Journal of Mathematical Sciences, Vol. 82, No. 1, 1996

POSITION OF SINGULARITIES OF HYPERSURFACES AND THE TOPOLOGY OF THEIR COMPLEMENTS

A. Libgober

UDC 512.761, 512.73, 515.165.4

We study the mixed Hodge structure on certain homotopy groups of the complements to algebraic hypersurfaces in complex projective space. As part of this we describe how these homotopy groups depend on the position of singular points of hypersurfaces in the ambient space. We also obtain the regularity of certain linear systems with base points at the singularities of the hypersurfaces.

0. Introduction

A relationship between the position of singularities of the branching locus and topology was discovered by O. Zariski [29]. He showed that the irregularity of a surface which is a resolution of singularities of a cyclic covering of $\mathbf{CP}^2$, branched over a curve of degree d and having only nodes or cusps as its singularities, is equal to zero unless both the degree d and the degree of the covering is divisible by 6. In the remaining cases, the irregularity is equal to the difference between actual and expected dimensions (i.e., the superabundance) of the family of curves of degree $d-3-\frac{d}{6}$ passing through the cusps of the curve. For example, a cover of $\mathbf{CP}^2$ which has a degree divisible by six branched over a sextic curve with six cusps has its irregularity equal to 1 or 0 depending on whether these cusps belong to a conic or not. On the other hand, Zariski also proved that if the fundamental group of the complement to an irreducible curve is abelian, then the irregularity of the cyclic branched covering of any degree is zero. The precise way in which the irregularity of a cyclic covering depends on the fundamental group of the complement to the branching locus was described in [12] (cf. [13, 14]), where it was shown that the irregularity of a cyclic covering (or its first Betti number) depends on the quotient of the commutator of the fundamental group of the complement by its second commutator. Moreover, in the case where the singularities are more complicated than cusps or nodes, the irregularity of cyclic branched coverings can also be expressed via dimensions of certain linear systems which are determined by the local structure of the curve's singular points (cf. [13, 18]).

The present work concerns a high-dimensional generalization of these results. We consider the Hodge number $h^{n,0}$ of a resolution of a cyclic covering of $\mathbf{CP}^{n+1}$, the branch locus of which belongs to the union of an irreducible hypersurface V with isolated singularities and the hyperplane at infinity. (The inclusion of this hyperplane into the branching locus allows us to consider coverings of arbitrary degrees, which is quite essential in our arguments, while the degree of a covering of $\mathbf{CP}^{n+1}$ branched only over V must be a divisor of the degree of V.) The Hodge number $h^{n,0}$ is a birational invariant and hence depends only on the branch locus and the degree of the covering. We show that it can be determined in terms of the actual and expected dimensions of certain linear systems of hypersurfaces in $\mathbf{CP}^{n+1}$. The latter have degrees depending on the degree and type of singularities of V and have base points at the singularites of V. These linear systems are specified by assigning "prescribed behavior" at these base points or, more precisely, by requiring the local equations to belong to certain sheaves of ideals depending on the local type of singularities. On the other hand, $h^{n,0}$ of a desingularization of a cyclic branched covering can be related to the homotopy group $\pi_n(\mathbf{CP}^{n+1}-(V\cup H)\otimes\mathbf{Q}$, which was studied in [16]. In particular, the triviality of the latter implies the vanishing of $h^{n,0}$ of cyclic coverings of an arbitrary degree. This actually can be made more precise. We study the mixed Hodge structure on $\pi_n(\mathbf{CP}^{n+1}-V\cup H)\otimes\mathbf{C}$ for which both the weight and Hodge filtrations are invariant under the canonical action of $\pi_1(\mathbf{CP}^{n+1}-V\cup H)=\mathbf{Z}$ on π_n. Next we use it to define a certain $\mathbf{Q}[t,t^{-1}]$-submodule $L_{n,0}$ in $\pi_n(\mathbf{CP}^{n+1}-V\cup H)\otimes\mathbf{C}$ which is trivial if and only if $h^{n,0}$ vanishes for any

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 34, Algebraic Geometry–5, 1996.

cyclic covering with branching locus $V \cup H$. In fact, we calculate this module in terms of the aforementioned linear systems of hypersurfaces determined by the singularities and the degree of V. Note that the space $\mathbf{CP}^{n+1} - V \cup H$ is not nilpotent, so the construction of J. Morgan and R. Hain (cf. [22, 10, 11]) of the mixed Hodge structure on the homotopy groups is not directly applicable.

To illustrate the consequences we note, for example, the following (cf. Corollary 4.4 and Example 4.5):

Corollary. *Let V be a hypersurface in $\mathbf{CP}^{n+1}$ transversal to the hyperplane at infinity, all singularities of which are locally isomorphic to $x_1^{q_1} + ... x_{n+1}^{q_{n+1}}$ ($q_i \geq 2$ for any i) and the degree of which is divisible by $D = \prod q_i$. Let $\tilde{X}_D$ be a nonsingular model of a cyclic covering of $\mathbf{CP}^{n+1}$ branched over V having degree D, and let $s_V(N)$ be the superabundance of hypersurfaces of degree N passing through the singularities of V. Then*

$$\frac{1}{2} \dim_{\mathbf{C}} \pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{C} \geq \dim_{\mathbf{C}} L_{n,0} = h^{n,0}(\tilde{X}_D) \geq s_V\left(\sum_i \frac{D}{q_i} - n - 2\right).$$

The content of the paper is as follows. In Sec. 1, we discuss models of cyclic coverings of arbitrary degree of $\mathbf{CP}^{n+1}$ as hypersurfaces blown up $\mathbf{CP}^{n+2}$'s and in weighted projective spaces. We describe the n-dimensional homology of unbranched coverings of $\mathbf{CP}^{n+1} - V \cup H$ in terms of $\pi_n(\mathbf{CP}^{n+1} - V \cup H)$. Note that the order of $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{Q}$ can be viewed (in the case where V is transversal to H) as the characteristic polynomial of the monodromy acting on the Milnor fibre of a nonisolated singularity (of a hypersurface with a one-dimensional singular locus). The latter is just the cone over the hypersurface V (cf. [16]). These characteristic polynomials were studied independently in [3]. Next we describe a classical method of dealing with holomorphic forms using adjoint ideals. As a slight generalization of the classical case, however, we express the dimension of the space of holomorphic form of a manifold in terms of ideal sheaves determined by the image of the manifold in a *weighted* projective space. Finally, we discuss local invariants of isolated singularities, namely, the constants of quasiadjunction which were introduced in the context of curves in $\mathbf{C}^2$ in [13] and their dependence on an embedded resolution of singularity.

In Sec. 2, we define the mixed Hodge structure of $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{C}$ in the case where V has isolated singularities (including infinity in the sense of [16]). This allows us to define certain $\mathbf{C}[t, t^{-1}]$-torsion modules $L_{p,q}$ arising from this mixed Hodge structure. In particular, the orders $\lambda_{p,q}$ of $L_{p,q}$ are the polynomials in terms of which one can describe the Hodge numbers of the coverings. This is done in this section for unbranched coverings of $\mathbf{CP}^{n+1} - V \cup H$.

Section 3 concerns the Hodge theory of branched coverings and shows how to calculate $h^{n,0}$ in terms of the modules $L_{n,0}$ from Sec. 2. A consequence of this, which is essential in Sec. 4, is that $h^{n,0}$ is a periodic function of the degree of the covering. The Hodge numbers $h^{p,n-p}$ depend on the resolution of singularities if $p \neq 0$, but in the case of weighted homogeneous singularities, one has a preferred resolution (in the category of V-manifolds). Using this, we interpret $L_{p,q}$ for any p, q using the Hodge numbers of branched coverings. In the last section, we calculate $h^{n,0}$ in terms of linear systems of hypersurfaces determined by constants of quasiadjunction of singularities defined in Sec. 2. As a result, one obtains the regularity of these linear systems (cf. Corollary 4.2). Note that these linear systems (for almost all local types of singularities) include the system of hypersurfaces passing through singular points of V. This also gives the calculation of irregularity in the case of coverings of $\mathbf{CP}^2$. In this case another proof was given in [18] which is based on the approach developed in [6]. Our proof is close to the original one of O. Zariski, where the regularity of linear systems defined by the base points follows from the fact that the collection of irregularities of nonsingular projective models of cyclic branched covers of $\mathbf{C}^2$ branched over a fixed curve is a bounded set. However, here we use weighted projective spaces (instead of working with not normal hypersurfaces, cf. [29]). A calculation of the Hodge numbers of cyclic branched coverings was also considered in [28].

Acknowledgments. The author gratefully acknowledges the hospitality of the Institute for Advanced Study, where a part of this work was carried out. While this paper was circulating I received numerous useful suggestions for improvements. I am greatly indebted to all those who commented on an earlier version. I

would particularly like to thank J. Steenbrink. Note that most of the results in this paper were presented in [15].

1. Preliminaries

In this section, we discuss the birational models of cyclic branched coverings, a calculation of the homology of unbranched coverings, the theory of adjoints in weighted projective spaces, and a certain situation in which the Hodge numbers $h^{p,q}$, $p \neq q$, of a divisor with normal crossings are zeros. We work with a hypersurface V of degree d and a hyperplane H which we call the hyperplane at infinity. We say that V has only isolated singularities including infinity iff both V and $V \cap H$ have only isolated singularities (cf. [16]). The study of the first nontrivial (in the sense [16]) homotopy group of the complement to a hypersurface with a singular locus of dimension s, i.e., $\pi_{n-s}(\mathbf{CP}^{n+1} - V \cup H)$, can be reduced to this case. Indeed, this group is isomorphic to the first nontrivial homotopy group of the complement to a generic section of the hypersurface by a linear subspace of codimension s, as follows from the Lefschetz theorem (cf. [13, 16]).

Definition 1.1. The m-fold cover of $\mathbf{CP}^{n+1}$ branched along $V \cup H$ is a complex algebraic variety X_m such that

(1) X_m is a normal pseudomanifold (i.e., $H_{\dim_{\mathbf{R}} X_m}(X_m, X_m - x) = \mathbf{Z}$ for any $x \in X_m$ cf. [7]);

(2) there is a map $\pi : X_m \to \mathbf{CP}^{n+1}$ such that the restriction of π to $X_m - \pi^{-1}(V \cup H)$ is isomorphic to the covering of $\mathbf{CP}^{n+1} - (V \cup H)$ corresponding to the canonical map of the fundamental group of the latter space (which is isomorphic to $\mathbf{Z}$) onto $\mathbf{Z}/m \cdot \mathbf{Z}$.

