DIFFERENTIAL FORMS ON COMPLETE INTERSECTIONS AND A RELATED QUOTIENT MODULE

A.Libgober

Department of Mathematics
University of Illinois at Chicago

To Professor F.Hirzebruch with admiration

Abstract. The Hodge filtration on the cohomology of a non-singular complete intersection of hypersurfaces in a projective space is described in terms of meromorphic forms with poles along the union of these hypersurfaces. The components of the Hodge decomposition of a complete intersection are identified with the components of a graded module, which in the case of hypersurfaces reduces to the Jacobian algebra.

1.Introduction

Explicit constructions of differential forms for studying cohomology classes of non-singular hypersurfaces in a projective space can be traced back at least to Gauss. More recently, differential forms on non-singular hypersurfaces were investigated in the works of P.Griffths, B.Dwork and N.Katz (cf. also [ABG]) in the late 60's and were used for rather diverse purposes. Among other things, it was shown in Griffiths' work that the cohomology classes of meromorphic forms with poles along a non singular hypersurface correspond to the cohomology classes of the forms on the latter. Moreover the Hodge filtration on the middle dimensional cohomology of a non-singular hypersurface corresponds to the filtration of the space of meromorphic forms given by the order of the pole. Griffiths' theory is quite explicit and it allows one to identify the Hodge spaces $H^{p,q}$ of middle dimensional cohomology of a hypersurface $Q(z_0,...,z_{n+1})=0$ in $\mathbf{CP}^{n+1}$ with the graded components of the Jacobian algebra of $Q(z_0,...,z_{n+1})$, i.e.

$$\mathbf{C}[z_0,...,z_{n+1}]/(\frac{\partial Q}{\partial z_0},...,\frac{\partial Q}{\partial z_{n+1}}) \tag{1.1}$$

Recently, Griffiths' theory was used successfully for the calculations of the Picard-Fuchs equations of families of hypersurfaces (cf. [M]) in weighted projective spacs and for partial verification of conjectures of mirror symmetry. Also, motivated by mirror symmetry, in

1991 *Mathematics Subject Classification.* 14D, 14M.

Key words and phrases. Complete intersections, Hodge filtration, Jacobian algebra.

[HY] this correspondence between the Hodge spaces and the components of the Jacobian algebra was used to a) interpret the duality of components of Hodge decompostion as the duality in the Jacobian algebra (1.1) and b) construct on the latter, in the case when $Q(z_0,...,z_{n+1})=0$ is a Calabi Yau manifold, a natural representation of the Lie algebra $sl_2(\mathbf{C})$.

A relation between the holomorphic forms on a complete intersection in a complex manifold and the forms with logarithmic singularities on the complement contains in a work of Deligne (cf. [D]). For a complete intersection V_n of dimenision n given in $\mathbf{CP}^{n+1}$ by:

$$Q_1(z_0,...,z_{n+r})=0,...,Q_r(z_0,...,z_{n+r})=0, \tag{1.2}$$

one can interpret $H^{n,0}(V_n)$ as the space of forms on the complement to the union of r hypersurfaces, each given by one of r equations (1.2) which have the pole of order 1 along each of r components of the union. In fact, Griffiths's theory can be generalized to complete intersections and this was outlined in [LT] in order to obtain Picard Fuchs equations for the families of complete intersections (cf. also [KT] for further calculations using this method).

One of the purposes of this paper is to provide a justification of the method used in [LT]. Another is to describe a module associated with a complete intersection in a projective space (over $\mathbf{C}[z_0,...,z_{n+r}]$ in the case when the latter is given by equations (1.2)) the components of which, similar to the components of the Jacobian algebra in hypersurface case, correspond to the components of the Hodge decomposition of the complete intersection. This module for a complete intersection (1.2) is a quotient of $\mathbf{C}[u_1,..,u_r,z_0,..z_{n+r+1}]$ where the weights of the variables are determined by the degrees of defining equations. In particular for $r=1$ the algebra is different from the Jacobian algebra (though is closely related to it cf. Example 1 in section 4).

We shall start by describing the correspondence between the cohomology classes from components of Hodge decomposition of complete intersections (1.2) and classes of meromorphic forms on the complement to the union of r hypersurfaces, each of which is given by one of the defining equations of a complete intersection (1.2). We also give a proof of Dolbeault lemma (cf. [G1] App.). Then we shall describe the module whose components correspond to the Hodge spaces of a complete intersection (1.2). Finally we shall give examples of calculations of the components of Hodge decomposition in some particular cases to illustrate the above. The issues of duality and the $sl(2,\mathbf{C})$ representation mentioned above will be discussed elsewhere.

I am grateful to J.Teitelbaum for useful discussions of material in this work which came out from our joint paper [LT].

