only one singular point, which is unibranched, and has one characteristic pair (m, k) $(k \leq m)$. If $d^2 > m \cdot k$, then $\pi_1(\mathbb{CP}^2 - C)$ is abelian. In particular the Alexander module of C is trivial.*

Proof. The greatest common divisor of m and k, which is equal to the number of branches of the singularity, is equal to 1. The resolution of singularities of plane curves can be described in terms of the euclidian algorithm for finding this greatest common divisor (see [BKn]). Let $m = a_1 k + r_1$, and $k = a_2 \cdot r_1 + r_2, \ldots, r_{s-1} = a_{s+1} \cdot r_s + 1$ be the steps of the euclidian algorithm. Then the sequence of blowups that results in the embedded resolution produces the following: each of the first a_1 blowups gives an exceptional curve with multiplicity k, the intersection index of which, with the strict transform of the curve in question, is equal to k. The result is that the self-intersection index of the strict transform drops by k^2. Subsequent blowups have a similar effect with the blowups corresponding to the last step in the euclidian algorithm resulting in a nonsingular strict transform with the tangency order with the exceptional curve equal to r_s. The strict transform of additional r_s blowups is a curve, the union of which, with the exceptional locus, is a divisor with normal crossings. The self-intersection of the strict transform is equal to

$$\begin{aligned} d^2 - a_1 \cdot k^2 - a_1 \cdot r_1^2 - \cdots - a_{s+1} \cdot r_s^2 - r_s \\ = d^2 - (m - r_1) \cdot k - \cdots - (r_{s+1} - 1) \cdot r_s - r_s \\ = d^2 - m \cdot k > 0. \end{aligned}$$

Hence our claim follows from Nori's theorem in [N]. □

4. The divisibility theorems

In this section we prove two theorems relating the order of the homotopy group of the complement to a hypersurface V, to the type of singularities of V including the singularities at infinity. We will discuss the relation of these results to the divisibility theorem for Alexander polynomials in [L2]. We assume, for the reason described in Lemma 1.5, that all singularities of V are isolated.

First recall that if $c \in V$ is a singular point of V, then one can associate with it the characteristic polynomial P_c of the monodromy operator in the Milnor fibration of the singularity c. By a certain abuse of language we will call P_c the polynomial of the singularity c. This polynomial also can be obtained as follows: Let us consider the cyclic cover U_c of $B_c - B_c \cap V$, where B_c is a small ball about c in some Riemannian metric in $\mathbb{CP}^{n+1}$, corresponding to the kernel of the homomorphism $lk_c : \pi_1(B_c - B_c \cap V) \to \mathrm{H}_1(B_c - B_c \cap V, \mathbb{Z}) = \mathbb{Z}$ (this homomorphism is given by the evaluation of the linking number with $V \cap B_c$). Clearly U_c is homotopy equivalent to the Milnor fiber and the characteristic polynomial of the automorphism of $\mathrm{H}_n(U_c, \mathbb{Q})$, induced by the deck transformation, coincides with the polynomial of the singularity c. This is a consequence of the existence of the fibration of $B_c - B_c \cap V$. On the other hand, if $c \in \mathrm{Sing}_\infty$ and B_c is as above, then $\mathrm{H}_1(B_c - (V \cup H) \cap B_c, \mathbb{Z}) = \mathrm{H}_2(B_c, B_c - (V \cup H), \mathbb{Z}) = \mathrm{H}^{2n}(B_c \cap T(V \cup H), S_c \cap T(V \cup H))$, where S_c is the boundary of B_c and $T(\ )$ denotes the regular neighborhood. The latter is isomorphic to $\mathrm{H}^{2n}(V \cap B_c, \partial(V \cap B_c), \mathbb{Z}) \oplus \mathrm{H}^{2n}(H \cap B_c, \partial(H \cap B_c), \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. The map $lk_c : \pi_1(B_c - (V \cup H) \cap B_c) \to \mathbb{Z}$ corresponding to the projection of $\mathrm{H}_1(B_c - B_c \cap (V \cup H))$ onto $\mathrm{H}^{2n}(B_c \cap V, \partial(B_c \cap V), \mathbb{Z}) = \mathbb{Z}$ geometrically is the linking number with V. For $c \in \mathrm{Sing}_\infty$ let U_c be the infinite cyclic cover of $B_c - (V \cup H) \cap B_c$ corresponding to the kernel of the homomorphism lk_c.

Definition 4.1. For $c \in \mathrm{Sing}_\infty V$ the order $P_c(t)$ of $\mathrm{H}_n(U_c, \mathbb{Q})$, where U_c is the infinite cyclic cover constructed in the last paragraph and where the homology group, considered as the module over $\mathbb{Q}[t, t^{-1}]$ via the action induced by the deck transformations on U_c, is called the polynomial of the singular point c.