Remark 1.2. If, as above, $d = \deg(V)$ and $f(z_0, \ldots, z_{n+1}) = 0$ is a defining equation of V, then the d-fold covering is just the hypersurface in $\mathbf{CP}^{n+2}$ given by $z_{n+2}^d = f(z_0, \ldots, z_{n+1})$. In general, a hypersurface model of X_m with isolated singularities in a nonsingular space can be obtained as follows. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which has isolated singularities including singularities at infinity. Let $f(z_0, ..., z_{n+1}) = 0$ be an equation of V and $z_0 = 0$ be the equation of the hyperplane H at infinity. Then there is a manifold $\widetilde{\mathbf{CP}}^{n+2}$ which is obtained from $\mathbf{CP}^{n+2}$ by a series of blow ups with centers at codimension 2 subspaces over the subspace $\hat{H} \subset \mathbf{CP}^{n+1}$ given by $z_0 = z_{n+2} = 0$, a point $p \in \widetilde{\mathbf{CP}}^{n+2}$, and a map $\pi : \widetilde{\mathbf{CP}}^{n+2} \to \mathbf{CP}^{n+2}$, which have the following properties.

(1) If the variety X'_m is the projective closure in $\mathbf{CP}^{n+2}$ of the affine hypersurface $z_{n+2}^m = f(1, z_1, \ldots, z_{n+1})$, then the proper π-preimage X_m of X'_m in $\widetilde{\mathbf{CP}}^{n+2}$ is an m-fold branched covering of $\mathbf{CP}^{n+1}$ branched over $V \cup H$.

(2) The covering $\pi_m : X_m \to \mathbf{CP}^{n+1}$, where $\pi_m = \pi|_{X_m}$, is totally ramified over V and the restriction of π_m over the hyperplane at infinity H: $\pi_m^{-1}(H) \to H$ is a covering of H of degree $\mathrm{g.c.d.}(d, m)$.

(3) The singularities of X_m are only at the points $\pi_m^{-1}(\mathrm{Sing}(V))$. In particular X_m is nonsingular along $\pi_m^{-1}(H - H \cap \mathrm{Sing}(V \cap H))$. (Here Sing( ) denotes the singular locus of a variety.)

Remark 1.3. We shall need a birational model of a cyclic branched covering X_m which is a hypersurface in a weighted projective space (cf. [4]). Recall that the weighted projective space $\mathbf{CP}_{\mathbf{w}}^{n+2}$ of weight $\mathbf{w} = (q_0, \cdots, q_{n+2})$ is a quotient of $\mathbf{C}^{n+3} - 0$ by the action of $\mathbf{C}^*$ given as follows: $t \cdot (z_0, \ldots, z_{n+2}) = (t^{q_0} \cdot z_0, \ldots, t^{q_{n+2}} \cdot z_{n+2})$. Alternatively $\mathbf{CP}_{\mathbf{w}}^{n+2}$ is a quotient of the standard projective space by the action of the product of $n+3$ cyclic groups $\mathbf{Z}/q_0 \cdot \mathbf{Z} \times \cdots \times \mathbf{Z}/q_{n+2} \cdot \mathbf{Z}$. The map $\mathbf{CP}^{n+2} \to \mathbf{CP}_{\mathbf{w}}^{n+2}$ is given by $(z_0, \ldots, z_{n+2}) \to (z'_0, \ldots, z'_{n+2})$, where $z'_i = z_i^{q_i}$.

Let $F(z_0, \ldots, z_{n+1}) = 0$ be an equation of a hypersurface of degree d in $\mathbf{CP}^{n+1}$ which has only isolated singularities including infinity. Let $l = \mathrm{l.c.m.}(d, m)$ and $g = \mathrm{g.c.d.}(d, m)$. Let $\mathbf{CP}_{\mathbf{w}}^{n+2}$ be the weighted projective space of weight $\mathbf{w} = (1, m/g, m/g, ..., m/g, d/g)$, where the weight of z_{n+2} is d/g, the weight of each of $z_1, ..., z_{n+1}$ is m/g, and the weight of z_0 is 1. Such a choice of weights yields that

(a) The weighted projective space $\mathbf{CP}^{n+2}_{\mathbf{w}}$ contains $\mathbf{C}^{n+1}$ as an open set (cf. [4, 1.2.4]). The projective closure of the hypersurface $z^m_{n+2} = F(1, z_1, \dots, z_{n+1})$ in $\mathbf{CP}^{n+2}_{\mathbf{w}}$ is an m-fold cyclic branched covering of $\mathbf{CP}^{n+1}_{(1,m/g,\dots,m/g)}$ branched over the union of the closure of V and the subspace H_0 given in the latter projective space by $z_0 = 0$, i.e., provides a model of X_m;

(b) the restriction of this covering over H_0 has degree g (as one can check directly).

The degree of this hypersurface is equal to l. If V is transversal to the hyperplane at infinity, then this projective closure has only isolated or quotient singularities. Alternatively, X_m can be described in a similar way as a hypersurface of degree $d \cdot m$ in weighted projective space of weight $(1, m, \dots, m, d)$, with the same conclusions about singularities (recall that projective spaces of weights $(1, \frac{m}{g}, \dots, \frac{m}{g}, \frac{d}{g})$ and $(1, m, \dots, m, d)$ are isomorphic (cf. [4, 1.3.1])). If m divides d, then there is no ramification over the points of H_0, and using the identification of weighted projective spaces of weights $(1, m, \dots, m, d)$ and $(1, 1, \dots, 1, \frac{d}{m})$ we obtain the model of the cover discussed in [4, 3.5.4].

Next we shall consider the homology of unbranched coverings.

Lemma 1.4. *Let V be a hypersurface in $\mathbf{CP}^{n+1}$ having isolated singularities, including singularities at infinity. Then*

1. *For $2 \leq i \leq n-1$, the homology of the k-fold cyclic covering of $(\mathbf{CP}^{n+1} - V \cup H)_k$ of $\mathbf{CP}^{n+1} - V \cup H$ satisfies*

$$H_i((\mathbf{CP}^{n+1} - V \cup H)_k, \mathbf{Q}) = 0.$$

2. *The homology group $H_n((\mathbf{CP}^{n+1} - V \cup H)_k, \mathbf{Q})$ is isomorphic to the cokernel of the multiplication by $t^k - 1$ on $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{Q}$ if $n \geq 2$.*

3. *The eigenspace of the linear map of $H_n((\mathbf{CP}^{n+1} - V \cup H)_k, \mathbf{C})$, induced by the deck transformation corresponding to the eigenvalue ζ, is isomorphic to the cokernel of the multiplication of $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{C}$ by $t - \zeta$ provided $n \geq 2$.*

4. *If $n = 1$, let $\pi_1 = \pi_1(\mathbf{CP}^2 - (V \cup H))$. Then the cokernel of multiplication by $t^k - 1$ on $\pi_1'/\pi_1'' \otimes \mathbf{Q}$ is isomorphic to a subspace of codimension 1 in $H_1((\mathbf{CP}^2 - V \cup H)_k, \mathbf{Q})$.*

Proof. The first statement follows from the $(n-1)$-connectedness of the universal cover of $(\mathbf{CP}^{n+1} - V \cup H)$, which is proven in [16]. To verify the rest let us consider the sequence of chain complexes with compact support and rational coefficients (cf. [16]) in which $(\mathbf{CP}^{n+1} - V \cup H)_\infty$ denotes the infinite cyclic cover:

$$0 \to C_*((\mathbf{CP}^{n+1} - V \cup H)_\infty) \to C_*((\mathbf{CP}^{n+1} - V \cup H)_\infty) \to C_*((\mathbf{CP}^{n+1} - V \cup H)_k) \to 0. \quad (1.1)$$

We consider these complexes as complexes of finitely generated $\mathbf{Q}[t, t^{-1}]$-modules with the action of t given by the action of the generator of the group of deck transformations of the coverings. The left homomorphism in (1.1) is multiplication by $t^k - 1$. The corresponding sequence of the homology groups gives

$$\begin{aligned} &\to H_n((\mathbf{CP}^{n+1} - V \cup H)_\infty, \mathbf{Q}) \to H_n((\mathbf{CP}^{n+1} - V \cup H)_\infty, \mathbf{Q}) \to \\ &\qquad \to H_n((\mathbf{CP}^{n+1} - V \cup H)_k, \mathbf{Q}) \to H_{n-1}((\mathbf{CP}^{n+1} - V \cup H)_\infty) \to \end{aligned} \quad (1.2)$$

The right term in this sequence is trivial according to the first part of this proposition. The left term is canonically isomorphic to $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{Q}$, and parts 2 and 3 follow. In the case $n = 1$ (i.e., part 4) the homomorphism of the right term in (1.2) is the map of 0-dimensional homology groups. This map (i.e., $t - 1$) is trivial since the deck transformation acts as the identity on H_0. The result hence follows since the infinite cyclic cover is connected and hence $H_0((\mathbf{CP}^2 - (V \cup H))_\infty, \mathbf{Q}) = \mathbf{Q}$.

The following sufficient condition for the vanishing of $h^{p,q}$ when $p \neq q$ for divisors with normal crossings will be used in the study of the Hodge numbers of compactifications of unbranched coverings just considered.

Definition 1.5 (cf. [24]). A complex space X has V-singularity at a point $p \in X$ if p admits a neighborhood U_p in X which is analytically equivalent to $\mathbf{C}^l/G_p$ ($l = \dim X$) for a finite subgroup of $GL(l, \mathbf{C})$. A complex space is V-manifold if it has only V-singularities at all its points. A compact complex space X is a V-variety with normal crossings if at any $p \in X$ it admits a neighborhood U_p which is isomorphic to a quotient by a finite group $G_p \subset GL(l, \mathbf{C})$ of a union of coordinate planes: $\{(z_1, \dots, z_l) \in \mathbf{C}^l | z_1 \cdot \dots \cdot z_k = 0, |z_i| < \epsilon, 1 \le i \le l\}$.

Lemma 1.6. *Let $\bigcup D_i$ be a V-variety with V-normal crossings with $n = \dim D_i$ for any i. Suppose that*

(1) *$Gr_F^p Gr_{p+q}^W (H^{p+q}(D_i), \mathbf{C}) = 0$ for $p + q \ge n + 1$ and $p \ne q$ for any i.*

(2) *The mixed Hodge structure on $H^{p+q}(\bigcup D_i, \mathbf{C})$ is pure of weight $p + q$ provided $p + q \ge n + 1$.*

Then $Gr_F^p Gr_{p+q}^W (H^{p+q}(\bigcup D_i, \mathbf{C})) = 0$ for $p + q \ge n + 1$ and $p \ne q$.