2. Meromorphic forms on projective space with poles along a divisor with normal crossings

Let $D=\cup V_i$ $(i=1,...r)$ be a divisor in $\mathbf{CP}^{n+r}$ for which V_i are irreducible components given by the equations $Q_i(z_0,...,z_{n+r})=0$ $(i=1,..,r)$. Let $A^s(\mathbf{CP}^{n+r}-D)$ be the space of meromorphic forms of degre s on $\mathbf{C}P^{n+r}$ holomorphic outside of D. Forms from $A^s(\mathbf{CP}^{n+r}-D)$ can be written in homogeneous coordinates $z_0,....,z_{n+r}$ of $\mathbf{CP}^{n+r}$ as a sum of the forms (cf. [G], th.2.9):

$$\frac{1}{Q_1^{s_1}...Q_r^{s_r}}\Sigma_{0\le j_1<...<j_{n+r+1-s}\le n+r, 0\le i_1<...<i_s\le n+r, i_p\ne j_q}[\Sigma(-1)^\alpha z_{j_\alpha}P_{j_1,...\hat{j_\alpha}..,j_{n+r+1-s}}(z_0,..$$

$$...,z_{n+r+1})](-1)^{j_1+...j_{n+r+1-s}}dz_{i_1}\wedge...\wedge dz_{i_s} \quad (2.1)$$

where

$$\Sigma_i s_i deg Q_i = deg P_{j_1,...,\hat{j_\alpha},j_{n+r-s}}+s+1 \quad (2.2)$$

In particular a form of top degree $n+r$ is a sum of the forms

$$\frac{P(z_0,...,z_{n+r})}{Q_1^{s_1}...Q_r^{s_r}}\Omega \quad ,\Omega=\Sigma(-1)^\alpha z_\alpha dz_0\wedge...\hat{dz_\alpha}\wedge dz_{n+r},$$

$$deg P=\Sigma_i s_i deg Q_i-(n+r+1). \quad (2.3)$$

Similarly a form of degree $n+r-1$ is a sum of the forms:

$$\frac{1}{Q_1^{s_1}...Q_r^{s_r}}\Sigma_{i<j}(-1)^{i+j}[z_iP_j-z_jP_i]dz_0\wedge...\wedge\hat{dz_i}\wedge...\wedge\hat{dz_j}\wedge...\wedge dz_{n+r},$$

$$\Sigma_i s_i deg Q_i = deg P_l+(n+r) \quad (2.4)$$

The spaces A^s have a natural filtration defined by the "order of the pole" defined as follows. Let

$$A^s_l(\mathbf{CP}^{n+r}-D)$$

be the space of forms of degree s the sum of the orders of which along irreducible components of D does not exceed l. A form (2.1) belongs to $A^s_l(\mathbf{CP}^{n+r}-D)$ iff $\Sigma s_i\le l$. Since

$$H^{n+r}(\mathbf{CP}^{n+r}-D,\mathbf{C})=A^{n+r}(\mathbf{CP}^{n+r}-D)/dA^{n+r-1}(\mathbf{CP}^{n+r}-D) \quad (2.5)$$

according to algebraic deRham theorem (cf. [Gr]), the filtration on $A^{n+r}(\mathbf{CP}^{n+r}-D)$ defines an increasing filtration on the cohomology of the complement to D by setting

$$F^lH^{n+r}(\mathbf{CP}^{n+r}-D)$$

to be the image of $A^{n+r}_l(\mathbf{CP}^{n+r}-D)$ in (2.5).

Now let us assume that $V\subset\mathbf{CP}^{n+r}$ is a non-singular complete intersection of non-singular hypersurfaces $V_1,...,V_r$. Let $H^n(V_n,\mathbf{C})_0$ be the primitive part of the cohomology i.e. the kernel of the cup product $H^n(V,\mathbf{C})\to H^{n+2}(V,\mathbf{C})$ with the cohomology class $h\in H^2(V,\mathbf{C})$ dual to the hyperplane section of V.

The following lemma provides a link between the vector spaces of meromorphic forms with poles along certain components of $\cup_{k=1}^{k=r}V_k$ and cohomology of V:

Lemma 2.1. The following exact sequence takes place:

$$\oplus_{k=1}^{k=r} A^{n+r}(\mathbf{CP}^{n+r} - V_1 \cup ... \cup V_{k-1} \cup V_{k+1} \cup ... \cup V_r) \oplus A^{n+r-1}(\mathbf{CP}^{n+r} - V_1 \cup ... \cup V_r) \to$$

$$\to A^{n+r}(\mathbf{CP}^{n+r} - V_1 \cup ... \cup V_r) \to H^n(V, \mathbf{C})_0 \to 0 \tag{2.6}$$

The left homomorphism in (2.6) is the direct sum of the r maps induced by inclusion $\mathbf{CP}^{n+r} - V_1 \cup ... \cup V_r \subset \mathbf{CP}^{n+r} - V_1 \cup ... \cup V_{k-1} \cup V_{k+1} \cup ... \cup V_r$ and the differential $\omega \to d\omega$.