Remark 4.2. Note that U_c has a homotopy type of a finite-dimensional complex. In particular $\mathrm{H}_i(U_c, \mathbb{Q})$ is a torsion $\mathbb{Q}[t, t^{-1}]$-module for any i. So the above polynomial $P_c(t)$ is nonzero. For singularities at infinity this follows from the following realization of the infinite cyclic cover U_c. Let ϕ_c be a holomorphic function in a neighborhood N_c of c in $\mathbb{CP}^{n+1}$ such that $\phi_c = 0$ coincides with V in this neighborhood. Let $V_c(s)$ be given by the equation $\phi_c = s$ in N_c. Then U_c is homotopy equivalent to $V_c(s) - H \cap V_c(s)$ for s sufficiently close to 0. Indeed the union of hypersurfaces $\phi_c = s$ ($s \leq \epsilon$ and N_c is sufficiently small) is homeomorphic to the ball (see [Mi]), and the function ϕ_c provides the locally trivial fibration of this ball over a punctured disk. Because the singularity of $V \cap H$ is isolated, H will be transversal to all hypersurfaces $\phi_c = s$ (ϵ sufficiently small). Therefore $\phi_c(s)$ also provides the locally trivial fibration of the complement in this ball to $V \cap H$. Hence $V_c(s) - V_c(s) \cap H$ is homotopy equivalent to U_c.

THEOREM 4.3. *If V and $V \cap H$ have only isolated singularities, then the order P_V of $\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ as the module over $\mathbb{Q}[t, t^{-1}]$ divides the product $\Pi_c P_c \cdot (t-1)^\kappa$ (for some $\kappa \in \mathbb{Z}$) of the polynomials P_c of all singularities of V including those in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$ as well as those at infinity. The factor $(t-1)^\kappa$ can be omitted if one of the following conditions takes place:*

(a) $\mathrm{H}^{n+1}(V, H \cap V, \mathbb{Q}) = 0$.

(b) *the $\mathbb{Q}[t, t^{-1}]$-submodule of $\pi_n(\mathbb{CP}^{n+1} - (V \cap H)) \otimes \mathbb{Q}$ consisting of elements annihilated by a power of $t-1$ is semisimple.*

Remark 4.4. One of the consequences of the theorem is that the order of $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))$ is not 0. Hence this $\mathbb{Q}[t, t^{-1}]$-module is a torsion module.

Proof. Let $T(V)$ be a regular neighborhood of V in $\mathbb{CP}^{n+1}$. First let us observe that $\pi_n(T(V) - T(V) \cap (H \cup V))$ surjects onto $\pi_n(\mathbb{CP}^{n+1} - (V \cap H))$. Indeed $T(V)$ contains a generic hypersurface W of the same degree as V, which is transversal to $V \cup H$. According to the Lefschetz theorem, $\pi_i(W - W \cap (V \cup H))$ maps (by the map j_* induced by the inclusion) isomorphically to $\pi_i(\mathbb{CP}^{n+1} - (V \cup H))$ for $1 \leq i \leq n-1$ and surjects for $i = n$. Now our claim follows from the fact that j_* can be factored as

$$\begin{aligned} \pi_n(W - W \cap (V \cup H)) &\to \pi_n(T(V) - T(V) \cap (V \cup H)) \\ &\to \pi_n(\mathbb{CP}^{n+1} - (V \cup H)). \end{aligned} \tag{4.1}$$

Next let us consider a collection of nonintersecting balls B_c about the singular points of V, $c \in \mathrm{Sing}(V) \cup \mathrm{Sing}_\infty(V)$. For a sufficiently small regular neighborhood $T(V)$ of V the complement to the union of B_c ($c \in \mathrm{Sing}(V) \cup$

$\mathrm{Sing}_\infty(V)$),

$$(4.2)\quad B_0 = T(V) - (T(V)\cap(V\cup H)) - \bigcup_c B_c \cap (T(V) - T(V)\cap(V\cup H)),$$

fibers over $V-(H\cap V\cup_c(B_c\cap V))$ with a punctured disk as a fiber. A generator of H_1 of such a fiber maps onto the generator of $\mathrm{H}_1(\mathbb{C}^{n+1}-V,\mathbb{Z})$. Hence one has the surjection

$$lk_T : \mathrm{H}_1(T(V)-T(V)\cap(V\cup H),\mathbb{Z}) \to \mathrm{H}_1(\mathbb{C}^{n+1}-V,\mathbb{Z}) = \mathbb{Z}$$

(see Lemma 1.6; this map is just the linking number). The kernel of lk_T defines the infinite cyclic cover U_T of $T(V)-T(V)\cap(V\cup H)$. For any c the map lk_c coincides with the composition of the map of the fundamental groups induced by the embedding $B_c - B_c\cap V \to T(V)-T(V)\cap(V\cup H)$ and the map lk_T. A similar factorization takes place for the linking number homomorphism of $\pi_1(B_0)$, which defines the infinite cyclic cover U_0 of B_0. We obtain

$$(4.3)\qquad U_T = U_0 \cup_c U_c.$$

We claim that U_T is $n-1$-connected and in particular $\pi_n(T(V)-T(V)\cap(V\cup H)) = \mathrm{H}_n(U_T,\mathbb{Z})$. Indeed the fundamental group of U_T can be obtained from the fundamental group of U_0 by an induction corresponding to adding U_c one by one using the van Kampen theorem (on π_1 of the union). Each time, the fundamental group is replaced by the quotient by the image of the fundamental group of the link of the corresponding singularity (in the case that $c\in\mathrm{Sing}_\infty(V)$, one should rather take the quotient by the image of the fundamental group of the complement to the intersection of the link of the singularity with the hyperplane at infinity inside this link). But the fundamental group of the affine portion of the smoothing of V, which is transversal to H, is calculated by the same procedure. Because the latter is simply connected, we obtain that the fundamental group of U_T is trivial.