Proof. Recall that the weight filtration on $H^{p+q}(\bigcup D_i, \mathbf{C})$ is the filtration induced on the abutment of the Mayer–Vietoris spectral sequence $E_1^{p,q} = H^p(D^{[q]}) \Rightarrow H^{p+q}(\bigcup D_i, \mathbf{C})$ (cf. [9, Sec. 4]). This spectral sequence degenerates in the term E_2. Therefore the weight filtration W_i on $H^{p+q}(\bigcup D_i, \mathbf{C})$ satisfies $W_i/W_{i-1} = 0$ for $i \ne p + q$ and $W_{p+q} = E_\infty^{p+q,0} = E_2^{p+q,0} = \operatorname{Ker} \bigoplus H^{p+q}(D_i) \to \bigoplus H^{p+q}(D_i \cap D_j)$. The F-filtration on W_{p+q} is the restriction of the F-filtration on the E_1 term of the spectral sequence above, i.e., the restriction from the F-filtration on $\bigoplus H^{p+q}(D_i, \mathbf{C})$. Therefore, the lemma follows from our assumption on $Gr^F Gr_W$ for each component D_i.

Next let us consider the calculation of the Hodge numbers of a hypersurface F in a weighted projective space $\mathbf{CP}_{\mathbf{w}}^N$ using the sheaf of adjoint ideals Adj_F (cf. [1] for the case of standard projective space). Let D be the degree of F and $Q = \sum q_i$ be the sum of weights of $\mathbf{CP}_{\mathbf{w}}^N$ where the weight $\mathbf{w}$ is $(q_0, \dots, q_N)$. Let $\pi : \mathcal{O}_{\mathbf{CP}_{\mathbf{w}}^N} \to \mathcal{O}_F$ be the restriction map and let $f : \tilde{F} \to F$ be a resolution of singularities of F. Then we put $\mathrm{Adj}_F = \pi^{-1}\mathcal{A}$ for $\mathcal{A} = f_*(\Omega_{\tilde{F}})(-D + Q)$. Here $\Omega_{\tilde{F}}$ is the dualizing sheaf of the resolution $\tilde{F}$, i.e., the sheaf of holomorphic $N-1$-forms on $\tilde{F}$. In particular, one has

$$0 \to \mathcal{O}_{\mathbf{CP}_{\mathbf{w}}^N}(-D) \to \mathrm{Adj}_F \to \mathcal{A} \to 0. \tag{1.3}$$

Note that in this definition one can take, instead of $\tilde{F}$, a V-manifold $\bar{F}$ admitting a birational morphism $\bar{f} : \bar{F} \to F$, and replace $\Omega_{\tilde{F}}$ by $\tilde{\Omega}_{\bar{F}} = i_* \left(\Omega^{N-1}_{\bar{F} - \mathrm{Sing}(\bar{F})}\right)$, where i is the embedding $i : \bar{F} - \mathrm{Sing}(\bar{F}) \to \bar{F}$ ($\mathrm{Sing}\, \bar{F}$ is the singular locus of $\bar{F}$). Then $\mathcal{A} = \bar{f}_*(\tilde{\Omega}_{\bar{F}})$ because, if $f' : \tilde{F} \to \bar{F}$ is a resolution of $\bar{F}$, one has $\tilde{\Omega}_{\bar{F}} = f'_*(\Omega_{\tilde{F}})$ (cf. [24, Lemma (1.1)]). The stalk $\mathrm{Adj}_F(p)$ of the sheaf Adj_F at a singular point $p \in F$ can be described as follows. Let $\phi \in \mathcal{O}_{\mathbf{CP}_{\mathbf{w}}^N}(p)$ be a germ of a holomorphic function at p and ω_0 be a nonvanishing holomorphic $(N-1)$-form on $F - \mathrm{Sing}(F)$. Then $\phi \in \mathrm{Adj}_F(p)$ iff $\bar{f}(\phi \cdot \omega_0)$ extends to a holomorphic form on $\bar{F} - \mathrm{Sing}(\bar{F})$ (cf. [19]). An equivalent description is that $\phi \in \mathrm{Adj}_F$ iff for any $(N-1)$-chain γ in $F - \mathrm{Sing}(F)$ one has

$$\int_\gamma (\phi \cdot \omega_0) < \infty \tag{1.4}$$

(cf. [19]).

Using adjoint ideals of the tower of cyclic covers, one can define the invariants (constants of quasiadjunction) of an isolated singularity $f(z_1, \dots, z_{n+1})$ (which we shall assume to be at the origin of $\mathbf{C}^{n+1}$) as a consequence of the following (cf. [13, 18]):

Proposition 1.7 (cf. [13; 18, Définition–Proposition 3.2]). *Let us fix a germ ϕ of a holomorphic function at the origin and let us consider the following function ψ_ϕ of $m \in \mathbf{N}$: $\psi_\phi(m) = \min\{l \mid z_{n+2}^l \cdot \phi \in \mathrm{Adj}(z_{n+2}^m = f(z_1, \dots, z_{n+1})\}$. Then $\psi(m) = [\kappa_\phi \cdot m]$ for some rational number κ_ϕ called the constant of quasiadjunction of the singularity f corresponding to the germ ϕ.*

We shall recall a proof for convenience. Let us first note that a V-manifold $\bar{F}_m$ and a birational map $\bar{f} : \bar{F}_m \to F_m$, where F_m is a germ at the origin of the complex space given in $\mathbf{C}^{n+2}$ by $z_{n+2}^m = f(z_1, \dots, z_{n+1})$, can be obtained as follows. Let $\rho : (Y_f, \bigcup_i E_i) \to (\mathbf{C}^{n+1}, 0)$ be an embedded resolution of the singularity $f(z_1, \dots, z_{n+1}) = 0$. Then the normalization of the fibre product $F_m \times_{\mathbf{C}^{n+1}} Y_f$ (the latter is defined using

ρ and the projection of F_m onto $\mathbf{C}^{n+1}$) has only quotient singularities (cf. [17]) and can be taken as $\bar{F}_m$. For each component E_i of the exceptional locus of the embedded resolution ρ, let a_i be the multiplicity of E_i in the full preimage of the hypersurface $f(z_1, \dots, z_{n+1}) = 0$ and let c_i (resp. f_i) be the multiplicity of $\rho^*(dz_1 \wedge \dots \wedge dz_{n+1})$ (resp. $\rho^*(\phi)$) along E_i. Finally let $g_i(m) = \text{g.c.d.}(a_i, m)$. The composition of the normalization $\bar{F}_m \to F_m \times_{\mathbf{C}^{n+1}} Y_f$ and the projection of the latter on the second factor identifies $\bar{F}_m$ as a cyclic cover of Y_f. On the other hand, $\bar{F}_m$ is a regular neighborhood of the exceptional locus of a resolution of the singularity $z_{n+2}^m = f(z_1, \dots, z_{n+1})$. Each component of this exceptional locus is a branched covering of one of the E_i's. The ramification index at the points of components which cover E_i is equal to $\frac{m}{g_i}$, as can be seen from the normalization of the covering of a transversal to E_i (which in terms of a local parameter u in this transversal looks like the normalization of $u^{a_i} = z_{n+2}^m$). One can take $\dfrac{dz_1 \wedge \dots \wedge dz_{n+1}}{z_{n+2}^{m-1}}$ as a nonvanishing form on F_m. Hence $z_{n+2}^l \cdot \phi$ is in the adjoint ideal of the singularity $z_{n+2}^m = f(z_1, \dots, z_{n+1})$ iff

$$(l - m + 1) \cdot \operatorname{mult} \bar{f}^*(z_{n+2}) + \operatorname{mult} \bar{f}^*(\phi) + \operatorname{mult} \bar{f}^*(dz_1 \wedge \dots \wedge dz_{n+1}) \geq 0 \tag{1.5}$$

for any component of the exceptional set of $\bar{F}_m \to F_m$ and where *mult* is the multiplicity along such a component. For components over E_i one has $\operatorname{mult} \bar{f}^*(z_{n+2}) = \frac{a_i}{g_i}$, $\operatorname{mult} \bar{f}^*(\phi) = \dfrac{f_i m}{g_i}$, $\operatorname{mult}(dz_1 \wedge \dots \wedge dz_{n+1}) = \dfrac{c_i m}{g_i} + \dfrac{m}{g_i} - 1$. (To verify the latter it suffices to note that in the local coordinates $(u_1, \dots, u_{n+1})$ in which $\bar{F}_m \to Y_f$ is given by $u_1 = v^{\frac{m}{g_i}}$, the pull back of the form $u_1^{c_i} du_1 \wedge du_2 \wedge \dots \wedge du_{n+1}$ is equal to $v^{\frac{mc_i}{g_i} + \frac{m}{g_i} - 1} dv \wedge du_2 \wedge \dots du_{n+1}$). Hence the inequality (1.5) in this case shows that $\psi_\phi(m)$ is the minimum over all i of solutions of the inequality: $(l - m + 1)\left(\dfrac{a_i}{g_i} + \dfrac{f_i m}{g_i} + \dfrac{c_i m}{g_i} + \dfrac{m}{g_i} - 1\right) \geq 0$. The latter is equivalent to $l + 1 > m\left(1 - \dfrac{f_i + c_i + 1}{a_i}\right)$. The minimal solution of this inequality is $\left[m\left(1 - \dfrac{f_i + c_i + 1}{a_i}\right)\right]$. Hence $\psi_\phi(m) = [\kappa_\phi \cdot m]$, where $\kappa_\phi = \max_i \left(1 - \dfrac{f_i + c_i + 1}{a_i}\right)$.

Remark 1.8. The proof shows in particular the following. Let a_i be the multiplicity of a component of an embedded resolution of an isolated hypersurface singularity and let c_i be the multiplicity along this component of the pull back of a nonvanishing top-dimensional differential form. Then $\min(\frac{c_i+1}{a_i})$ is an invariant of the singularity. More precisely, this number is $1 - \kappa_1$, where κ_1 is the constant of quasiadjunction described above corresponding to $\phi = 1$. This proof also shows how the constants of quasiadjunction are related to the *geometric weights* from [27] (cf. also [5]).