The proof is based on the following:

Lemma 2.2. Let V be a complete intersection in $\mathbf{CP}^{n+1}$. Then there is a spectral sequence $E_1^{p,q} = \oplus H^q(\mathbf{CP}^{n+r} - V_{i_0} \cup ... \cup V_{i_p})$ abutting to the primitive cohomology $H^n(V)_0$ where the direct sum is over all collections of indices $1 \le i_0 < ... < i_p \le n$.

Proof of lemma 2.2. Recall that the cohomology of a union of spaces $\mathcal{V}_i$ $(i = 1, ..., n)$ can be found using Gysin spectral sequence ([D],[GS]):

$$E_1^{p,q} = \oplus H^q(\mathcal{V}_{i_0} \cap ... \cap \mathcal{V}_{i_p}) \Rightarrow H^{p+q}(\mathcal{V}_1 \cup ... \cup \mathcal{V}_n) \tag{2.7}$$

where direct sum is taken over all collections of i_j's $(1 \le i_0 < ... < i_p \le n)$. Applying this to $\mathcal{V}_i = \mathbf{CP}^{n+1} - V_i$ we obtain the spectral sequence as in 2.7 abutting to $H^{p+q}(\mathbf{CP}^{n+r} - V, \mathbf{C})$. On the other hand the sequence:

$$\to H^i(\mathbf{CP}^{n+r}) \to H^i(\mathbf{CP}^{n+r} - V) \to H^{i-2r}(V) \to H^{i+1}(\mathbf{CP}^{n+r}) \to \tag{2.8}$$

shows that the primitive part of $H^i(V)$ is isomorphic to $H^{i-2r}(\mathbf{CP}^{n+r} - V)$ ((2.8) can be viewed as exact sequence of the pair $(\mathbf{CP}^{n+r}, \mathbf{CP}^{n+r} - V)$ in which $H^i(\mathbf{CP}^{n+r}, \mathbf{CP}^{n+r} - V)$ is replaced using excision and Poincare duality by $H^{i-2r}(V)$).

Proof of lemma 2.1. One has $E_1^{p,q} = 0$ in the following cases: a) $q > n + r$ (because $\mathbf{CP}^{n+r} - \cup V_i$ is an $n + r$-dimensional Stein manifold and b) if $p > r$. This implies that $E_2^{r,n+r} = E_\infty^{r,n+r} = H^{n+2r}(\mathbf{CP}^{n+r} - V) = H^n(V)$. On the other hand $E_2^{r,n+r} = H^{n+r}(\mathbf{CP}^{n+1} - V_1 \cup ... \cup V_r)/(\oplus H^{n+r}(\mathbf{CP}^{n+1} - V_1 \cup ... V_{k-1} \cup V_{k+1} ... \cup V_r)$ and the result follows from (2.5).

The Hodge filtration on $H^n(V, \mathbf{C})_0$ i.e. the filtration: $F^p H^n(V, \mathbf{C})_0 = \oplus_{i \ge p} H^{i,n-i}(V)_0$ and the filtration by the order of the pole on $A^{n+r}(\mathbf{CP}^{n+r} - D)$ in (2.6) correspond to each other. More precisely one has the following:

Theorem 2.3. In the sequence (2.6)

$$Im(F^{q+r}(A^{n+r}(\mathbf{CP}^{n+r} - \cup V_i)) = F^{n-q}(H^n(V, \mathbf{C})_0 \tag{2.9}$$

The subspace of $H^{n+r}(\mathbf{CP}^{n+r} - \cup_{i=1}^{i=r} V_i)$ of cohomology classes the image of which belongs to $F^{n-q+1} H^n(V, \mathbf{C})$ is generated by the subspace $F^{q+r-1} H^{n+r}(\mathbf{CP}^{n+r} - \cup_{i=1}^{i=r} V_i)$ and the classes of forms which are holomorphic along one of the hypersurfaces V_i.

