Automorphisms of desingularizations of quotients and equivariant McKay correspondence

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Abstract *We propose a formula for calculation of the Lefschetz number of an automorphism acting on a crepant resolution of a quotient space. It derives from an equivariant form of McKay correspondence which is verified for automorphisms of finite subgroups of $SL_2(\mathbf{C})$ or finite abèlian subgroups of $SL_3(\mathbf{C})$. This is applied to comparison of the Lefschetz numbers of involutions of pairs of mirror manifolds.*

Les automorphismes de désingularisation des quotients et la correspondance de McKay équivariante

Résumé. *Nous proposons une formule pour le calcul du nombre de Lefschetz d'un automorphisme d'une résolution crépante d'un quotient. Cette formule est dérivée d'une forme équivariante de la correspondance de McKay qui est vérifiée pour les automorphismes des sous-groupes de $SL_2(\mathbf{C})$ ou des sous-groupes finis et abéliens de $SL_3(\mathbf{C})$. On l'emploie pour comparer les nombres de Lefschetz des variétés "miroir".*

Version française abrégée. Soit X une variété kählerienne, G un groupe fini d'automorphismes biholomorphes de X qui préservent le volume (holomorphe) de X et soit $\widetilde{X/G} \to X/G$ une désingularisation crépante (i.e. pour laquelle $\pi^*(\Omega_{X/G}) = \Omega_{\widetilde{X/G}}$ où Ω_Z est le faisceau dualisant de Z).

Soit h un automorphisme biholomorphe de X pour lequel le groupe H engendré par h et G est fini et pour lequel G est normal dans H. Pour $g \in G$, soit X^g le lieu des points fixes de g, $[g]$ la classe de conjugaison de g et, pour $x \in X$, soit $Stab_x$ le stabilisateur de x. Soit $\mathcal{C}(h) = \{g \in G | hgh^{-1} = g\}$ le centralisateur de h dans G. Nous supposons que l'une des conditions suivantes est vérifiée:

1. L'action de G commute avec l'action de h et $(X^h)/G = (X/G)^h$,
ou
2. G est abélien.

Alors nous montrons que l'égalité

$$L(h, \widetilde{X/G}) = \Sigma_{[g], g \in \mathcal{C}(h)} L(h, X^g/Cent(g)) \tag{1}$$

est un corollaire de la forme suivante de la "correspondance de McKay" :

(**) *Soient H un sous-groupe fini de $GL_n(\mathbf{C})$ et G un sous-groupe normal de H inclus dans $SL_n(\mathbf{C})$ pour lequel H/G est un groupe cyclique de générateur h. Soit $\widetilde{\mathbf{C}^n/G} \to \mathbf{C}^n/G$ une désingularisation crépante. Alors le nombre des classes de conjugaison de G qui sont fixées par h est égal au nombre de Lefschetz de l'action de h sur $\widetilde{\mathbf{C}^n/G}$.*

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La démonstration de (1) utilise la stratification naturelle de X/G et l'additivité du nombre de Lefschetz.

Si $n = 1$, alors l'égalité (1) se réduit à la formule de Dixon, Harvey, Vafa et Witten ([1], [4], [7]).

Si $n = 2$, on vérifie (**) par calcul du nombre de Lefschetz et du nombre des classes de conjugaison pour les sous-groupes $\mathbf{Z}_n$, D_n et E_6 de $SL_2(\mathbf{C})$.

Pour G abélien et $n = 3$, on vérifie (1) en utilisant la désingularisation crépante torique *équivariante* de $\mathbf{C}^3/G$.

Si V est une quintique dans $\mathbf{CP}^4$, $\tilde{V}$ est le miroir correspondant et h est une permutation des coordonnées: $h(x_0, x_1, x_2, x_3, x_4, x_5) = (x_1, x_0, x_2, x_3, x_4, x_5)$, alors on montre, en employant (1), que $L(h, V) = L(\tilde{h}, \tilde{V}) = 56$ où $\tilde{h}$ est une involution "mirroir" sur $\tilde{V}$. Pour $h : (x_0, x_1, x_2, x_3, x_4, x_5) \to (x_1, x_0, x_2, x_4, x_3)$, on montre que $L(h, V) = -8$ et $L(\tilde{h}, \tilde{V}) = 8$.