Examples 1.9. Weighted homogeneous singularities. Let $g(z_1, \dots, z_{n+1}, z_{n+2})$ be a weighted homogeneous singularity. According to [19], a monomial $z_1^{i_1} \cdots z_{n+2}^{i_{n+2}}$ belongs to the adjoint ideal of $g(z_1, \dots, z_{n+2})$ if and only if $(i_1 + 1, \dots, i_{n+2} + 1)$ is inside the Newton polytope of $g(z_1, \dots, z_{n+2})$. This immediately implies the proposition in the case of weighted homogeneous singularities. If the singularity is locally equivalent to $z_1^{q_1} + \dots + z_{n+1}^{q_{n+1}} = 0$, then the Newton polytope of $z_{n+2}^m = f(z_1, \dots, z_{n+1})$ is

$$\left\{(x_1, \dots, x_{n+2}) \in \mathbf{R}^{n+2} \,\middle|\, \sum_{i=1}^{i=k+1} \frac{m \cdot D}{q_i} \cdot x_i + D \cdot x_{n+2} > D \cdot m\right\},$$

where $D = \prod_{i=1}^{i=k+1} q_i$. Hence the minimal l such that $z_{n+2}^l \cdot z_1^{i_1+1} \cdots z_{n+1}^{i_{n+1}+1}$ is in the adjoint ideal of $z_{n+2}^m = f(z_1, \ldots, z_{n+1})$ equals $\left[m\left(1 - \sum_k \frac{i_k+1}{q_k}\right)\right]$. Therefore, the constants of quasiadjunction are $\left(1 - \sum \frac{i_k+1}{q_k}\right)$, provided this expression is positive.

Note that, using Proposition 1.7, one can, of course, always determine the constants of quasiadjunction from an embedded resolution of the singularity.

Definition 1.10. The ideal of quasiadjunction of a hypersurface F with isolated singularities is the ideal $\mathcal{A}_\kappa$ such that $\Gamma(U, \mathcal{A}_\kappa) = \{\phi \in \Gamma(U, \mathcal{O}_{\mathbf{CP}^{n+1}}) | \kappa = \min \lambda \in \Theta, \kappa_\phi < \lambda \in \Theta\}$, where Θ is the collection of constants of quasiadjunction of all singularities of F.

Example 1.11. Let us consider the ideal of quasiadjunction in the local ring of a point $p \in \mathbf{CP}^{n+1}$ in which $F(z_0, \ldots, z_{n+1}) = 0$ has a quasihomogeneous singularity. Such an ideal corresponding to the constant of quasiadjunction of $\phi = 1$ which equals $\kappa_1 = \left(1 - \sum_{i=1}^{n+1} \frac{1}{q_i}\right)$ is just the maximal ideal in $\mathcal{O}_{\mathbf{CP}^{n+1}}$. In general, a germ $\phi = 0$ belongs to an ideal $\mathcal{A}_\kappa$ if the curve satisfies a certain geometric condition. For example, for a plane curve singularity $x^2 = y^5$ the ideal $\mathcal{A}_{\frac{3}{10}}$ consists of germs the set of zeros of which has the tangent cone belonging to the tangent cone of the singularity.

2. Mixed Hodge Structure on π_n

In this section, we shall introduce the mixed Hodge structure on the first nontrivial homotopy group of the complement to a hypersurface. We shall use it to define $\mathbf{C}[t, t^{-1}]$-modules $L_{p,q}$ providing a link between π_n and the Hodge numbers of the cyclic branched coverings (Proposition 3.3).

Let V be a hypersurface of degree d with isolated singularities in $\mathbf{CP}^{n+1}$ and H be a hyperplane at infinity such that V and $V \cap H$ have, at most, isolated singularities (cf. [16]). It follows from the divisibility theorem ([16, Sec. 4]) that the order $\Delta_V(t)$ of $\pi_n(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{C}$ as a $\mathbf{C}[t, t^{-1}]$-module is a cyclotomic polynomial and that the action of t is quasi-unipotent. Indeed $\Delta_V(t)$ divides the product of polynomials associated with singularities of V, including singularities at infinity. For singular points of V which are outside the hyperplane at infinity, the associated polynomial is the characteristic polynomial of local monodromy and hence is cyclotomic. Let us consider a singular point at infinity near which V (resp. H) is given by $f(z_0, z_1, \ldots, z_n) = 0$ (resp. $z_0 = 0$). Let M_1 (resp. M_2) be the portions of $f(z_0, \ldots, z_n) = s$ (resp. $f(0, z_1, \ldots, z_n) = s$) in a small ball about the singular point. The polynomial associated with this singular point is the characteristic polynomial of the monodromy acting on $M_1 - M_2$ (cf. [16, Remark 4.2]). Its cyclotomic property follows again from the monodromy theorem and *equivariant,* with respect to the action of the monodromy operator, exact sequence: $H_{n-1}(M_2) \to H_n(M_1 - M_2) \to H_n(M_1)$ (i.e., the exact sequence of the pair $(M_1, M_1 - M_2)$ in which $H_{n+1}(M_1, M_1 - M_2)$ is replaced by $H_{n-1}(M_2)$).

Let $N(V)$ be an integer such that any root of $\Delta_V(t)$ is a root of unity of degree $N(V)$. The action of the fundamental group of the $N(V)$-fold cover $(\mathbf{CP}^{n+1} - V \cup H)_{N(V)}$ of $\mathbf{CP}^{n+1} - V \cup H$ on $\pi_n((\mathbf{CP}^{n+1} - V \cup H)_{N(V)})$ is nilpotent.

On the other hand, the homology of $(\mathbf{CP}^{n+1} - V \cup H)_{N(V)}$ can be identified with the cokernel of the endomorphism of $\pi_n \otimes \mathbf{C}$ which is the multiplication by $t^{N(V)} - 1$. In the cases where one knows that $\pi_n(\mathbf{C}^{n+1} - V \cup H) \otimes \mathbf{C}$ is a semisimple $\mathbf{C}[t, t^{-1}]$ module (for example, if H is transversal to V, cf. [16]), the multiplication by $t^{N(V)} - 1$ is trivial (cf. [16]) and the cokernel of this map coincides with $\pi_n \otimes \mathbf{C}$. In general, we shall call this cokernel *the semisimple part* of $\pi_n \otimes \mathbf{C}$ and shall denote it π_n^s.

Definition 2.1. The mixed Hodge structure on $\pi_n(\mathbf{CP}^{n+1} - V \cup H))$ is the one obtained from the MHS on the homotopy Lie algebra (cf. [10, 11]) via isomorphism: $\pi_n((\mathbf{CP}^{n+1} - V \cup H)_{N(V)} = \pi_n(\mathbf{CP}^{n+1} - V \cup H)$. Alternatively, the induced mixed Hodge structure on π_n^s is the one obtained from the canonical mixed Hodge structure existing on the homology of a quasiprojective variety (cf. [2]) via the isomorphism $\pi_n^s(\mathbf{CP}^{n+1} - V \cup H) \otimes \mathbf{C}) = H_n((\mathbf{C}^{n+1} - V \cup H)_{N(V)}, \mathbf{C}))$ described in Lemma 1.4.

Note that the deck transformations acting on $(\mathbf{C}^{n+1} - V \cup H)_{N(V)}$ are algebraic maps and in particular preserve both filtrations on $\pi_n \otimes \mathbf{C}$. Therefore

$$L_{p,q}(V) = Gr_F^{-p} Gr_{-p-q}^W \tag{2.1}$$

has the natural structure of $\mathbf{C}[t, t^{-1}]$-module. We will need the cyclic decomposition of this module:

$$L_{p,q}(V) = \oplus \mathbf{C}[t, t^{-1}]/(\lambda_{i,(p,q),V}(t)), \tag{2.2}$$

where $\lambda_{i+1} | \lambda_i$.

These polynomials $\lambda_{i,(p,q),V}(t)$ are well defined up to a unit of $\mathbf{C}[t, t^{-1}]$.

Definition 2.2. The (p,q)-part of the order of $\pi_n \otimes \mathbf{C}$ is the $\mathbf{C}[t, t^{-1}]$-order of the module (2.2). We denote this polynomial $\Delta_V^{p,q}(t)$.

Remark 2.3. One can define similar groups (2.1) and corresponding polynomials using π_n^s instead of $\pi_n \otimes \mathbf{C}$. By abuse of notation we shall use the same symbols in both cases. The statements refer to either case unless otherwise stated. I do not know examples of hypersurfaces for which $\pi_n^s \neq \pi_n \otimes \mathbf{C}$.

Remark 2.4. The mixed Hodge structure on π_n^s is independent of a choice of the base point used in the definition of the homotopy group.

Remark 2.5. In the case $n = 1$, the same arguments give the mixed Hodge structure on π_1'/π_1'' (resp. semisimple part of it), where π_1 is the fundamental group of the complement to an affine curve. In the case where the line H at infinity is transversal to the projective closure C of the curve, the polynomial $\Delta^{1,0}(t) \cdot \Delta^{0,1}(t)$ coincides with the Alexander polynomial of C (cf. [12]). Indeed, if $\tilde{X}_d$ is a resolution of the singularities of the cyclic branched covering of $\mathbf{CP}^2$ branched over C, $\tilde{C}$ is the preimage of the curve C in $\tilde{X}_d$, and E is the exceptional locus, then one has the exact sequence of the mixed Hodge structures:

$$\to H^2(\tilde{X}_d) \to H^2(E) \to H^3(\tilde{X}_d, E) \to H^3(\tilde{X}_d) \to \tag{2.3}$$

The left homomorphism in this sequence is surjective. Indeed, the composition $H^2(\tilde{X}_d, \tilde{X}_d - E, \mathbf{C}) \to H^2(\tilde{X}_d, \mathbf{C}) \to H^2(E, \mathbf{C})$ can be identified with the homomorphism

$$H^2(T(E), \partial T(E), \mathbf{C}) \to H^2(T(E), \mathbf{C}),$$

where $T(E)$ is a regular neighborhood of E in $\tilde{X}_d$ and $\partial T(E)$ is its boundary. The latter can be interpreted as the map $\mathrm{Hom}(H^2(T(E), \mathbf{C}), \mathbf{C}) \to H^2(T(E), \mathbf{C})$ which is an isomorphism because it corresponds to the intersection form $H^2(E, \mathbf{C}) \times H^2(E, \mathbf{C}) \to \mathbf{C}$ which is nondegenerate (cf. [21]). Hence (2.3) implies that the Hodge structure on $H^3(\tilde{X}_d, E)$ is pure of weight 3 and hence the Hodge structure on $H_1(\tilde{X}_d - E)$ is pure of weight -1. On the other hand, one can easily verify using the sequence (1.2) specialized to the case $n = 1$ that the composition of the embedding $\pi_1'/\pi_1'' \otimes \mathbf{C}$ into $H_1(\tilde{X}_d - (\tilde{C} \cup E))$ with the homomorphism $H_1(\tilde{X}_d - (\tilde{C} \cup E)) \to H_1(\tilde{X}_d - E)$ induced by inclusion is an isomorphism. Therefore the claim follows.