On the other hand, we have the following Mayer–Vietoris sequence:

$$(4.4)\qquad \bigoplus_c \mathrm{H}_i(U_0\cap U_c) \to \bigoplus_c \mathrm{H}_i(U_c)\oplus\mathrm{H}_i(U_0) \to \mathrm{H}_i(U_T) \to .$$

For $2\le i\le n-1$ we have $\mathrm{H}_i(U_c) = \mathrm{H}_i(V_c(s)-V_c(s)\cap H) = 0$ (where, as in Remark 4.2, $V_c(s)$ is a smoothing of a singularity $c\in\mathrm{Sing}\,V\cup\mathrm{Sing}_\infty V$). This follows from the standard connectivity of the Milnor fiber for finite singularities and, for singularities at infinity, it follows from the latter, the exact sequence of the pair and $\mathrm{H}_i(V_c(s),V_c(s)-V_c(s)\cap H) = \mathrm{H}_{i-2}(V_c(s)\cap H) = 0$ for $2\ne i\le n$ (the first isomorphism is a consequence of excision and the Poincaré duality). On the other hand, the map $\bigoplus_c \mathrm{H}_i(U_0\cap U_c)\to\mathrm{H}_i(U_0)$ is surjective for $i\le n-1$. To see this let us consider excision of the union of small balls about all singular points of V. This shows that

$$\mathrm{H}_i\Big(U_0 \cap \Big(\bigcup_c U_c\Big)\Big) = \mathrm{H}_i\Big(\bigcup_c \partial V_c(s) - \partial V_c(s) \cap H\Big) \to \mathrm{H}_i(U_0)$$
$$= \mathrm{H}_i\Big(V' - H \cap V' - \bigcup_c (V_c(s) - V_c(s) \cap H)\Big)$$

is surjective for $i \leq n-1$ and is an isomorphism for $i \leq n-2$, because

$$\begin{aligned}
&\mathrm{H}_i\Big(V' - H \cap V' - \bigcup_c (V_c(s) - V_c(s) \cap H), \bigcup_c \partial V_c(s) - \partial V_c(s) \cap H\Big) \\
&= \mathrm{H}_i\Big(V', \bigcup_c V_c(s) \cup H \cap V'\Big) \qquad \text{(by excision)} \\
&= \mathrm{H}_i(V, V \cap H) \qquad \text{(by retraction)} \\
&= 0
\end{aligned}$$

in this range by the Lefschetz theorem. Moreover from $\mathrm{H}_i(U_0 \cap U_c) = \mathrm{H}_i(\partial V_c(s)) = 0$ for $0 < i < n-1$ we obtain $\mathrm{H}_i(U_T) = 0$ for $0 < i < n$. □

For $i = n$ the sequence (4.4), being equivariant with respect to the action of $\mathbb{Z}$ by the deck transformations, implies that if

$$\begin{aligned}
Q &= \mathrm{ord}\Big(\mathrm{Ker} \bigoplus_c \mathrm{H}_{n-1}(U_0 \cap U_c) \to \bigoplus_c \mathrm{H}_{n-1}(U_c) \oplus \mathrm{H}_{n-1}(U_0)\Big) \\
&= \mathrm{ord}(\mathrm{Ker} \bigoplus_c (\mathrm{H}_{n-1}(U_0 \cap U_c) \to \mathrm{H}_{n-1}(U_0))
\end{aligned}$$

and

$$R = \mathrm{ord}\Big(\mathrm{Coker} \bigoplus_c \mathrm{H}_n(U_0 \cap U_c) \to \bigoplus_c \mathrm{H}_n(U_c) \oplus \mathrm{H}_n(U_0)\Big),$$

then $\mathrm{ord}(\mathrm{H}_n(U_T)) = Q \cdot R$. The orders of $\mathrm{H}_{n-1}(U_0 \cap U_c)$, $\mathrm{H}_n(U_0 \cap U_c)$ and $\mathrm{H}_n(U_0)$ are powers of $t-1$, because those are the cyclic covers of the (trivial) circle bundles. The order of $\bigoplus \mathrm{H}_n(U_c)$ is equal to $\Pi_c P_c$ and, hence, R divides this product multiplied by a power of $t-1$. Therefore the order of $\mathrm{H}_n(T(V) - T(V) \cap (V \cup H))$ divides $\Pi_c P_c$ multiplied by $(t-1)^\kappa$ for some κ. It follows from the exact sequence (4.1) that the same is true for order $P_V(t)$ of $\pi_n(\mathbb{CP}^{n+1} - (V \cup H))$. To conclude the proof we need to show that the order of 0 of $P_V(t)$ at 1 does not exceed the sum of the orders of the 0 at 1 of $P_c(t)$.