Pour V intersection complète de bidegré $(3, 3)$ dans $\mathbf{CP}^5$, avec $\tilde{V}$ le miroir correspondant et h la permutation de coordonnées $(x_1, x_2, x_3, x_4, x_5, x_6) \to (x_2, x_1, x_3, x_5, x_4, x_6)$, on a $L(h, V) = -16$ et $L(\tilde{h}, \tilde{V}) = 16$.

1. Introduction. Let X be a Kähler manifold and G be a finite group of biholomorphic automorphisms of X preserving holomorphic volume. Let $\pi : \widetilde{X/G} \to X/G$ be a crepant resolution of the orbit space X/G (i.e. the pullback of the dualizing sheaf of X/G is the canonical bundle of $\widetilde{X/G}$). The Euler characteristic of $\widetilde{X/G}$ can be found as follows: (cf. [1],[4],[7]):

$$e(\widetilde{X/G}) = \frac{1}{|G|}\Sigma_{hg=gh} e(X^g \cap X^h) = \Sigma_{[g]} e(X^g/C(g)) \qquad (0)$$

Here X^g, for $g \in G$, is the fixed point set of an automorphism, $C(g)$ is the centralizer of g and the summation in is over all conjugacy classes $[g]$ of elements $g \in G$. This formula plays a key role in validating the relation $e(X) = -e(X')$ for Calabi Yau manifolds X, X' which constitute mirror pairs (cf. [2]).

In this note we propose a formula designed for the study of manifolds for which the pair (X, G) admits a symmetry h. This formula allows one to calculate the Lefschetz number of an automorphism acting on a crepant resolution of the quotient (for $h = 1$ this formula becomes (0)). We show that it can be derived from an equivariant form of McKay correspondence between the data from a crepant resolution of a quotient singularity $\mathbf{C}^n/G$, where G admits an outer automorphism h and the action of h on the conjugacy classes of G. This version of McKay correspondence is valid in dimension $n = 2$ and if G is abelian for $n = 3$ (cf. sect. 4). In particular this yields an intriguingly simple relation among the Lefschetz numbers of involutions on some Calabi Yau threefolds (considered in [2] and [5]) and their mirrors (sect. 3).

More precisely we have the following (here $L(h, Z)$ denotes the Lefschetz number $\Sigma_i(-1)^i tr(h, H^i(Z))$ of a transformation h acting on a topological space Z):

Theorem. Let X be a Kähler manifold on which a finite group G of holomorphic volume preserving automorphisms acts holomorphically and let h be a biholomorphic automorphism of X such that the group H generated by G and h is finite and contains G as a normal subgroup. Let $\widetilde{X/G}$ be a crepant resolution of X/G. Let $\mathcal{C}_G(h)$ be the centralizer of h in G. If (**) below takes place and either

(A) h commutes with the action of G and $(X^h)/G = (X/G)^h$,

or

(B) G is abelian.

then

$$L(h, \widetilde{X/G}) = \Sigma_{[g], g \in \mathcal{C}_G(h)} L(h, X^g/C(g)) \tag{1}$$

Note that h leaves invariant X^g (in the case B for $g \in \mathcal{C}_G(h)$) and acts on $C(g)$-orbits. In the case (A) $x \to l \cdot x, l \in G$ provides an h-equivariant isomorphism $X^g \to X^{lgl^{-1}}$ and hence $L(h, X^g/C(g))$ is the same for all g in the same conjugacy class.