The proof (as in [G]) is based on the following lemma of Dolbeault (cf. [G1]):

Lemma 2.4. Let X be a complex manifold and $\hat{\Omega}^p_X$ be the sheaf of closed holomorphic p-forms. There is a canonical isomorphism:

$$\psi^{p,q} : H^q(\hat{\Omega}^p) \to F^q H^{p+q}(X, \mathbf{C}) \tag{2.10}$$

onto the subspace of classes in $H^{p+q}(X, \mathbf{C})$ of Hodge filtration q. Moreover the coboundary operator $\delta_q : H^{p-1}(\hat{\Omega}^{q+1}_X) \to H^p(\hat{\Omega}^q_X)$ of the exact sequence:

$$0 \to \hat{\Omega}^q_X \to \Omega^q_X \to \hat{\Omega}^{q+1}_X \to 0 \tag{2.11}$$

(the right homomorphism is the differential) corresponds to the embedding:

$$F^{q+1}H^{p+q}(X, \mathbf{C}) \to F^q H^{p+q}(X, \mathbf{C})$$

.

Proof of Lemma 2.4. The truncated holomorphic deRham complex

$$\mathcal{D}^q : \Omega^q_X \to \Omega^{q+1}_X \to ... \to \Omega^{dimX}_X \to 0 \tag{2.10}$$

has hypercohomology $\mathbf{H}^r = F^q(H^r(X, \mathbf{C})$ (cf. [D], ch. 3). On the other hand the complex (2.10) is quasiisomorphic to the complex:

$$\hat{\Omega}^q \to 0 \to ... \tag{2.11}$$

and the hypercohomology of the latter is isomorphic to $H^q(\hat{\Omega}^p_X)$. The second part of the lemma follows from surjectivity of the map of the hypercohomology $\mathbf{H}^{p+q}(\mathcal{D}^q) \to \mathbf{H}^{p+q}(\Omega^q_X) = H^p(\Omega^q)$ correspondingto the morphism of the complexes:

$$\mathcal{D}^q \to \Omega^q_X \to 0 \tag{2.12}$$

(by abuse of notation we denote by Ω^q_X the complex which has a single non zero term Ω^q_X).

Remark 2.5 If X is Kahler and $h \in H^2(X, \mathbf{C})$ is the Kahler class then h belongs to $H^1(\hat{\Omega^1_X})$ because a Kahler class is clearly in $F^1H^2(X)$. Hence the primitive part of $H^q(\hat{\Omega}^p_X)$ can be defined as the kernel of the multiplication by h which is the composition of the cup product in cohomology and the exterior product on coefficients: $H^q(\hat{\Omega}^p) \otimes H^1(\hat{\Omega}^1) \to H^{q+1}(\hat{\Omega}^{p+1})$.

Proof of the Theorem 2.3 Let $D = \cup_{i=1}^{i=r} D_i$ be a divisor with normal crossings in a compact complex manifold X such that each of r components D_i $(i = 1..r)$ is smooth. Let $\hat{\Omega}^p_X(kD_{(i_1,...i_s)})$ be the sheaf of closed meromorphic p-forms holomorphic outside of $D_{i_1} \cup ... \cup D_{i_s}$ and having a total order of poles along the components $D_{i_1}, ..., D_{i_s}$ of divisor D at most k.

Step 1. One has the following exact sequence:

$$0 \to \hat{\Omega}^{p+r}((0)D_{(0)}) \to \oplus\hat{\Omega}^{p+r}((1)D_{(1)}) \to ... \to \oplus\hat{\Omega}^{p+r}((r-1)D_{(r-1)}) \to$$

$$\to \hat{\Omega}^{p+r}((r)D) \to \hat{\Omega}^{p}_{V} \to 0 \tag{2.13}$$

Here $\oplus\hat{\Omega}^{p+r}((k)D_{(s)})$ denotes the direct sum over all possible choices of s components of D of the sheafs of meromorphic $p+r$ forms with a) the total order of poles along the components of D not exceeding k and b) having the order of the poles not exceeding 1 over each of chosen s components of D. The middle maps $\oplus\hat{\Omega}(sD(s)) \to \oplus\hat{\Omega}((s+1)D_{s+1})$ are defined by assigning to a collecton of the forms $w_{i_1,..,i_s}$ the collection of the forms $\omega_{j_1,...,j_{s+1}} = \Sigma(-1)^k\omega_{j_1,...,\hat{j}_k,..,j_{s+1}}$. To define the last non-trivial map in (2.13) note that a form from $\hat{\Omega}^{p+r}((s)D_{i_1,..,i_s})$ is closed only if it is logarithmic (i.e. for some choice of coordinates $z_1,..,z_{n+r}$ belongs to $\mathbf{C}[z_1,...,z_{n+r}]$ submodule generated by $\frac{dz_i}{z_i}$ cf. [D] ch.3). Indeed a form $\omega = \Sigma\frac{\phi_i}{z_i}$ where ϕ_i has no summand divisible by dz_i has

$$d\omega = \Sigma\frac{dz_i \wedge \phi_i}{z_i^2}. \tag{2.14}$$

Hence $d\omega = 0$ implies that its component having a pole of order 2 along $z_i = 0$ is zero. That is $\phi_i = 0$ i.e. the form ω is logarithmic. The right homomorphism in (2.13) is just the residue of a logarithmic form defined by:

$$Res\frac{dz_{i_1}}{z_{i_1}}...\frac{dz_{i_s}}{z_{i_s}}\phi(z_1,...,z_{n+r}) = \phi|_{z_{i_1}=0,...,z_{i_s}=0} \tag{2.15}$$