Remark 2.6. This mixed Hodge structure, in the case where V is a weighted homogeneous hypersurface given by $f(z_1, \ldots, z_{n+1}) = 0$, coincides with the canonical mixed Hodge structure on a nonsingular algebraic variety which is the affine hypersurface $f(z_1, \ldots, z_{n+1}) = 1$. To see this we let $w_1, \ldots, w_{n+1}$ be the weights of the weighted homogeneous polynomial $f(z_1, \ldots, z_{n+1})$ (i.e., f is a linear combination of monomials $z_1^{i_1} \cdots z_{n+1}^{i_{n+1}}$ such that $i_1/w_1 + \cdots + i_{n+1}/w_{n+1} = 1$ (cf. [20])). Let $N = \mathrm{g.c.d.}(w_1, \ldots, w_{n+1})$. If $V_{f=1}$ (resp. $V_{f=0}$) is the hypersurface given by $f = 1$ (resp. $f = 0$), then $\pi_n(\mathbf{C}^{n+1} - V_{f=0})$ is isomorphic to

$H_n(V_{f=1}, \mathbf{Z})$ as a module over $\mathbf{Z}[t, t^{-1}]$ with the module structure given by defining the action of t to be the same as the action of the monodromy operator. The order of $\pi_n(\mathbf{C}^{n+1} - V_{f=0}) \otimes \mathbf{C})$ as a $\mathbf{C}[t, t^{-1}]$-module is the characteristic polynomial of the monodromy operator acting on $H_n(V_{f=1}, \mathbf{Q})$. Hence any root of the $\mathbf{Q}[t, t^{-1}]$-order of $\pi_n(\mathbf{C}^{n+1} - V_{f=0}) \otimes \mathbf{Q}$ is a root of a unity of degree N. Moreover, because the monodromy operator acting on the homology of the Milnor fibre of a weighted homogeneous singularity is semisimple ([25]), we see that $\pi_n^s(\mathbf{C}^{n+1} - V_{f=0}) = \pi_n(\mathbf{C}^{n+1} - V_{f=0})$. Next we claim that the N-fold cyclic covering of $\mathbf{C}^{n+1} - V_{f=0}$ is biregularly equivalent to $\mathbf{C}^* \times V_{f=1}$. Indeed, the cyclic covering in question is isomorphic to the hypersurface $V_{f=z_0^N}$ in $\mathbf{C}^{n+2}$ given by the equation $f(z_1, \dots, z_{n+1}) = z_0^N$ with a removed intersection of this hypersurface with the hyperplane $z_0 = 0$. Let us consider the map $\Phi : \mathbf{C}^* \times V_{f=1} \to V_{f=z_0^N}$ given by $(t, z_1, \dots, z_{n+1}) \to (t, t^{N/w_1} z_1, \dots, t^{N/w_{n+1}} z_{n+1})$. Φ clearly is a biregular map and has as its image the complement to the intersection of $V_{f=z_0^N}$ with the hyperplane $z_0 = 0$. The mixed Hodge structure on the product can easily be calculated using the fact that the Kunneth formula is consistent with mixed Hodge structures. This, combined with the calculation of the mixed Hodge structure on $\mathbf{C}^*$ (cf. [2]), gives the assertion on the mixed Hodge structure on the homotopy group of the complement to a weighted homogeneous hypersurface. The Hodge numbers of an affine quasihomogeneous hypersurface were completely calculated by J. Steenbrink ([25]). For examples, the calculations on p. 223, ibid., give the following Hodge numbers for the mixed Hodge structure of $\pi_2 \otimes \mathbf{C}$ of the complement to the hypersurface $x^3 + y^3 + z^3 + 3\lambda xyz = 0$ ($\lambda^3 \neq 1$): $h^{2,0} = h^{0,2} = 0, h^{1,1} = 6, h^{1,2} = h^{2,1} = 1$.

Lemma 2.7. *The Hodge number $h_{p,q}((\mathbf{CP}^{n+1} - V \cup H)_k)$ of the k-fold cover of $\mathbf{CP}^{n+1} - V \cup H$ is equal to the number of common roots of $t^k - 1$ and the polynomials $\lambda_{i,(p,q),V}$ which are the orders of the cyclic decomposition of the module* (2.2).

Proof. First notice that if $l = \text{g.c.d.}(k, N(V))$, then the homology of k-fold and l-fold cyclic covers are canonically isomorphic in dimensions not exceeding n. Indeed, we have the following exact sequence (cf. Lemma 1.4):

$$\to H_n((\mathbf{CP}^{n+1} - V \cup H)_k) \to H_n((\mathbf{CP}^{n+1} - V \cup H)_k) \to H_n((\mathbf{CP}^{n+1} - V \cup H)_l) \to 0. \qquad (2.4)$$

The homomorphism between two left terms in the last exact sequence is the multiplication by $t^l - 1$. A root η of an order of a $\mathbf{Q}[t, t^{-1}]$- cyclic summand of $H_n((\mathbf{CP}^{n+1} - V \cup H)_k, \mathbf{Q})$ considered as a $\mathbf{Q}[t, t^{-1}]$-module is a root of unity which satisfies $\eta^{N(V)} = \eta^k = 1$. Hence such a root is a root of unity of degree l. Therefore the left homomorphism in (2.4) is trivial. Note also that the right homomorphism in the sequence (2.4) is induced by an algebraic map which is the projection map of the covering and therefore is a morphism of mixed Hodge structures.

Let us now show the lemma in the case where $k = l$, i.e., when $k | N(V)$. To this end, let us consider the exact sequence (2.4) with $k = N(V)$. Both middle maps in (2.4) are the maps of the mixed Hodge structures because the deck transformation and the projection both are algebraic maps. The result follows.

If $k \neq l$, then the homology of the k and l-fold covers are isomorphic and the multiplication by $t^k - 1$ and $t^l - 1$ have the same cokernels since none of the roots of $\dfrac{t^k - 1}{t^l - 1}$ is a root of $t^{N(V)} - 1$.

3. Hodge Numbers of Cyclic Branched Coverings

In this section, we shall compare the Hodge numbers of branched and unbranched coverings and obtain the restriction on the weights of the mixed Hodge structure on the homotopy groups considered in the last section.

Proposition 3.1. *Let V be a hypersurface in $\mathbf{CP}^{n+1}$ such that all singularities of V are isolated ($n \geq 2$). Let $\tilde{X}_m$ be a nonsingular model of the m-fold cyclic cover of $\mathbf{CP}^{n+1}$ branched along $V \cup H$. Then the Hodge number*

$h^{n,0}(\tilde{X}_m)$ is equal to the number of common roots of the $\mathbf{C}[t,t^{-1}]$-orders $\lambda_{i,(n,0),V}$ of the cyclic decomposition of the $\mathbf{C}[t,t^{-1}]$ module (2.2) and t^m-1. In particular, the Hodge number $h^{n,0}(\tilde{X}_m)$ does not exceed the sum over i of the numbers of common roots of t^m-1 and the $\mathbf{C}[t,t^{-1}]$ orders $\lambda_i(t)$ of the direct summands of the cyclic decomposition of $\pi_n(\mathbf{C}^{n+1}-V)\otimes\mathbf{C}=\oplus\mathbf{C}[t,t^{-1}]/(\lambda_i(t))$.

Proof. Let X_m be the m-fold cyclic cover of

$$\mathbf{CP}^{n+1}-(V\cup H). \tag{3.1}$$

First note that $rkH_n(X_m,\mathbf{C})$ is equal to the sum over i of the numbers of common roots of $\lambda_i(t)$ and t^m-1 (cf. Lemma 2.7). Next let us consider the branched covering $\pi:\bar{X}_m\longrightarrow\mathbf{CP}^{n+1}$ branched over $V\cup H$ constructed in Remark 1.2. Let E be the exceptional locus of the resolution of singularities $\tilde{X}_m$ of $\bar{X}_m$ which are in $\pi^{-1}(SingV)$. Let $\tilde{H}$ be the union of components in $\tilde{X}_m$ over $\bar{H}$ and let $\tilde{V}$ be the proper preimage of V in $\tilde{X}_m$, which we may assume is transversal to all components of E and $\tilde{H}$. We have the following exact sequence:

$$\begin{aligned}\to H^{n+1}(E\cup\tilde{V}\cup\tilde{H},\mathbf{C})\to H^{n+2}(\tilde{X}_m,E\cup\tilde{V}\cup\tilde{H},\mathbf{C})&\\ \to H^{n+2}(\tilde{X}_m,\mathbf{C})\to H^{n+2}(E\cup\tilde{V}\cup\tilde{H},\mathbf{C})\to&\end{aligned}\tag{3.2}$$

The group $H^{n+2}(\tilde{X}_m,E\cup\tilde{V}\cup\tilde{H},\mathbf{C})$ in (3.2) is dual to $H_n(X_m,\mathbf{C})$, as follows from the excision and the Poincaré duality. On the other hand, the sequence (3.2) is a sequence of mixed Hodge structures. The groups $H^i(E\cup\tilde{V}\cup\tilde{H},\mathbf{C})$ support for $i\geq n+1$ the Hodge structure for which

$$F^{n+1}(H^i(E\cup\tilde{V}\cup\tilde{H},\mathbf{C}))=0. \tag{3.3}$$

To show (3.3), recall that the Hodge filtration on the cohomology of a variety $E\cup\tilde{V}\cup\tilde{H}=D=\bigcup D_i$ with normal crossings is obtained from the Hodge filtration on a complex $A^p(D^{[q]})$. The latter is the complex of differential p-forms on the disjoint union of q-fold intersections of components of D, and the cohomology of D can be interpreted as the total cohomology of this double complex (cf. [9, Sec. 4]). Because $F^k(A^p(D^{[q]}))=0$ for $k\geq n+1$ ($n=\dim D_i$) we obtain that $F^{n+1}(H^p(D))=0$. Now it follows from (3.2) and (3.3) that $Gr_F^{n+1}Gr_{n+2}^W(H^{n+2}(\tilde{X}_m,E\cup\tilde{V}\cup\tilde{H}))=Gr_F^{n+1}Gr_{n+2}^W(H^{n+2}(\tilde{X}_m))=H^{n+1,1}(\tilde{X}_m)$. Therefore the proposition is a consequence of duality.