The proof of (1) is based on (a use of special cases of) the following "equivariant form of the weak McKay correspondence" (cf. [3],[1])

(**) *Let H be a finite subgroup of $GL_n(\mathbf{C})$, G a normal subgroup of H which belongs to $SL_n(\mathbf{C})$ and such that H/G is a cyclic group with a generator h. (In particular h induces automorphisms of $\mathbf{C}^n/G$ and G). Then the Lefschetz number of h acting on an h-equivariant crepant resolution of $\mathbf{C}^n/G$ is equal to the number of h-invariant conjugacy classes in G.*

This will be proven below when $n = 2$ or $n = 3$ and G abelian. One of the consequences of the theorem 1 is the equality $L(X, h) = \pm L(\tilde{X}, \tilde{h})$ for some mirror pairs $X, \tilde{X}$ and involutions $h, \tilde{h}$ on respective manifolds.

2.Proof of the theorem. Let us consider the case of abelian G. Let $\mathcal{S} = \{S\}$ be the collection of subgroups of G which are the stabilizers of points in X. The quotients X^S/G provide a stratification of X/G. Let $\pi^{-1}(X^S/G) \subset \widetilde{X/G}$ be the preimages of these strata in a crepant resolution $\pi : \widetilde{X/G} \to X/G$. We have $L(h, \widetilde{X/G}) = \Sigma_{S \in \mathcal{S}, h(S)=S} L(h, \pi^{-1}(X^S/G))$ since $L(h, \bigcup_{h(S) \neq S} \pi^{-1}(X^S/G)) = 0$ because h has no fixed points on the latter.

For an h-invariant S let $inv(h, S)$ be the set of h-invariant elements in S. A neighbourhood of a point $\bar{x} \in X^S/G$ corresponding to an h-invariant orbit Gx (i.e. such that $hx = gx$) is isomorphic to U_x/S for some neighbourhood U_x of x. We can assume that the symmetry $h_g = g^{-1}h$ preserves U_x and different choices of g yield equivariantly isomorphic actions of h_g's for which the number of invariant elements in S is the same. We have from (**):

$$\Sigma_{S \in \mathcal{S}, h(S)=S} L(h, \pi^{-1}(X^S/G)) = \Sigma_{h(S)=S} L(h, X^S/G) inv(h, S) =$$

$$\Sigma_{S, h(S)=S} \Sigma_{g \in S, h(g)=g} L(h, X^S/G) = \Sigma_{g, h(g)=g} \Sigma_{S, h(S)=S, g \in S,} L(h, X^S/G) =$$

$$\Sigma_{g, g \in S, h(g)=g} L(h, X^S/G) \tag{2}$$

(since there is no fixed points outside strata corresponding to subgroups for which $h(S) \neq S$). Hence (1) follows in the case B.

In the case when the action of h commutes with the action of G let $X^{[S]} = \cup_T X^T$ where T is a subgroup of G conjugate to $S \in \mathcal{S}$. h acts on $X^{[S]}$ and

$$L(h, \widetilde{X/G}) = \Sigma_{[S]} L(h, \widetilde{X^{[S]}/G}) \tag{3}$$

Now $\widetilde{X^{[S]}/G}$ fibres over $X^{[S]}/G$ with a resolution of the singularity of U_x/S where $x \in X^{[S]}$ and U_x is a neighbourhood of x which supports the action of h (since all points of an h-invariant orbit are fixed by h as follows from A). The Lefschetz number of the resolution is equal to the number $con(S)$ of the conjugacy classes in S (cf. (**)). Hence

$$L(h, \widetilde{X^{[S]}/G}) = L(h, X^{[S]}/G)con(S) = \Sigma_{[g]\in S} L(h, X^{[S]}/G) =$$

$$\Sigma_{S\in[S],[g]\in S} L(h, X^S)\frac{|S|}{|G|} = \Sigma_{S\in[S],g\in S} L(h, X^S)\frac{|C(g)\cap S|\cdot|S|}{|G|\cdot|S|}$$

Substituting this in (3) yields

$$L(h, \widetilde{X/G}) = \Sigma_g \Sigma_{g\in S} L(h, X^S))\frac{|C(g)\cap S|}{|G|} \tag{4}$$

If $\mathcal{S}_g$ denotes the set of $C(g)$-congugacy classes of subgroups S containing g then (4) can be replaced by

$$\Sigma_{g\in G}\Sigma_{\{S\}_g\in\mathcal{S}_g}\Sigma_{S\in\{S\}_g} L(h, X^S)\frac{|C(g)\cap S|}{|G|} =$$

$$\Sigma_{g\in G} L(h, X^g/C(g))\frac{|C(g)|}{|G|} = \Sigma_{[g]} L(h, X^g/C(g))$$