The exactness of (2.13) in the middle terms is straightforward. In order to show exactness of (2.13) at the term $\hat{\Omega}^{p+r}((r)D)$ (the only part of (2.13) which is used below) we need to verify that if $Res\omega = 0$ and $d\omega = 0$ then ω is holomorphic along one of components of D. It follows from (2.15) that $Res\omega = \phi(z_1,...,z_{n+r}) = 0$ on V i.e. ϕ belongs to the ideal generated by $z_1,...,z_r$. Hence it is a sum of the forms divisible by one of z_i $i = 1,...,r$. Therefore ω is sum of the forms holomorphic along one of components of D.

Step 2. We are going to show that on $X = \mathbf{CP}^{n+r}$ the sequence (2.13) applied to $D = \cup V_i$ implies that

$$H^q(\oplus\hat{\Omega}^{p+r}_{\mathbf{CP}^{n+r}}(r-1)D)_0 \to H^q(\hat{\Omega}^{p+r}_{\mathbf{CP}^{n+r}}((r)D)_0 \to H^q(\Omega^p_V)_0 \to 0 \tag{2.16}$$

where subscript 0 designates the primitive part of the cohomology group. That is, the kernel multiplication by $h \in H^1(\hat{\Omega}^1_{\mathbf{CP}^{n+r}})$ induced by the product $H^q(\hat{\Omega}^p(sD))\otimes H^1(\hat{\Omega}^1) \to H^{q+1}(\hat{\Omega}^{p+1}(sD))$. Let $W_m(\hat{\Omega}^{p+r}(sD)$ be the m-th term of the filtration on $\hat{\Omega}^{p+r}(sD)$ induced by the weight filtration on logaritmic forms. The exactness of (2.16) follows from the vanishing:

$$H^{q+1}(W_{r-1}(\hat{\Omega}^{p+r}_{\mathbf{CP}^{n+r}}((r-1)(V_{i_1} \cup ... \cup V_{i_{r-1}})))_0 = 0 \quad (p+q=n). \tag{2.17}$$

We shall prove (2.17) by showing that $N_l = H^{q+1}(W_l(\hat{\Omega}^{p+r}_{\mathbf{CP}^{n+r}}(logD)))$ satisfies $N_l \geq N_{l+1}$ ($(logD)$ indicates logarithmic along D forms). For $l = 0$ we have $N_0 = H^{q+1}(\hat{\Omega}^{p}_{\mathbf{CP}^{p+r}})_0 = 0$ which proves (2.17). To verify the equality among N_l let us consider the exact sequence:

$$W_l(\hat{\Omega}^{p+r}(logD)) \to \oplus W_{l+1}(\hat{\Omega}^{p+r}(logD) \to \oplus_{1 \leq i_1 < \ldots < i_{l+1} \leq r} \hat{\Omega}^{p+r-l}_{V_{i_1} \cap \ldots \cap V_{i_{l+1}}} \to 0 \qquad (2.18)$$

(cf. [D] 3.1.5). From (2.18) we obtain:

$$H^{q+1}(W_l(\hat{\Omega}^{p+r}(logD))_0 \to H^{q+1}(W_{l+1}(\hat{\Omega}^{p+r}(logD))_0 \to H^{q+1}(\hat{\Omega}^{p+r-l-1}_{V_{i_1} \cap \ldots \cap V_{i_{l+1}}})_0$$

The latter group is zero because the primitive cohomology of a complete intersection is trivial in all dimensions except the middle one (and $p + q + r - l \neq n + r - l - 1 = dim V_{i_1} \cap \ldots \cap V_{i_{l+1}}$). Hence $N_l \geq N_{+1}$.

Step 3. Let us consider the exact sequence:

$$0 \to \hat{\Omega}^l((k)D) \to \Omega^l((k)D) \to \hat{\Omega}^{l+1}((k+1)D) \to 0 \qquad (2.19)$$

in which the right homomorphism is the exterior differential. The corresponding exact cohomology sequence gives the isomorphism:

$$H^q(\hat{\Omega}^{l+1}((k+1)D)) = H^{q+1}(\hat{\Omega}^l((k)D), \qquad (2.20)$$

for any $q \geq 0$ because of vanishing $H^q(\Omega^l((k)D))$ (cf. [B] p.246).