If one assumes case (a) above, then according to Lemmas 1.12 and 1.6, $P_V(1) \neq 0$ and the theorem follows. Moreover, in case (b), the order of the 0 of $P_V(t)$ at 1 is equal to the rank of $\mathrm{H}^{n+1}(V, H \cap V, \mathbb{Q})$, as follows from the sequence (1.3) and the assumption on semisimplicity of the submodule of

$\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ annihilated by a power of $(t-1)$. The order of the 0 of $P_c(t)$ at 1 is greater than or equal to the rank $\mathrm{H}_{n-1}(L_c, \mathbb{Q})$, where L_c is the link of the singularity c (see [Mi]). Hence, in case (b), the theorem follows from the inequality

$$rk\,\mathrm{H}^{n+1}(V, V \cap H, \mathbb{Q}) \leq \Sigma_c rk\,\mathrm{H}_{n-1}(L_c, \mathbb{Q}). \tag{4.5}$$

To show this, note that

$$\begin{aligned}\mathrm{H}^{n+1}(V, V \cap H, \mathbb{Q}) &= \mathrm{H}^{n+1}(V, V \cap H \cup S, \mathbb{Q}) \\ &= \mathrm{H}^{n+1}(V - T(H) \cap V, (\partial T(H) \cap V) \cup T(S), \mathbb{Q}),\end{aligned}$$

where S is the collection of the singular points of V outside H, $T(S)$ is a small regular neighborhood of this finite set in V and $T(H)$ is the regular neighborhood of $V \cap H$ in V. The last cohomology group is dual to $\mathrm{H}_{n-1}(V - V \cap H - S, \mathbb{Q})$ (use excision of $T(S)$). Inequality (4.5) will follow from the exact sequence of the pair and the vanishing of $\mathrm{H}_{n-1}(V - S - \dot{V} \cap H, \bigcup L_c, \mathbb{Q})$. This group is isomorphic to $\mathrm{H}_{n-1}(\widetilde{V} - H \cap V, \bigcup M_c, \mathbb{Q})$, where M_c is the Milnor fiber of the singularity c and $\widetilde{V}$ is a generic hypersurface in the pencil of hypersurfaces containing V and having $V \cap H$ as the base locus. The vanishing of the last group is a consequence of the $n-1$ connectedness of the Milnor fibers M_c, and the vanishing of $\mathrm{H}_{n-1}(\widetilde{V} - V \cap H)$ follows from the exact sequence of the pair $(\widetilde{V}, \widetilde{V} - H \cap V)$.

THEOREM 4.5. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ having only isolated singularities including infinity. Let H be the hyperplane at infinity. Let $\mathbb{S}_\infty$ be a sphere of a sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$ (or equivalently the boundary of a sufficiently small tubular neighborhood of H in $\mathbb{CP}^{n+1}$). Let U_∞ be the infinite cyclic cover of $\mathbb{S}_\infty - V \cap \mathbb{S}_\infty$ corresponding to the kernel of the homomorphism $\pi_1(\mathbb{S}_\infty - V \cap \mathbb{S}_\infty) \to \mathbb{Z}$ given by the linking number (see remark below). Let P_∞ be the order of $\mathrm{H}_n(U_\infty, \mathbb{Q})$ as the $\mathbb{Q}[t, t^{-1}]$-module. Then P_V divides P_∞.*

Remark 4.6. Note that $V \cap \mathbb{S}_\infty$ is a connected manifold if $n \geq 2$. If $n = 1$, the number of connected components of $V \cap \mathbb{S}_\infty$ is "the number of places of the curve at infinity." By Alexander duality $\mathrm{H}_1(\mathbb{S}_\infty - V \cap \mathbb{S}_\infty, \mathbb{Z}) = \mathrm{H}^{2n-1}(V \cap \mathbb{S}_\infty, \mathbb{Z})$. The latter is isomorphic to $\mathbb{Z}$, provided that $n \geq 2$. For curves (i.e., $n = 1$), $\mathrm{H}_1(\mathbb{S}_\infty - \mathbb{S}_\infty \cap V, \mathbb{Z})$ is a free abelian group of the rank equal to the number of places at infinity.

Remark 4.7. Note that $\mathrm{H}_n(U_\infty, \mathbb{Q})$ is a torsion module. Indeed let $L_c(V)$ (resp. $L_c(V \cap H)$) be the link of the singularity c of V (resp. $V \cap H$) in $\mathbb{CP}^{n+1}$ (resp. H). Let B_c be a polydisk in $\mathbb{CP}^{n+1}$ of the form $D_c^{2n} \times D_c^2$ about c such that part of its boundary $\mathbb{S}^1_c \times D^{2n}_c \subset \partial T(H) = \mathbb{S}_\infty$. Then

$$\mathbb{S}_\infty - \mathbb{S}_\infty \cap V = \partial T(H) - \partial T(H) \cap V = \bigcup_c (\mathbb{S}^1_c \times D^{2n}_c - V) \cup U, \tag{4.6}$$

where U fibers over $H - H \cap V$ with a circle as a fiber and, hence, the homology of the infinite cyclic cover of U is a torsion $\mathbb{Q}[t, t^{-1}]$-module (of the order that is a power of $t^d - 1$, where d is the degree of V, because the circle about H is homologous to d multiplied by the generator of $\mathrm{H}_1(\mathbb{CP}^{n+1} - (V \cup H))$). Then $\mathbb{S}^1_c \times D^{2n}_c - V = B_c - (V \cup H) \cap B_c$, and the homology groups of the infinite cyclic cover in question are $\mathbb{Q}[t, t^{-1}]$-torsion modules, as follows from Remark 4.2. Finally the intersection in the union (4.6) of U with $\mathbb{S}^1_c \times D^{2n}_c - V \cap \mathbb{S}^1_c \times D^{2n}$ fibers over $L_c(V \cap H)$. Hence the homology of its infinite cyclic cover is a torsion $\mathbb{Q}[t, t^{-1}]$-module as well. Hence the Mayer–Vietoris sequence yields the claim.