Proposition 3.2. *Let V be a hypersurface with isolated singularities, including infinity, and H be the hyperplane at infinity. Then $Gr_k^W(\pi_n^s(\mathbf{CP}^{n+1}-V\cup H)))=0$ for $k\neq -n,\ -n-1$.*

Proof. The $N(V)$-fold covering $\bar{X}_{N(V)}$ of $\mathbf{CP}^{n+1}$ branched over $V\cup H$ is a complex space with isolated singularities, and therefore its cohomology groups support a pure Hodge structure in dimensions greater than $\dim\bar{X}_{N(V)}$ ([26, Theorem 1.13]).[1] We are going to show that for $r\neq k, k\geq n+1$, one has

$$Gr_r^W(H^k(\bar{V}\cup\bar{H}))=0, \tag{3.4}$$

where $\bar{V}$ (resp. $\bar{H}$) is the preimage of V (resp. H) under the projection $\bar{X}_{N(V)}\to\mathbf{CP}^{n+1}$ of the $N(V)$-fold cover of $\mathbf{CP}^{n+1}$ branched over $V\cup H$. If we assume the purity (3.4) of the mixed Hodge structure, then the sequence

$$\to H^{n+1}(\bar{V}\cup\bar{H})\to H^{n+2}(\bar{X}_{N(V)},\bar{V}\cup\bar{H})\to H^{n+2}(X_{N(V)})\to \tag{3.5}$$

implies that the relative cohomology group $H^{n+2}(\bar{X}_{N(V)},\bar{V}\cup\bar{H})$ supports the mixed Hodge structure of weights $n+2$ and $n+1$. Hence the weights of $H^n(\bar{X}_{N(V)}-(\bar{V}\cup\bar{H}))=\mathrm{Hom}(H^{n+2}(\bar{X}_{N(V)},\bar{V}\cup\bar{H}),\mathbf{Q}(-n-1))$ are n and $n+1$ and the proposition follows.

[1] Here a slightly more general situation than the one considered in [26] may occur: $\bar{X}_{N(V)}$ may be a V-manifold with isolated singularities (i.e., each singularity is a quotient of an isolated singularity by an action of a finite group). This will take place if one uses the model of a cyclic branched cover from Remark 1.3 rather than Remark 1.2. The proof is similar to that in [26, 1.13].

To show (3.4), note that the restriction map $H^k(\bar{V}) \to H^k(\bar{V} \cap \bar{H})$ is surjective for $k \geq n+1$ as a consequence of the isolatedness of the singularities of $\bar{V} \cap \bar{H}$. Indeed, for $k \geq n+1$ the group $H^k(\bar{V} \cap \bar{H}, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ (resp. to 0) if k is even (resp. odd). To see this, let us denote by $\widetilde{\bar{V} \cap \bar{H}}$ a smoothing of the singularities of $\bar{V} \cap \bar{H}$. Let M be the union of the Milnor fibres of all singularities of $\bar{V} \cap \bar{H}$. Then $H^k(\bar{V} \cap \bar{H}) = H^k(\bar{V} \cap \bar{H}, \mathrm{Sing}(\bar{V} \cap \bar{H})) = H^k(\widetilde{\bar{V} \cap \bar{H}}, M)$ and the claim follows from the exact sequence $H^{k-1}(M) \to H^k(\widetilde{\bar{V} \cap \bar{H}}, M) \to H^k(\widetilde{\bar{V} \cap \bar{H}}) \to H^k(M)$ and the standard results on connectivity of the Milnor fibres and cohomology of nonsingular hypersurfaces.

The surjectivity of this restriction map implies that the map $H^{n+2}(\bar{V} \cup \bar{H}, \mathbf{Q}) \to H^{n+2}(\bar{V}, \mathbf{Q}) \oplus H^{n+2}(\bar{H}, \mathbf{Q})$ is injective. Now the mixed Hodge structures on both summands in the last direct sum are pure because $n+2$ is bigger than the dimension of $\bar{V}$ and the dimension of $\bar{H}$ since both have at most *isolated* singularities (cf. [26, Theorem 1.13]). Therefore we obtain the purity of $H^{n+2}(\bar{V} \cup \bar{H})$, which proves the proposition.

Let $f(z_1, \ldots, z_{n+1})$ be a polynomial having an isolated singularity at the origin. Suppose that f is a weighted homogeneous singularity of weights $w_1, \ldots, w_{n+1}$ (with the convention used in Remark 2.6, recall that $N = \text{g.c.d.}(w_1, \ldots, w_{n+1})$). Note that a resolution of this singularity in the category of V-manifolds can be obtained as follows. Let $\Gamma \subset \mathbf{C}^{n+1} \times \mathbf{CP}^n_{N/w_1, \ldots, N/w_{n+1}}$ be the closure of the incidence correspondence of the action $(t, (z_1, \ldots, z_{n+1})) \to (z_1 t^{N/w_1}, \ldots, z_{n+1} t^{N/w_{n+1}})$. In other words, set theoretically Γ is

$\{(P, Q) \mid P \in \mathbf{C}^{n+1}, Q \in \mathbf{CP}^n_{w_1, \ldots, w_{n+1}},$ *where P belongs to the closure in $\mathbf{C}^{n+1}$ of the orbit Q*$\}$.

Γ is a V-manifold since $\mathbf{CP}^n_{N/w_1, \ldots, N/w_{n+1}}$ is. If $p_1 : \Gamma \to \mathbf{C}^{n+1}$ is the natural projection and V_f is the affine hypersurface in $\mathbf{C}^{n+1}$ given by $f(z_1, \ldots, z_{n+1})$, then $p_1^{-1}(V_f) \to V_f$ is a resolution of the singularity of V_f at the origin. The exceptional set is the hypersurface in $\mathbf{CP}^n_{N/w_1, \ldots, N/w_{n+1}}$ defined by $f(z_1, \ldots, z_{n+1}) = 0$. We shall call this resolution the *canonical resolution* of the weighted homogeneous singularity V_f.

A property of the exceptional set E of such a resolution, which we shall use in the next proposition, is

$$h^{p,q} = 0 \quad \text{unless } p = q \text{ or } p + q = \dim E. \tag{3.6}$$

This follows from a similar property of the Hodge numbers of weighted projective spaces and the weak Lefschetz theorem in this context (cf. [4, (4.2.2)]).

Proposition 3.3. *Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which has isolated singularities including infinity. Assume that all singularities of V are weighted homogeneous.*

(1) *One has $Gr^k_F Gr^W_{-n}(\pi^s_n(\mathbf{CP}^{n+1} - V \cup H)) = 0$ for $2k \neq -n$.*

(2) *Let $\bar{X}_m$ be a branched cyclic m-fold cover of $\mathbf{CP}^{n+1}$ branched over $V \cup H$ (i.e., $\bar{X}_m$ has only isolated weighted homogeneous or V-singularities). For $p \neq q$ the Hodge numbers $h^{p,q}(\hat{X}_m)$ of the canonical resolution $\hat{X}_m$ of isolated weighted homogeneous singularities of $\bar{X}_m$ is equal to the sum over i of the numbers of common roots of $t^m - 1$ and the orders $\lambda_{i,p,q}(t)$ of all terms in the cyclic decomposition (2.2) of $L_{p,q}(V)$.*

Proof. Let $\hat{V}$ and $\hat{H}$ be preimages of V and H respectively in the canonical resolution $\hat{X}$ of the cyclic branched covering $\bar{X}_m$. First note that

$$h^{p,q}(E \cup (\hat{V} \cup \hat{H})) = 0, \quad p \neq q, \quad p + q \neq n + 1. \tag{3.7}$$

Indeed, $E \cap (\hat{V} \cup \hat{H})$ is a disjoint union of weighted homogeneous hypersurfaces of dimension $n-1$. This implies that $H^k(E \cap (\hat{V} \cup \hat{H}))$ is generated by the classes which are restrictions of the cohomology classes of weighted projective spaces containing these hypersurfaces. Therefore the left homomorphism in the Mayer–Vietoris sequence

$$\to H^k(E) \oplus H^k(\hat{V} \cup \hat{H}) \to H^k(E \cap (\hat{V} \cup \hat{H})) \to H^{k+1}(E \cup \hat{V} \cup \hat{H}) \to H^{k+1}(E) \oplus H^{k+1}(\hat{V} \cup \hat{H}) \tag{3.8}$$

is surjective. Hence $H^k(E \cup \hat{V} \cup \hat{H})$ supports a pure Hodge structure provided $k \geq n+1$. Therefore (3.7) follows from the purity of the mixed Hodge structure on $H^k(\hat{V} \cup \hat{H})$ for $k \geq n+1$ (cf. (3.4)) and Lemma 1.6.

Now (3.7) implies that the map on $Gr_F^k Gr_{n+2}^W$ induced by $H^{n+2}(\hat{X}_m, E \cup \hat{V} \cup \hat{H}) \to H^{n+2}(\hat{X}_m)$ for $k \neq n+2-k$ is an isomorphism. Therefore part 2 follows from the exact sequence (3.2) and Lemma 2.7. The claim 1 on $Gr_{-n}^W(\pi_n(\mathbf{CP}^{n+1} - V \cup H))$ follows from (3.7) as well.

Remarks 3.4. 1. This proof gives an alternative argument to the one used in Proposition 3.1, albeit in the case where the singularities are weighted homogeneous. It also shows that the dimension $h^{n,0}(\tilde{X}_m)$ of the space of holomorphic forms on a (nonsingular) resolution of $\hat{X}_m$ is equal to the sum over i of the numbers of common roots of $t^m - 1$ and the orders $\lambda_{i,(n,0)}(t)$ of all terms of the cyclic decomposition (2.2) of $L_{n,0}(V)$. Indeed, it suffices to note that the singularities of $\hat{X}_m$ are all rational. Hence, the isomorphism of $\tilde{\Omega}^{\cdot}_{\hat{X}_m}$ and the direct image of $\tilde{\Omega}^{\cdot}_{\tilde{X}_m}$ (cf. [24, Lemma 1.11]), combined with the Leray spectral sequence applied to the resolution map, gives $h^{n,0}(\tilde{X}_m) = h^{n,0}(\hat{X}_m)$, which implies the claim.

2. In the case $n = 1$, this proof actually works without any assumptions on the singularities of a branch locus. Indeed, the vanishing of the Hodge numbers (which restricted our proof to the weighted homogeneous case (cf. (3.7))) is valid for $n = 1$, since the cohomology group $H^2(E \cup (\hat{V} \cup \hat{H}))$ supports a pure Hodge structure of weight 2 as follows from (3.8) for $k = 1$ since $\dim(E \cup \hat{V} \cup \hat{H}) = 0$.

4. The Position of Singularities of Branching Loci and Hodge Numbers of Cyclic Coverings

In this section, we shall calculate the Hodge number $h^{n,0}$ of a cyclic covering, branched over the union of a hypersurface with isolated singularities and the hyperplane at infinity, in terms of the cohomology of ideals of quasiadjunction, introduced in Sec. 1. As a part of the proof we show the regularity of certain linear systems of hypersurfaces having base points at the singularities of the branching locus (cf. Corollary 4.2). This allows us, in some cases, to detect the nontriviality of the modules $L_{n,0}$ from Sec. 2 and hence the nonvanishing of $\pi_n(\mathbf{CP}^{n+1} - V \cup H)$.