Step 4. In follows from (2.20) that

$$H^q(\hat{\Omega}^{p+r}((r)D) = H^0(\hat{\Omega}^{p+q+r}((q+r)D)) = H^0(\hat{\Omega}^{n+r}((q+r)D)) = H^0(\Omega^{n+r}((q+r)D).$$

Hence we obtain the surjection:

$$F^{q+r}(A^{n+r}(\mathbf{CP}^{n+r} - \cup V_i)) \to F^p(H^{p+q}(V, \mathbf{C}))$$

and the theorem follows.

Remark 2.6.The case $q = 0$ of this theorem is a consequence of (3.1.5) of [D] (and does not require Bott's vanishing theorem). The latter establishes the following. Let X be a compact complex manifold, Y be a divisor with normal crossings and $\tilde{Y}_l$ be the disjoint union of l-fold intersections of components of Y and $i_{l*} : \tilde{Y} \to X$ the embedding. Let $\Omega^*_X(< Y >)$ be the logarithmic complex and let W be its weight filtration. Then the residue map provides the isomorphism of complexes (cf. [D] ch.3):

$$Res : Gr^W_l(\Omega^*_X(< Y >)) \to i_{l*}\Omega^*_{\tilde{Y}^l}[-l]$$

In particular if $V = \cap_{i=1}^{i=k} D_i$ is a non-singular complete intersection, then for $l = k$ we have an isomorphism of components of degree $n+1$: $Gr^W_k(\Omega^{n+1}_{\mathbf{CP}^{n+1}} < \cup D_i >) \to \Omega^{n+1-k}_V$. The left sheaf is isomorphic to the sheaf of meromorphic form on $\mathbf{CP}^{n+r} - \cup D_i$ having the total order of the pole not exceeding k modulo forms whose polar sets are a proper subsets of $\cup D_i$. Hence the space of its global sections is identified via the residue map with $H^0(\Omega^{n+1-k}_V)$.

3. A quotient module of a complete intersection.

We continue to work with a non-singular complete intersection V given by the equations (1.2) of degrees $d_i = degQ_i$.

Let $S = k[z_1, ..., z_{n+r+1}], R = k[z_1, ..., z_{n+r+1}] \otimes_k k[u_1, ..., u_r]$. For the most part R will be considered as a S-module. We shall call the grading by the degree relative to u (resp. relative to z) u- (resp. z-) grading. Let $L_k : R \to R$ $k = 1, ..., n + r + 1$ be given by

$$L_k : F \to \Sigma_{i=1}^{i=r} u_i^2 \partial(F)/\partial u_i \partial Q_i/\partial z_k \tag{3.1}$$

On a monomial in variables $u_1, .., u_r$ the operator L_k acts as follows:

$$L_k(u_1^{s_1}...u_r^{s_r} F_{s_1,...,s_r}(z_0, .., z_{n+r+1})) = \Sigma_{i=1}^{i=r} s_i u_1^{s_1}...u_i^{s_i+1}...u_r^{s_r} \frac{\partial Q_i}{\partial z_k} F(z_0, .., z_{n+r+1}) \tag{3.2}$$

Let $M_i : R \to R$ $(i = 1, ..r)$ be the operator of multiplication by Q_i. Operators L_k and M_k are homomorphisms of S-modules.

Let us define $\tilde{\mathcal{Q}}(V)$ as the cokernel of the map

$$\Phi = \oplus_{k=1}^{k=n+r+1} L_k \oplus_{i=1}^{i=r} M_i : R^{n+2r+1} \to R. \tag{3.3}$$

It follows from (3.2) that a) L_k are homomorphisms of graded S-modules (of (u, z) bidegree equal to $(1, d - 1)$) and b) clearly that M_i are homomorphisms of S-modules of degree 0. Hence $\tilde{\mathcal{Q}}(V)$ is a direct sum of graded S-module and can be viewed as a bigraded vector space which is a graded S-module with respect to z-grading.

Let us consider two gradings deg_1 and deg_2 on R. We put:

$$deg_1 u_i = 1, deg_1 x_j = 0, deg_2 u_i = -d_i, deg_2 x_j = 1 \tag{3.4}$$

(deg_1 is u-grading mentioned above). This bigrading on R defines the bigrading on $\tilde{\mathcal{Q}}(V)$ (L_k (resp. M_i) has $deg_1 = 1$ (resp. 0) and $deg_2 = -1$ (resp. d_i)).