3.Applications to mirror manifolds. Let V be a quintic in $\mathbf{P}^4$ given by the equation (cf. [2]): $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\lambda x_0x_1x_2x_3x_4 = 0$ and let h_1 be the involution $h_1(x_0, x_1, x_2, x_3, x_4) = (x_1, x_0, x_2, x_3, x_4)$. The mirror is a crepant resolution of $\widetilde{V/G}$ where $G = \mathbf{Z}_5^3$ is faithfully-acting quotient of the group of automorphisms of V given by $g(x_0, x_1, x_2, x_3, x_4) = (\omega_5^{\alpha_0}x_0, \omega_5^{\alpha_1}x_1, \omega_5^{\alpha_2}x_2, \omega_5^{\alpha_3}x_3, \omega_5^{\alpha_4}x_4)$ where $\Sigma\alpha_i \equiv 0 mod 5$ and ω_5 is a non-trivial root of unity of degree 5. The condition $g \in \mathcal{C}_G(h_1)$ implies that $\alpha_0 = \alpha_1$. Each 1-dimensional fixed point set of $g \in G$ is an intersection of V with a subspace given by vanishing of two coordinates and is fixed by the elements for which three exponents α_i coincide. Since there are 3 possibilities to select α_i coinciding with chosen $\alpha_0 = \alpha_1$ we obtain 12 elements ($\neq id$) in $\mathcal{C}_G(h_1)$ having 1-dimensional fixed point set. The Lefschetz number of h_1 acting on the quotient of each of these fixed point sets is equal to 2.

Similarly one can argue with the 0-dimensional fixed point sets. Each is the fixed point set of an element with two pairs of equal components ($\alpha_i = \alpha_j$ for two pairs of indices (i, j)). Since $\alpha_0 = \alpha_1$ we see that there are 12 elements ($\neq id$) in $\mathcal{C}_G(h_1)$ having 0 dimensional fixed point set. Since the quotient in the case of 0-dimensional fixed point sets has Euler characteristic 2 and h_1 acts trivially on it we see that the contribution of $g \neq id$ in (1) is $2 \times 12 + 2 \times 12 = 48$. To obtain the contribution of the identity element we identify the cohomology of V/G with the G-invariant part of the cohomology of V. The even dimensional cohomology of V/G in each dimension has rank 1. In dimension 3 the G-invariant part of $H^3(V, \mathbf{C})$ is generated by the residues of meromorphic forms $\omega_\lambda = \frac{\Sigma(-1)^i x_i dx_0\wedge\ldots d\hat{x}_i\ldots\wedge dx_4}{Q(\lambda,x_0,x_1,x_2,x_3,x_4)}$, $\frac{d\omega_\lambda}{d\lambda}$, $\frac{d^2\omega_\lambda}{d\lambda^2}$, $\frac{d^3\omega_\lambda}{d\lambda^3}$ via Griffiths theory (cf.[2], [6]). These forms are h_1-anti-invariant. Hence $L(h_1, V^{id}/G) = 8$. Therefore $L(h_1, \widetilde{V/G}) = 8 + 48 = 56$.

On the other hand $L(h,V)=e(V^h)$. The fixed point set of h_1 acting on V consists of the point $(1,-1,0,0,0)$ and the points of the quintic in the hyperplane $x_0=x_1$. i.e. a non singular quintic surface (having the euler characteristic 55). Hence we obtain: $L(h_1,V)=L(h_1,\widetilde{V/G})(=56)$.