Proof of Theorem 4.5. First note that $\mathbb{S}_\infty \cap V$ is homotopy equivalent to $T(H) - T(H) \cap (V \cup H)$, where $T(H)$ is the tubular neighborhood of H for which $\mathbb{S}_\infty$ is the boundary. If L is a generic hyperplane in $\mathbb{CP}^{n+1}$, which we can assume to be contained in $T(H)$, then once again by the Lefschetz theorem we find that the composition

$$\begin{aligned} \pi_n(L - L \cap (V \cup H)) &\to \pi_n(T(H) - T(H) \cap (V \cup H)) \\ &\to \pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \end{aligned} \tag{4.7}$$

is surjective. Now the theorem follows from the multiplicativity of the order in exact sequences. □

COROLLARY 4.8. *If V is a hypersurface transversal to the hyperplane H at infinity, then $\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ is a semisimple $\mathbb{Q}[t, t^{-1}]$-module. Any root of the order P_V of the homotopy group is a root of 1 of degree d.*

Proof. The surjectivity of the right homomorphism in the sequence (4.7) shows that the claim will follow from the semisimplicity of $\pi_n(T(H) - T(H) \cap (V \cup H) \otimes \mathbb{Q})$ as a $\mathbb{Q}[t, t^{-1}]$-module. But $T(H) - T(H) \cap (V \cup H)$ is homotopy equivalent to the link of the singularity $\mathcal{V}_0 : z_1^d + \cdots + z_{n+1}^d = 0$ ($d = \deg V$) in $\mathbb{C}^{n+1}$, provided that V is transversal to H. Indeed the projective closure of this hypersurface intersects H in a nonsingular hypersurface which is isotopic to $V \cap H$, and this isotopy can be extended to a neighborhood of $H \cap V$. The monodromy of $\mathcal{V}_0$ is semisimple (this is the case for any weighted homogeneous singularity, because this monodromy has a finite order, as one can see from the explicit description of it (cf. [M])). The last part in the statement of the corollary follows from Milnor's formula for the characteristic polynomial of the monodromy of weighted homogeneous polynomials applied to the singularity $\mathcal{V}_0$ (cf. [M]). □

COROLLARY 4.9. *Let V be a hypersurface in $\mathbb{CP}^{n+1}$ given by equation $f = 0$. Assume that the singularities of V have codimension k in V. If V is transversal to the hyperplane H at infinity (as a stratified space), then $\pi_k(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ is isomorphic as a $\mathbb{Q}[t, t^{-1}]$-module to $\mathrm{H}_k(M_f, \mathbb{Q})$, where M_f is the Milnor fiber of the singularity at the origin in $\mathbb{C}^{n+2}$ of the hypersurface $f(x_0, \ldots, x_{n+1}) = 0$, with the usual module structure given by the monodromy operator.*

Proof. We can assume that the singularities of V are isolated, because the general case can be reduced to this via the Lefschetz theorems (see Section 1). First notice that $\pi_n(\mathbb{CP}^{n+1} - V)$ is isomorphic to $\mathrm{H}_n((\widetilde{\mathbb{CP}^{n+1} - V})_d, \mathbb{Z})$, where $(\widetilde{\mathbb{CP}^{n+1} - V})_d$ is the $d = \deg V$-fold cyclic covering of $\mathbb{CP}^{n+1} - V$, because π_1 of the latter is $\mathbb{Z}/d\mathbb{Z}$. This d-fold covering is analytically equivalent to the affine hypersurface $f = 1$, which is diffeomorphic to M_f. The deck transformation of the covering $(\widetilde{\mathbb{CP}^{n+1} - V})_d \to \mathbb{CP}^{n+1} - V$ in this model corresponds to the transformation induced by multiplication of each coordinate of $\mathbb{C}^{n+2}$ by a primitive root of unity of degree d. It is well known that this is also a description of the monodromy of a weighted homogeneous polynomial (cf. [M]). Finally, according to Lemma 1.13

$$\pi_n(\mathbb{CP}^{n+1} - V) \otimes \mathbb{Q} = \pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}/(t^d - 1) \\ \cdot \pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q},$$

which is isomorphic to $\pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q}$ because of the results contained in Corollary 4.8. □

Remark 4.10. Corollary 4.9 is an extension to high dimensions of a result due to R. Randell [R], which gives a similar fact for Alexander polynomials. In the case of irreducible curves the divisibility theorem, Theorem 4.3, gives a somewhat different result than the one in [L2], where it is shown that the Alexander polynomial divides the product of the characteristic polynomials of all *branches* of the curve in all singular points.