Let $\mathcal{Q}(V) = \Phi(I) \subset \tilde{\mathcal{Q}}(V)$ be the image of the principal ideal I of R generated by the element $u_1 \cdot ... \cdot u_r$

Proposition 3.1. The Hodge space $H^{p,n-p}(V)$ canonically isomorphic to the space of the elements of $\mathcal{Q}$ which have deg_1 equal to $p + r$ and deg_2 is $-n - r - 1$.

Proof. The space $F^{p+r}(A^{n+r}(\mathbf{CP}^{n+r} - \cup V_i)$ can be identified with the component of $R = \mathbf{C}[u_1, ..., u_r, z_0, ..., z_{n+r+1}]$ the polynomials for which $deg_1 = p + r$ and $deg_2 = -(n + r + 1)$ via the correpondence:

$$\Sigma_{s_1+...s_r=p+r} \frac{P_{s_1,...,s_r}}{Q_1^{s_1}...Q_r^{s_r}} \Omega \to \Sigma u_1^{s_1}...u_r^{s_r} P_{s_1,...,s_r}. \tag{3.5}$$

According to theorem 2.3 the space $H^{p,n-p}$ is the quotient of $F^{p+r}(A^{n+r}(\mathbf{CP}^{n+r} - \cup V_i)$ by a subspace spanned by

a) forms holomorphic along a component V_i $(i = 1, ..r)$
b) exact forms; and
c) forms which have pole filtration $p + r - 1$.

The quotient of bigraded component $R_{p+r,-n-r-1}$ by the subspace corresponding to a) in (3.5) is naturally identified with the corresponding component of the principal ideal $I(R)$ generated by $u_1...u_r$. Similarly the quotient of $I(R)_{p+r,-n-r-1}$ by the subspace correponding to c) via identification of forms and the components of R becomes identified with the bigraded component of the quotient of $I(R)$ by the subspace spanned by the images of M_i $(i = 1, ..., r)$. Finally from the identity:

$$d(\Sigma_{s_1,...,s_r} \frac{1}{Q_1^{s_1}...Q_r^{s_r}} \Sigma_{k<l}(-1)^{k+l}[z_k P_{s_1,..,s_r,l} - z_k P_{s_1,..,s_r,l}] dz_0 \wedge ... \wedge \hat{dz_k} \wedge ... \wedge \hat{dz_l} \wedge$$

$$... \wedge dz_{n+r}) = \Omega \cdot \Sigma_{s_1,...,s_r} \Sigma_{i=1}^{i=r} \Sigma_{k=1}^{k=n+2} \frac{s_i P_{s_1,....,s_r,k} \cdot \partial Q_i / \partial z_k}{Q_1^{s_1} \cdots Q_i^{s_i+1} \cdots Q_r^{s_r}} + \Sigma_k \frac{\partial P_{s_1,...,s_r,k} / \partial x_k}{Q_1^{s_1} \cdots Q_i^{s_i} \cdots Q_r^{s_r}} \quad (3.6)$$

it follows that $H^{p,n-p}(V)$ is isomorphic to the component of the quotient of $I(R)$ by the space spanned by the images of M_i and L_k. The latter is the component of $\mathcal{Q}(V)$.

Remark. Though the theorem of section 2 provides an identification of the meromorphic forms with the cohomology of V in the middle dimension the proposition *does not* identify a bigraded component of $\mathcal{Q}(V)$ with a subspace in cohomology of V. The components of $\mathcal{Q}(V)$ are identified with the graded vector space associated with the Hodge filtration. In particular, vanishing of an element in $\mathcal{Q}(V)$ does not mean that the corresponding form defines a trivial cohomology class but rather that the corresponding form is cohomologous to one with a smaller order of the pole.

4. Examples

1. In a number of cases the quotient module defined above for a complete intersection (1.2) or some of its graded components can be identified with components of the related algebra:

$$\mathbf{C}[u_1, .., u_r, z_0, ..., z_{n+r}]/(\Sigma u_i \frac{\partial Q_i}{\partial z_0}, ..., \Sigma u_i \frac{\partial Q_i}{\partial z_{n+r}}, Q_1, ..., Q_r) \quad (4.1)$$

with bigrading defined by (3.4). This is so in the case $r = 1$ (hypersurfaces). Indeed the image of $L_k : R \to R$ coincides with the image of multiplication by $u \frac{\partial Q}{\partial z_k}$. The bigrading (3.4) is equivalent to the u, x bigrading (i.e. components of both bigrading coincide). Moreover the evaluation at 1 identifies the components of the algebra :

$$\mathbf{C}[u, z_0, .., z_{n+r}]/(..., u \frac{\partial Q}{\partial z_k}, ...) \quad (4.2)$$

with the components of the Jacobian algebra (1.1).