Now let $h_2(x_0,x_1,x_2,x_3,x_4)=(x_1,x_0,x_2,x_4,x_3)$. So $\mathcal{C}_G(h_2)$ consists of elements in G for which $\alpha_0=\alpha_1,\alpha_3=\alpha_4$. There are 4 such elements different from the identity and the Lefschetz number of h_2 acting on the quotient of the fixed point set of each of those is 2. The contribution of the identity is zero since the action of h_2 mentioned above deRham representatives is trivial. On the other hand the fixed point set of h_2 consists of the union of the line $(x_0,-x_0,0,x_3,-x_3)$ and a plane quintic. Hence $L(h_2,V)=-L(h_2,\widetilde{V}/G)(=-8)$.

Let us consider example from [5] in which $V_{3,3}$ is a complete intersection in $\mathbf{P}^5$ given by equations: $x_1^3+x_2^3+x_3^3=3\lambda x_4x_5x_6, x_4^3+x_5^3+x_6^3=3\lambda x_1x_2x_3$ with the group G_{81} of order 81 acting as follows: $(x_1,x_2,x_3,x_4,x_5,x_6)\to(\zeta_3^{\alpha_1}\zeta_9^{\mu}x_1,\zeta_3^{\alpha_2}\zeta_9^{\mu}x_2,\zeta_9^{\mu}x_3,\zeta_3^{\alpha_4}\zeta_9^{-\mu}x_4,$ $\zeta_3^{\alpha_5}\zeta_9^{-\mu}x_5,\zeta_9^{-\mu}x_6)$ where $\alpha_i\in\mathbf{Z}_3, i=1,2,3,4, \mu\in\mathbf{Z}_3$ and $\mu\equiv\alpha_1+\alpha_2\equiv\alpha_4+\alpha_5 mod 3$. Let $h_3(x_1,x_2,x_3,x_4,x_5,x_6)=(x_2,x_1,x_3,x_5,x_4,x_6)$ The condition $g\in\mathcal{C}_G(h_3)$ implies that $\alpha_1=\alpha_2,\alpha_4=\alpha_5$ i.e. $g\in\mathcal{C}(h)$ must have the form: $(x_1,x_2,x_3,x_4,x_5,x_6)\to(\zeta_3^{\alpha}\zeta_9^{\mu}x_1,\zeta_3^{\alpha}\zeta_9^{\mu}x_2,\zeta_9^{\mu}x_3,\zeta_3^{-\alpha}\zeta_9^{-\mu}x_4,\zeta_3^{-\alpha}\zeta_9^{-\mu}x_5,\zeta_9^{-\mu}x_6)$ where $\alpha\in\mathbf{Z}_3,\mu\in\mathbf{Z}_9$ and $2\alpha=\mu mod 3$. Hence there are 9 elements in $\mathcal{C}(h_3)$ out of which 8 non identity elements have zero dimensional fixed point set. The Lefschetz number of h_3 acting on the G_{81}-quotient of each of these is equal to 2. For $g=id$ the contribution in (1) is the Lefschetz number $L(h_3,V^{id}/G_{81})$ which is equal to zero. This follows from explicit expression for the forms representing G_{81}-invariant cohomology classes on $V_{3,3}$ (cf. [5] (8) on p.32) similar to the above in the case of quintic. Hence $L(h_3,\widetilde{V/G_{81}})=0+2\times 8=16$. On the other hand the fixed point set of h acting on V consists of the line: $(x,-x,0,y,-y,0)$ and the intersection of V with $x_1=x_2,x_4=x_5$ (the latter has the Euler characteristic -18). Hence $L(h_3,V)=-L(h_3,\widetilde{V/G_{81}})(=-16)$.

One can wonder if there is a physical reason for these simple equalities $L(h,V)=-sign(h)L(h,\widetilde{V/G})$ between Lefschetz numbers of automorphisms of a Calabi Yau manifold and its mirror which came out in these examples.

4.Actions on resolutions of low dimensional quotient singularities We can verify (**) in dimension 2 by checking the standard list of examples (cf.[8]). In the case $G=\mathbf{Z}_n$ (resp. $G=D_n$, resp. $G=E_6$)for the standard involutions one obtains that the cardinality of G^h and the Lefschetz number both are equal to 2 (n-even) or 1 (n-odd) (resp. $r-1$, resp. 3). In the case of the automorphism of order 3 of D_4 both numbers are equal to 3.