5. Nontrivial π_n

The purpose of this section is to prove two results that allow one to construct special classes of hypersurfaces with isolated singularities for which $\pi_n(\mathbb{C}^{n+1} - V) \otimes \mathbb{Q}$ is nontrivial. We start with the following lemma which may be of independent interest:

LEMMA 5.1. *Let $p(x_1, \ldots, x_{l+1})$ be a polynomial having a singularity of codimension k at the origin $\mathbf{0}$. Let $x_i = f_i(z_{i,1}, \ldots, z_{i,n_i+1})$ $(i = 1, \ldots, l+1)$ be a collection of polynomials, all of which are assumed to have at most isolated* *singularities at* $\mathbf{0}$. *Suppose that* $n_i \geq k+1$ *for every* i, *for which* f_i *has a singularity at the origin. Then the polynomial of* $N = \sum_{i=1}^{l+1}(n_i+1)$ *variables* $p(f) = p(f_1(z_{1,1},\dots,z_{1,n_1+1}),\dots,f_{l+1}(z_{l+1,1},\dots,z_{l+1,n_{l+1}+1}))$ *has the singularity at* $\mathbf{0}$ *of codimension* k. *If* M_p *and* $M_{p(f)}$ *denote the Milnor fibers of the singularities of* p *and* $p(f)$, *respectively, then* $\mathrm{H}_k(M_p,\mathbb{Z}) = \mathrm{H}_k(M_{p(f)},\mathbb{Z})$ *as* $\mathbb{Z}[t,t^{-1}]$*-modules, where the action of* t *in each case is given by the action of the monodromy operator.*

Proof. First note that the use of induction allows one to reduce this lemma to the case when $f_i(z_{i,1},\dots,z_{i,n_i+1}) = z_{i,1}$ for $i \geq 2$, i.e., when the change of variables takes place only in one of the x_i. Let us select $\epsilon_1 > 0$ and a small ball B_1 in $\mathbb{C}^{n_1+1}$ such that the intersection of the hypersurface $f(z_{1,1},\dots,z_{1,n_i+1}) = s$ with B_1, provided that $0 < |s| \leq \epsilon_1$, is equivalent to the Milnor fiber of f_1. Let us consider a ball B_0 centered at the origin $\mathbb{C}^{l+1}$ of a radius less than ϵ_1. Let $\eta > 0$ be such that for $0 < |s| < \eta$ the portion of $p(x_1,\dots,x_{l+1}) = s$, which belongs to B_0, is equivalent to the Milnor fiber of p. Let L be the intersection of M_η with the coordinate hyperplane $x_1 = 0$ in $\mathbb{C}^{l+1}$. Finally let us fix a polydisk $D \subset \mathbb{C}^{l+n_1+1}$, the projection of which on the subspace $x_2 = \cdots x_{l+1} = 0$ belongs to B_1, such that the intersection of D with $p(f) = s$ for $0 < |s| < \eta$ is equivalent to the Milnor fiber of $p(f)$. On a part D' of the polydisk D the formula

$$x_1 = f_1(z_{1,1},\dots,z_{1,n_1+1}), \quad x_i = z_{i,1}(i = 2,\dots,l+1) \tag{5.1}$$

defines a holomorphic map $F : D' \to B$. This map, when restricted on a Milnor fiber $M_{p(f)} \subset D'$ of $p(f)$, which is given by $p(f) = \eta$, takes $M_{p(f)}$ onto the Milnor fiber of p given by $p = \eta$. Let $\tilde{L}$ be the preimage of L: $F^{-1}(L)$. The restriction of F on $M_{p(f)} - \tilde{L}$ is a locally trivial fibration $F : M_{p(f)} - \tilde{L} \to M_p - L$. The fiber of this fibration is equivalent to the Milnor fiber M_{f_1} of $f_1(z_{1,1},\dots,z_{1,n_1+1})$. This Milnor fiber is (n_1-1) connected. The Leray spectral sequence $E^2_{p,q} = \mathrm{H}_p(M_p - L, \mathrm{H}_q(M_{f_1},\mathbb{Q})) \Rightarrow \mathrm{H}_{p+q}(M_{p(f)} - \tilde{L},\mathbb{Q})$ shows that the isomorphism $\mathrm{H}_i(M_{p(f)} - \tilde{L},\mathbb{Q}) = \mathrm{H}_i(M_p - L,\mathbb{Q})$ will take place for the set of the i, which includes k and $k+1$. The following diagram (which compares two exact sequences of a pair)

$$\begin{array}{ccccccc}
\mathrm{H}_{k+1}(M_{p(f)}, M_{p(f)} - \tilde{L},\mathbb{Q}) & \to & \mathrm{H}_k(M_{p(f)} - \tilde{L},\mathbb{Q}) & \to & \mathrm{H}_k(M_{p(f)},\mathbb{Q}) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\mathrm{H}_{k+1}(M_p, M_p - L,\mathbb{Q}) & \to & \mathrm{H}_k(M_p - L,\mathbb{Q}) & \to & \mathrm{H}_k(M_p,\mathbb{Q}) & \to \\
\to \mathrm{H}_k(M_{p(f)}, M_{p(f)} - \tilde{L},\mathbb{Q}) & \to & \mathrm{H}_{k-1}(M_{p(f)} - \tilde{L},\mathbb{Q}) & \to & & \\
\downarrow & & \downarrow & & & \\
\to \mathrm{H}_k(M_p, M_p - L,\mathbb{Q}) & \to & \mathrm{H}_{k-1}(M_p - L,\mathbb{Q}) & \to & &
\end{array} \tag{5.2}$$