The component of $\mathcal{Q}(V)$ of bidegrees $deg_1 = p, deg_2 = -n-1$ and $deg_1 = p+1, deg_2 = -n-2$ can be identified with the components of the same bidegree of the quotient algebra (4.1) because in this case L_k again coincidse with the multiplication. This can be used for a fast calculation of the basis of the Hodge spaces in the cohomology. For example for V given by

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$$
$$\lambda_0 z_0^2 + \lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 \lambda_4 z_4^2 = 0 \qquad (4.3)$$
$$\mu_0 z_0^2 + \mu_1 z_1^2 + \mu_2 z_2^2 + \mu_3 z_3^2 + \mu_4 z_4^2 = 0$$

one obtains the following basis (using for example Macaulay ([Mc]) command "k-basis"):

$$u_1 u_2 u_3^2 z_0 z_4^2, u_1 u_2 u_3^2 z_1 z_4^2, u_1 u_2 u_3^2 z_2 z_4^2, u_1 u_2 u_3^2 z_3 z_4^2, u_1 u_2 u_3^2 z_4^3 \qquad (4.4)$$

2. Let us consider the case $r = 2, n = 4$, $degQ_1 = degQ_2 = 3$ which was studied in [LT]. The subspace of $Q(V)$ of elements for which $deg_1 = p-2$ ($p \geq 2$) and $deg_2 = -6$ (and which according to the lemma can be identified with the Hodge component $H^{p-2,5-p}(V)$) is a quotient of the component of R of polynomials the degree of which relative to x_i's is $3p-6$ and relative to u_1, u_2 is p. The component of R of elements of having deg relative to u_1, u_2 equal to p can be identified with $\oplus_{i=0}^{i=p} S$ and its intesection with the ideal $I(\tilde{Q}(V))$ is isomorphic as an S-module to S^{p-1}. $\Phi^{-1}(I(R) \cap R_{p,*})$ belongs to $\oplus_{k=1}^{k=6} I(R)_{p-1,*} \oplus_{r=1}^{r=2} I(R)_{p,*}$. The latter can be considered as a free S-module of rank $6(p-2)+2(p-1) = 8p-14$. Hence the p-component of $Q(V)$ as S-module is the cokernel of the map $S^{8p-14} \to S^{p-1}$ given by the matrix $(B_p, Q_1 I_p, Q_2 I_p)$ where B_p is given on p.33 [LT]. Hence the p-component of $Q(V)$ as S-module is the cokernel of the map $S^{8p-14} \to S^{p-1}$ given by the matrix $(B_p, Q_1 I_p, Q_2 I_p)$ where B_p is given on p.33 [LT].

References

[ABG] M.Atyiah, R.Bott, L.Garding, Lacunas for hyperbolic differential operators with constant coefficients II, Acta Mathematica, 131 (1973), 145-206.

[B] R.Bott, Homogeneous Vector Bundles, Annals of Mathematics, 66 (1957), 203-255.

[D] P.Deligne, Theorie de Hodge II, Inst. Hautes Etudes Sci. Publ. Math. 35 (1971) vol. 40. p.5-57.

[G] Ph. Griffiths. On periods of certain rational integrals, I ,II, Annals of Mathematics, 90 (1969) 460-541.

[G1] Ph.Griffiths, Some results on subvarieties of algebraic manifolds. Proc. Tata Conf. on Alg. Geom. Tata Institute, Bombay, (1968), 93-191.

[GS] Ph. Griffiths, W. Schmidt, Recent developments in Hodge theory: a discussion of techniques and results, Descrete Subgroups of Lie Groups and Application to Moduli, Oxford Univ. Press, 1973.

[Gr] A.Grothendieck, On the deRham cohomology of algebraic varieties, Publ. Math. IHES. no. 29, (1966), 95-103.

[HY] T.Hubsch, S.T.Yau, An $SL(2, \mathbf{C})$ action on chiral rings and the mirror map, Mordern Physics Letters A, vol. 7, No. 35 (1992), 3277-3289.

[Mc] D.Bayer, M.Stillman,*Macaulay: A system for computation in algebraic geometry and commutative algebra.*Source and object code available for Unix and Macintosh computers. Contact the authors, or download from `zariski.harvard.edu` via anonymous ftp.

[KT] Klemm , Theisen, Preprint, 1993 Hep-th.

[LT] A.Libgober , J.Teitelbaum, International Mathematics Research Notices (in Duke Math. Journ), No 1, 1993, p. 29-39.

[M] D.Morrison, Picard Fuchs equations and the mirror map for hypersurfaces, Essays on Mirror Symmetry, ed. S-T. Yau, Int. Press Co, Hong Kong, 1992.

Department of Mathematics, University of Illinois 851 S. Morgan Sr. Chicago, Illinois, 60607

E-mail address: libgober@math.uic.edu