Theorem 4.1. *Let $\tilde{X}_m$ be a $\mathbf{Z}$-equivariant resolution of the m-fold cyclic covering of $\mathbf{CP}^{n+1}$ branched over a hypersurface of degree d with isolated singularities. Let ζ be a root of unity of degree m. Let $H^{n,0}(\tilde{X}_m)_\zeta$ be the eigenspace of the linear map induced by the deck transformation acting on the space of holomorphic n-forms on $\tilde{X}_m$ corresponding to the eigenvalue ζ. Then $\dim H^{n,0}(\tilde{X}_m)_\zeta$ is equal to $s_\zeta = \sum_\kappa \dim H^1(\mathbf{CP}^{n+1}, \mathcal{A}_\kappa(d - n - 2 - \kappa \cdot d))$, where the summation is over all constants of quasiadjunction of all singularities such that $exp(2\pi i\kappa) = \zeta$ and $\kappa \cdot m \in \mathbf{Z}$.*

Proof. We shall work with the model of X_m which is the hypersurface in the weighted projective space $\mathbf{CP}^{n+2}_{\mathbf{w}}$ where the weights are $\mathbf{w} = (1, m, \dots, m, d)$. The weight of z_0 is 1, the weight of z_{n+2} is d, and the weight of each of the variables $z_1, \dots, z_{n+1}$ is m (cf. Remark 1.3). The degree of X_m is equal to $D = d \cdot m$. Let $Q = d + (n+1) \cdot m + 1$, and so in these notations we have $\Omega^{n+1}_{X_m} = \mathcal{O}_{X_m}(D - Q)$.

Step 1. Let us consider the ideal sheaf $\mathrm{Adj}(X_m)(-k \cdot d)$ sections of which over an open set in $\mathbf{CP}^{n+1}_{\mathbf{w}}$ are the sections of $\mathrm{Adj}(X_m)$ having the order of vanishing along the hyperplane $z_{n+2} = 0$ equal at least to k. The multiplication of z_{n+2} by a fixed primitive root of unity (say $\exp(2\pi i/d)$) defines the action of the group $\mathbf{Z}/d \cdot \mathbf{Z}$ on the sheaf $\mathrm{Adj}(X_m)(-k)$ because the condition (1.4) in the case of X_m and $\omega_0 = \dfrac{dz_1 \wedge \dots \wedge dz_{n+1}}{z_{n+2}^{m-1}}$ is clearly invariant under this action. We claim that there is an equivariant isomorphism: $H^n(\tilde{X}_m, \mathcal{O}_{\tilde{X}_m}) = H^1(\mathrm{Adj}(X_m)(D - Q))$.

Indeed,

$$H^l(\mathrm{Adj}(X_m)(D - Q - k \cdot d)) = H^l\left(f_*(\Omega^{n+1}_{\tilde{X}_m}) \otimes \mathcal{O}_{X_m}(-k \cdot d)\right) = H^l\left(\Omega^{n+1}_{\tilde{X}_m} \otimes f^*(\mathcal{O}_{X_d}(-k \cdot d))\right). \quad (4.1)$$

This is a result of the degeneration of the Leray spectral sequence:

$$H^p\left(R^q f_*(\Omega^{n+1}_{\tilde{X}_m}\otimes \mathcal{O}_{X_d}(-k\cdot d)\right)\Rightarrow H^{p+q}\left(\Omega^{n+1}_{\tilde{X}_m}\otimes f^*(\mathcal{O}_{X_m}(-k\cdot d))\right).$$

The degeneration in turn is a consequence of the Grauert–Riemenschneider vanishing theorem: $R^q f_*(\Omega^{n+1}_{\tilde{X}_m}) = 0$ for $q \geq 1$) (cf. [8]). On the other hand, $H^1\left(\Omega^{n+1}_{\tilde{X}_m}(-k\cdot d)\right) = H^n\left(\mathcal{O}_{\tilde{X}_d}(k\cdot d)\right)$ by the Serre duality. Clearly all isomorphisms are equivariant. This proves our claim.

Step 2. One has the following vanishing: $H^l(\mathrm{Adj}(X_m)(D-Q-k\cdot d)) = 0$ for $l\geq 2$ for any k and if $l=1$ for k sufficiently large.

Indeed, using (4.1) we need to show that $H^i(f^*(\mathcal{O}(k\cdot d)) = 0$ for any k and $1\leq i\leq (n-1)$ as well as for $i=n$ and k sufficiently large. To obtain the first part, note that $f_*(\mathcal{O}_{\tilde{X}_m}) = \mathcal{O}_{X_m}$ (normality of X_m) and $R^q f_*(\mathcal{O}_{\tilde{X}_m}) = 0$ for $q\leq (n-1)$ (a consequence of the fact that X_m is Cohen–Macauley). At the same time, the Leray spectral sequence $H^p(R^q f_*(\mathcal{O}_{\tilde{X}_m}))\Rightarrow H^{p+q}(\mathcal{O}_{\tilde{X}_m})$ implies that $H^i(f^*(\mathcal{O}_{\tilde{X}_m}(k\cdot d)) = H^i(\mathcal{O}_{X_d}(k\cdot d))$ for $i\leq (n-1)$. The latter cohomology groups are trivial, as follows from the cohomology sequence corresponding to $0\to\mathcal{O}_{\mathbf{CP}^{n+2}_{\mathbf{w}}}(-D+k\cdot d)\to\mathcal{O}_{\mathbf{CP}^{n+2}_{\mathbf{w}}}(k)\to\mathcal{O}_{X_m}\to 0$. The second part follows from the Kawamata–Viehweg vanishing theorem because $f^*(\mathcal{O}_{X_m}(k\cdot m))\otimes\Omega^{n+1}_{\tilde{X}_m}$ is big and nef for a large k.

Step 3. We will work with the cohomology sequence corresponding to the exact sequence of sheaves:

$$\begin{aligned}0\to \mathrm{Adj}_{X_m}(D-Q-k\cdot d-d)&\to \mathrm{Adj}_{X_m}(D-Q-k\cdot d)\\ &\to \mathrm{Adj}_{X_m}(D-Q-k\cdot d)\otimes\mathcal{O}_{\mathbf{CP}^{n+2}_{(1,m,\dots,m)}}\to 0\end{aligned}\tag{4.2}$$

where $\mathbf{CP}^{n+2}_{(1,m,\dots,m)}$ is the hyperplane given by $z_{n+2}=0$. Note that the map

$$H^0(\mathrm{Adj}_{X_m}(D-Q-k\cdot d))\to H^0\left(\mathrm{Adj}_{X_d}(D-Q-k\cdot d)\otimes\mathcal{O}_{\mathbf{CP}^{n+1}_{(1,m,\dots,m)}}\right)$$

is surjective. Indeed, if $\phi_0(z_0,\dots,z_{n+1})$ is a form such that there exist $\phi_1(z_1,\dots,z_i,\dots,z_{n+1})$ for which $\phi(z_1,\dots,z_i,\dots,z_{n+1},z_{n+2}) = \phi_0(z_1,\dots,z_i,\dots,z_{n+1})+z_{n+2}\cdot\phi_1(z_1,\dots,z_i,\dots,z_{n+1})$ belongs to the adjoint ideal of X_m in a neighborhood of a point $p\in X_m\cap\mathbf{CP}^{n+1}_{(1,m,\dots,m)}$ (i.e., it satisfies condition (1.4)), then ϕ_0 belongs to the adjoint ideal of X_m in this neighborhood of p. Geometrically, this means that the cone over a set of zeros of a section from $H^0(\mathrm{Adj}_{X_m}(D-Q-k\cdot d)\otimes\mathcal{O}_{\mathbf{CP}^{n+1}_{(1,m,\dots,m)}}$ is a set of zeroes of a global section of the adjoint ideal of X_m. This is implied by the inequality (1.5) describing the adjoint ideal since $\mathrm{mult}(\bar{f}^*\phi)\leq\mathrm{mult}(\bar{f}^*\phi_0)$, as follows from a local calculation. The fact that ϕ_0 is an adjoint ideal of X_m in the case where a singularity of the branching locus is weighted homogeneous follows immediately from the description of the adjoint ideal in the case in Example 1.9.

Hence, using Step 2 we obtain

$$\begin{aligned}0\to H^1(\mathrm{Adj}_{X_m}(D-Q-k\cdot d-d))&\to H^1(\mathrm{Adj}_{X_m}(D-Q-k\cdot d))\\ &\to H^1\left(\mathrm{Adj}_{X_m}(D-Q-k\cdot d)\otimes\mathcal{O}_{\mathbf{CP}^{n+1}_{(1,m,\dots m)}}\right)\to 0.\end{aligned}\tag{4.3}$$

Step 4. We are going to express the cohomology of the sheaf $\mathrm{Adj}_{X_m}(D-Q-k\cdot d)\otimes\mathcal{O}_{\mathbf{CP}^{n+1}_{(1,m,\dots,m)}}$ on $\mathbf{CP}^{n+1}_{(1,m,\dots,m)}$ as the cohomology of a sheaf of ideals of quasiadjunction on $\mathbf{CP}^{n+1}$. Secondly, using (4.3) we shall express $\dim H^1(\mathrm{Adj}_{X_m}(D-Q))$ via dimensions of the cohomology of the sheafs of ideals of quasiadjunction on $\mathbf{CP}^{n+1}$.

Before we proceed, let us define, for a positive integer k, the integer $\mu_{m,d,n}(k)$ as the solution of the equation

$$\nu+\mu_{m,d,n}(k)\cdot m = d\cdot m-(n+1)\cdot m-d-1-k\cdot d(=D-Q-k\cdot d),\tag{4.4}$$

where $0 \leq \nu < m$ (i.e., as the result of division of the right-hand side of (4.4) by m with remainder ν). Note that

$$d \cdot m - (n+1) \cdot m - d - 1 - k \cdot d = D - Q - k \cdot d$$

(cf. the remark before Step 1). Let us also order the constants of quasiadjunction in an increasing fashion: $\dots \kappa_i > \kappa_{i-1} \dots$.

The weighted projective space $\mathbf{CP}^{n+1}_{\{1,m,\dots,m\}}$ can be identified with $\mathbf{CP}^{n+1}$ (cf. [4, 1.3.1]). We claim that under this identification:

$$\mathrm{Adj}_{X_m}(D - Q - k \cdot d) \otimes \mathcal{O}_{\mathbf{CP}^{n+1}_{\{1,m,\dots,m\}}} = \mathcal{A}_{\kappa_{k,m}}(\mu_{m,d,n}(k)), \tag{4.5}$$

where

$$\kappa_{k,m} = \{\min \kappa \in \Theta | k < [\kappa \cdot m]\} \tag{4.6}$$

Here, as above, Θ is the set of constants of quasiadjunction of singularities of the branching locus.