and the 5-lemma yield that the isomorphism of our lemma is a consequence of the isomorphism

$$\text{(5.3)} \qquad \mathrm{H}_i(M_{p(f)}, M_{p(f)} - \tilde{L}, \mathbb{Q}) \to \mathrm{H}_i(M_p, M_p - L, \mathbb{Q}) \quad \text{for } i = k, k+1.$$

Let $T(L)$ (resp. $T(\tilde{L})$) be the regular neighborhoods of L (resp. $\tilde{L}$) in M_p (resp. $M_{p(f)}$) and $\partial T(L)$ (resp. $\partial T(\tilde{L})$) be the boundary of $T(L)$ (resp. $T(\tilde{L})$). Then using excision, one obtains $\mathrm{H}_i(M_{p(f)}, M_{p(f)} - \tilde{L}, \mathbb{Q}) = \mathrm{H}_i(T(\tilde{L}), \partial T(\tilde{L}), \mathbb{Q})$ and $\mathrm{H}_i(M_p, M_p - L, \mathbb{Q}) = \mathrm{H}_i(T(L), \partial T(L), \mathbb{Q})$. But $\mathrm{H}_i(T(\tilde{L}), \mathbb{Q}) = \mathrm{H}_i(T(L), \mathbb{Q})$. Indeed $\tilde{L}$ fibers over L with a contractible fiber (i.e., the part of $f_1 = 0$ inside B_1, which is the cone over the link of the singularity f_1), and $\partial T(\tilde{L})$ is a fibration over $\partial T(L)$ with an $(n_1 - 1)$-connected fiber. Hence the last claimed isomorphism follows from the 5-lemma. □

Lemma 5.1 has the following implication:

THEOREM 5.2. *For an integer k let $g_k(z_0, \ldots, z_{n+1})$ be a generic form of degree k. Let $V_{d_1,\ldots,d_{n+1}}$ be a hypersurface of degree $D = d_1 \cdot d_2 \cdot \cdots \cdot d_{n+1}$ given by the equation*

$$\text{(5.4)} \qquad g^{d_1}_{d_2 \cdot \cdots \cdot d_{n+1}} + g^{d_2}_{d_1 \cdot d_3 \cdot \cdots \cdot d_{n+1}} + \cdots + g^{d_{n+1}}_{d_1 \cdot d_2 \cdot \cdots \cdot d_n} = 0;$$

$V_{d_1,\ldots,d_{n+1}}$ is a hypersurface in $\mathbb{CP}^{n+1}$ with D^n isolated singularities each of which is equivalent to the singularity at the origin of $q(x_1, \ldots, x_{n+1}) = x_1^{d_1} + \cdots + x_{n+1}^{d_{n+1}}$. For a generic hyperplane $H \subset \mathbb{CP}^{n+1}$ there is the isomorphism

$$\text{(5.5)} \qquad \pi_n(\mathbb{CP}^{n+1} - (V \cup H)) \otimes \mathbb{Q} = \mathrm{H}_n(M_q, \mathbb{Q}),$$

where M_q is the Milnor fiber of the singularity of q at the origin. This isomorphism is the isomorphism of $\mathbb{Q}[t, t^{-1}]$-modules, where the module structure on the right is the one for which t acts as the monodromy operator.

Proof. The hypersurface $V_{d_1,\ldots,d_{n+1}}$ is a section by a generic linear subspace of dimension $n+1$ of a hypersurface $\widetilde{V}_{d_1,\ldots,d_{n+1}}$ in $\mathbb{CP}^{(n+2)(n+1)-1}$ given by

$$\text{(5.6)} \qquad \begin{aligned} G_{d_1,\ldots,d_{n+1}} = {} & \tilde{g}^{d_1}_{d_2 \cdot \cdots \cdot d_{n+1}}(z_{1,0}, z_{1,1}, \ldots, z_{1,n+1}) \\ & + \cdots + \tilde{g}^{d_{n+1}}_{d_1 \cdot \cdots \cdot d_n}(z_{n+1,0}, \ldots, z_{n+1,n+1}) = 0, \end{aligned}$$