Now let us consider the symmetries of toric crepant resolutions of abelian quotient singularities. Let $\mathbf{T}$ be the maximal torus of $SL_3(\mathbf{C})$ consisting of diagonal matrices with determinant 1, let H be a subgroup of $\mathbf{T}$ and let h be an element of $GL_3(\mathbf{C})$ which normalizes $\mathbf{T}$ and H. It can be viewed as an automorphism of both $\mathbf{T}$ and H. h also acts on the quotient $\mathbf{C}^3/H$. The latter is a toric variety and the lattices of 1-parameter subgroups M and N of the dense tori of $\mathbf{C}^3$ and $\mathbf{C}^3/H$ are related by the following sequence: $0\to M\to N\to H\to 0$. It is well known that toric crepant resolutions of $\mathbf{C}^3/H$ are determined by triangulations of the unit simplex $\Delta=\{(x_1,x_2,x_3)\subset M\otimes\mathbf{R}|x_1+x_2+x_3=1,x_i\geq 0\}$ with vertices $N\cap\Delta$ such that all simplices of a triangulation have volume relative to $N\cap\Delta$ equal to 1. The automorphism h, considered as an element of the Weyl group of $\mathbf{T}$ (i.e.

the symmetric group S_3), has the order equal to either 3 or 2. If $ord\ h = 2$ then h fixes a vertex P. If the intersection of $L = \{Q \in \Delta | hQ = Q\}$ with the side O_P of Δ opposite to P belongs to the lattice N, then we can split Δ by L into the union of two triangles, T and hT, take a triangulation of T in which all vertices are in $N \cap \Delta$ and have area 1 and then take h image of this triangulation in hT. If $ord\ h = 2$ but $L \cap O_P$ is not in N, consider the triangle T_1 of area 1 with vertex in the point $L \cap N$ closest to O_P and two vertices of N on O_P, then triangulate $T - T \cap T_1$ by triangles of area 1 with vertices in N and take its h image to triangulate $hT - hT \cap T_1$.

If $ord\ h = 3$ one similarly extends using h a triangulation of a fundamental domain of h acting on Δ (resp. the complement in Δ to the h-invariant triangle centered at $L \cap \Delta$) if $L \cap \Delta \in N$ (resp. if $L \cap \Delta$ not in N). One verifies directly the relation (**) for these triangulations.

Example. i) Let us consider the action of the subgroup $\mathbf{Z}_5^2$ of $\mathbf{T}$ consisting of matrices of the form: $\begin{pmatrix} \omega_5^a & 0 & 0 \\ 0 & \omega_5^b & 0 \\ 0 & 0 & \omega_5^c \end{pmatrix}$ where $a + b + c \equiv 0 mod 5$ where ω_5 is a root of unity of degree 5 (cf. quintic in sect.3). Let $h \in GL_3(\mathbf{C})$ permute coordinates cyclically. A crepant resolution can be obtained by triangulating the base of the unit simplex for which all triangles have area 1 relative to the lattice with the vertices at the points $(a, b, c), a+b+c = 1, 5a = 5b = 5c = 0 mod 1$ The Lefschetz number of the automorphism of the resolution of $\mathbf{C}^3/\mathbf{Z}^2$ constructed above is equal to 1 (coming from the only invariant triangle of area 1 having $(1/3, 1/3, 1/3)$ as its center) and the only h invariant element of $\mathbf{Z}_5^2$ is the identity.

ii) For the same G-action but h instead interchanging the first two coordinates one has 5 h-invariant elements in $\mathbf{Z}_5^2$. For the triangulation constructed above and which has two $h-$invariant 1-simplices $[(0, 0, 1), (\frac{1}{5}, \frac{1}{5}, \frac{3}{5})]$, $[(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}), (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})]$ (for each of which the Lefschetz number of equals 2) and one h invariant 2-simplex $[(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}), (\frac{2}{5}, \frac{3}{5}, 0), (\frac{3}{5}, \frac{2}{5}, 0)]$ for which the Lefschetz number is 1. Hence the total Lefschetz number of h acting on such crepant resolution is 5.

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