where $\tilde{g}_k$ are generic forms of a disjoint set of variables. The hypersurface (5.6) has a singular locus containing the component of codimension $n+1$ in the ambient space, which is given by $\tilde{g}_1 = \cdots \tilde{g}_{n+1} = 0$ (i.e., having codimension n inside the hypersurface) as well as possibly some components of larger codimensions. According to Corollary 4.9, for a generic hyperplane $\widetilde{H}$ the module $\pi_n(\mathbb{CP}^{(n+2)(n+1)-1} - (\widetilde{V}_{d_1,\ldots,d_{n+1}} \cup \widetilde{H})) \otimes \mathbb{Q}$ is isomorphic to $\mathrm{H}_n(M_{G_{d_1,\dots,d_{n+1}}},\mathbb{Q})$ with the usual $\mathbb{Q}[t,t^{-1}]$-module structure. The forms $\tilde{g}_{d_2\cdot\dots\cdot d_{n+1}},\dots,\tilde{g}_{d_1\cdot\dots\cdot d_n}$ have isolated singularities at the origins of corresponding $\mathbb{C}^{n+2}$ because of the assumption that g_k are generic. Hence the last lemma implies that $\mathrm{H}_n(M_{G_{d_1,\dots,d_{n+1}}},\mathbb{Q})$ is isomorphic to $\mathrm{H}_n(M_q,\mathbb{Q})$. □

PROPOSITION 5.3. *Let $f_i=0$ $(i=1,2)$ be the equation of a hypersurface V_{f_i} of a degree d in $\mathbb{CP}^{n_i+1}$. Assume that the codimension of the singular locus of V_{f_i} is k_i. Then the hypersurface $V_{f_1+f_2}$ in $\mathbb{CP}^{n_1+n_2+3}$ given by $f_1+f_2=0$ has the singular locus of codimension k_1+k_2+1, and*

$$\begin{aligned}\pi_{k_1+k_2+1}(\mathbb{CP}^{n_1+n_2+3}-V_{f_1+f_2})\otimes\mathbb{Q}\\ =(\pi_{k_1}(\mathbb{CP}^{n_1+1}-V_{f_1})\otimes\mathbb{Q})\otimes_{\mathbb{Q}}(\pi_{k_2}(\mathbb{CP}^{n_2+1}-V_{f_2})\otimes\mathbb{Q}).\end{aligned}\tag{5.7}$$

Proof. This is an immediate consequence of Corollary 4.9 and the Sebastiani–Thom theorem in [ST]. □

Examples 5.4. *Example* 1. Let $f(x_0,x_1,x_2)=0$ be an equation of a curve C of degree 6, which has 6 cusps on a conic curve. The homology of the infinite cyclic cover of $\mathbb{CP}^2-(C\cup L)$, for a generic line L, is isomorphic to $\mathbb{Q}[t,t^{-1}]/(t^2-t+1)$ (see [L2]). Recall that $\pi_1(\mathbb{CP}^2-(C\cup L))$ is the braid group on 3 strings, i.e., the group of the trefoil knot, and hence the homology in question is the Alexander module of the trefoil. Let $g(y_0,y_1,y_2)=0$ be an equation of another sextic with six cusps also on a conic curve. According to Proposition 5.3 (in which the homotopy groups in the case of curves are replaced by the Alexander modules), the generic section by $\mathbb{CP}^4$ of the hypersurface in $\mathbb{CP}^6$ given by

$$f(x_0,x_1,x_2)+g(y_0,y_1,y_2,)=0\tag{5.8}$$

is a threefold W, which has isolated singularities (the number of which is 6^2), and the order of the homotopy group $\pi_3\otimes\mathbb{Q}$ of the complement is (t^2-t+1). If one takes as f in equation (5.8) the equation of a sextic with 9 cusps, which is dual to a nonsingular cubic, then one obtains W, for which the order of $\pi_3\otimes\mathbb{Q}$ of the complement is $(t^2-t+1)^3$. Indeed the Alexander module of the complement to a sextic with 9 cusps is $(\mathbb{Q}[t,t^{-1}]/(t^2-t+1))^{\oplus 3}$. This is a consequence of calculating the fundamental group of the complement to such a curve, due to Zariski. He found that the fundamental group of this curve is the kernel of the canonical map of the braid group of the torus $\mathbb{S}^1\times\mathbb{S}^1$ onto $\mathrm{H}_1(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{Z})$ (see [Z]) Combining this with the calculation that uses the Fox calculus and the presentation for the braid group of the torus, one arrives at the Alexander module, as above. Therefore one obtains the threefold W with $\pi_3(\mathbb{CP}^4-W)\otimes\mathbb{Q}$ being the same as the Alexander module just mentioned. Iteration of this construction, obtained by replacing g in (5.8) by the equation of an n-dimensional hypersurface with isolated singularities and a nonvanishing homotopy group $\pi_n \otimes \mathbb{Q}$ of the complement, gives examples of hypersurfaces of degree 6 and dimension n for which $\pi_n \otimes \mathbb{Q}$ has an arbitrary large rank for sufficiently large n.

Example 2. Let us consider the equation (5.8) in which $f(x_0, x_1, x_2)$ is the form giving the equation of a sextic with 6 cusps not on a conic curve. The fundamental group of the complement to such a curve is abelian (see [Z]) and, therefore, the homology of the universal cyclic cover of the complement is trivial. In this case, the construction of Example 1 yields a threefold W', which has $\pi_3(\mathbb{CP}^4 - W') \otimes \mathbb{Q} = 0$, but which has the same degree (i.e., 6) and the same number of singularities (i.e., 36) of the same type as W (i.e., locally given by $x^2 + y^2 + u^3 + v^3 = 0$).

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