Both sheaves in (4.5) are twisted sheaves of ideals quotients by which the corresponding structure sheaves have 0-dimensional support. Moreover, the twisting sheaves are invertible and correspond to each other under the aforementioned identification of $\mathbf{CP}^{n+1}_{\{1,m,\dots,m\}}$ and $\mathbf{CP}^{n+1}$ (compare the dimensions of the spaces of global sections). The following identity follows from repeated use of (4.3) and (4.5):

$$\dim H^1(\mathrm{Adj}_{X_m}(D-Q)) = \Sigma_{\mu,i} s(m,d,n,\mu_{m,d,n}(k),\kappa_i) \cdot \dim H^1(\mathcal{A}_{\kappa_i}(\mu_{m,d,n}(k))). \tag{4.7}$$

Here the summation is over all integers i and μ which are solutions of (4.4) for those k for which the dimension of the cohomology group in (4.7) is nonzero (in particular this sum is finite). The constant $s(m,d,n,\mu,\kappa_i)$ in (4.7) is the number of solutions of Eq. (4.4) (for which $0 \leq \nu < m$) which also satisfy

$$[\kappa_{i-1} \cdot m] \leq k < [\kappa_i \cdot m]. \tag{4.8}$$

Clearly it is enough to check (4.5) locally. We have $z^k \phi \in \Gamma(U, \mathrm{Adj}_{X_m}(D - Q - k \cdot d))$ (U an open set in $\mathbf{CP}^{n+2}_{\mathbf{w}}$, $\phi \in \Gamma(U, \mathbf{CP}^{n+1}_{\{(1,m,\dots,m)\}})$) if and only if $k \geq [\kappa_\phi \cdot m]$. This takes place iff $[\kappa_{k.m} \cdot m] > [\kappa_\phi \cdot m]$, where $\kappa_{k.m}$ is defined in (4.6) and therefore $\phi \in \mathcal{A}_{\kappa_{k.m}}$.

Moreover, the sections in $H^0\left(\mathcal{O}_{\mathbf{CP}^{n+2}_{\mathbf{w}}}(D - Q - k \cdot d)\right)$ cut on the zero locus of $z_{n+2} \in H^0(\mathcal{O}_{\mathbf{CP}^{n+2}_{\mathbf{w}}}(d))$ the hypersurfaces $F_{\nu,\mu,\dots,\mu}(z_0, z_1, \dots, z_{n+1})$ for which the defining equation has degree ν in z_0 and μ in $z_1, \dots, z_{n+1}$, where μ and ν satisfy the relation (4.4). The degree of the projective closure in the standard $\mathbf{CP}^{n+1}$ of the portion of such a hypersurface $F_{\nu,\mu,\dots,\mu}$ in $\mathbf{C}^{n+1} \subset \mathbf{CP}^{n+1}_{\mathbf{w}}$ given by $z_0 \neq 0$ is the maximal μ satisfying (4.4). The maximal μ occurs if $0 \leq \nu < m$. The local equations of these hypersurfaces are such that $z^k \cdot \phi \in \mathrm{Adj}(z_{n+2}^m = f(z_1, \dots, z_{n+1}))$, i.e., they belong to the ideal $\mathcal{A}_{\kappa_i}$, where κ_i defined by the inequality (4.6). Therefore, repeated use of the sequence (4.3) expresses $\dim H^1(\mathrm{Adj}_{X_m}(D-Q))$ as the sum of terms of the form $\dim H^1(\mathcal{A}_\kappa(\mu))$, where μ and κ are determined from (4.4) and (4.6). Hence our claim follows.

Step 5. We have the following: if m is sufficiently large, then $s(m,d,n,\mu(k),\kappa)$ can be arbitrarily large unless $\mu = d - (n+2) - \kappa_i \cdot d$. In this case, $s(m,d,n,\mu,\kappa)$ is zero unless $\kappa \cdot m$ is an integer, and then $s(m,d,n,\mu,\kappa) = 1$.

Indeed, the solutions to the inequality $0 \leq \nu < m$ pictured in the (m,k)-plane represent the angle with slopes of the sides equal to $1 - \dfrac{n+2+\mu}{d}$ and $1 - \dfrac{n+1+\mu}{d}$. The solutions of the inequality (4.8) are solutions of the inequality $\kappa_{i-1} \cdot m \leq k \leq \kappa_i \cdot m - 1$. The latter are represented by the angle with slopes κ_{i-1} and κ_i. If m is sufficiently large, then the number of points in both angles for fixed m is either 0, infinitely large, or equal to 1 (if two angles have a common side). The last possibility takes place if and only if $\dfrac{n+2+\mu}{d} = \kappa$, i.e., $\mu = d - (n+2) - \kappa \cdot d$. The existence of the solution implies that $\kappa \cdot m$ is an integer.

Step 6. Since, according to Proposition 3.1, the sequence of integers $h^n(\tilde{X}_1), \dots, h^n(\tilde{X}_m), \dots$ is bounded, we obtain that the only nonzero terms in (4.7) are those for which $\mu = d - (n+2) - \kappa \cdot d$ provided $\kappa_i \cdot m$ are integers for all i. In this case we obtain the following expression for the Hodge number in question:

$$h^{n,0}(\tilde{X}_m) = \sum_{\kappa} \dim H^1(\mathcal{A}_\kappa(d-(n+2)-\kappa\cdot d) \tag{4.9}$$

provided m is sufficiently large.

Moreover, we see that $\dim H^1(\mathcal{A}_{\kappa_i}(\mu)) = 0$ if $s(m,d,n,\mu,\kappa_i)$ can be arbitrarily large. Therefore we have:

Corollary 4.2. *If F is a hypersurface with isolated singularities in $\mathbf{CP}^{n+1}$, then for $k < \kappa d$ one has $H^1(\mathcal{A}_\kappa(d-n-2-k)) = 0$.*

Step 7. Now in order to identify the summands in (4.9) as the dimensions of the eigenspaces of the automorphism induced by the deck transformation, let us calculate the Hodge number $h^n(\tilde{X}_d)$ by applying the sequence (4.3) for $m = d$ in the standard projective space, using the fact that $X_d \subset \mathbf{CP}^{n+2}$ is a hypersurface with isolated singularities corresponding to singularities of the branching locus V. Note that any element in $\Gamma(U, \mathrm{Adj}_{X_d}(d-(n+2)-k))$ can be represented as a combination of $z_{n+2}^l \cdot \phi_l(\ldots, z_i, \ldots)$ $(i \neq n+2)$ with $k \leq l < d$ using the defining equation of X_d to lower degree in z_{n+2}. In particular, this implies that the eigenvalues of the action on $H^n(\tilde{X}_d)$ induced by multiplication of z_{n+2} by $\exp(2\cdot\pi\cdot i/d)$ have the form $\exp(2\cdot\pi\cdot i\cdot s/d)$ with $d > s \geq k$. Therefore, if d_l is the dimension of the eigenspace corresponding to the eigenvalue $\exp(2\cdot\pi\cdot i\cdot l/d)$, then we have the inequality

$$d_l \geq \sum_{\kappa \geq l/d} \dim H^1(\mathcal{A}_\kappa(d-(n+2)-\kappa\cdot d). \tag{4.10}$$

However, the sum of the terms on the left in (4.10) is equal to the sum of the terms on the right. Hence, $d_l = 0$ for $l/d \neq \kappa$ for $\kappa \in \Theta$ and $d_\zeta = \sum_{\kappa} \dim H^1(\mathbf{CP}^{n+1}, \mathcal{A}_\kappa(d-(n+2)-\kappa\cdot d)$, where the summation is over all κ for which $\zeta = \exp(2\pi i\kappa)$. The eigenspaces in $H^n(\tilde{X}_m)_\zeta$ are independent of m (cf. Sec. 3), which concludes the proof of the theorem.

Theorem 4.3. *The module $L_{n,0}$ is isomorphic to*

$$\bigoplus (t-\zeta)^{s_\zeta},$$

where s_ζ is defined in Theorem 4.1.

Proof. This follows immediately from the theorem above and Proposition 3.1.

Corollary 4.4. *Let V be a hypersurface of degree d in $\mathbf{CP}^{n+1}$ and κ be the smallest among the constants of quasiadjunction of all singularities corresponding to 1. Then the rank of $L_{n,0}$ (and hence of $\pi_n(\mathbf{CP}^{n+1} - V \cup H)$) is not smaller than the difference between the actual and expected dimensions of hypersurfaces of degree $d-n-2-\kappa\cdot d$ passing through the singularities of V which have κ as a constant quasiadjunction corresponding to 1.*

Proof. This follows from the theorem above and Example 1.9.

Example 4.5. Let f_s denote a homogeneous form of degree s of $n+2$ variables. Let $q_1, \ldots, q_{n+1}$ be integers and $p_i = \dfrac{\prod_{j=1}^{n+1} q_j}{q_j}$. The hypersurface

$$f_{p_1}^{q_1} + \cdots + f_{p_{n+1}}^{q_{n+1}} = 0 \tag{4.11}$$

for generic forms f_{q_i} is a hypersurface of degree $Q = \prod q_j$ with isolated singularities which is a collection of points in $\mathbf{CP}^{n+1}$ forming a complete intersection of hypersurfaces $f_{p_1} = \cdots = f_{p_{n+1}} = 0$. Each singularity is locally isomorphic to a weighted homogeneous singularity of the type $x_1^{q_1} + \cdots + x_{n+1}^{q_{n+1}} = 0$, and the constant of quasiadjunction corresponding to 1 is $\left(1-\sum_{i=1}^{n+1}\frac{1}{q_i}\right)$ (cf. Example 1.9). According to the generalized Cayley–Bacharach theorem ([23, p. 120]), the difference between the actual and expected dimensions of hypersurfaces of degree $Q-n-2-\left(1-\sum\frac{1}{q_i}\right)\cdot Q$ having as a base locus a set of points which form a complete intersection of hypersurfaces $f_{p_1}=0,\dots,f_{p_{n+1}}=0$ equals 1. Hence $\operatorname{rk} L_{n,0}>1$ for this hypersurface. In fact, in [16] it was shown that $\pi_n\otimes\mathbf{Q}$ of the complement to this hypersurface is isomorphic to the middle-dimensional homology of the Milnor fibre of a singular point of the hypersurface $x_1^{q_1}+\dots+x_{n+1}^{q_{n+1}}=0$. In particular, it is a cyclic $\mathbf{Q}[t,t^{-1}]$-module and has order equal to $\prod_{1<i<n+1}\left(t-\prod_{1<j_i<q_i-1}\omega_{q_i}^{j_i}\right)$ (cf. [20]).

Example 4.6. The family of hypersurfaces (4.11) can be extended to the family

$$f_{p_1\cdot l}^{q_1}+\dots+f_{p_{n+1}\cdot l}^{q_{n+1}}=0,$$

where l is a positive integer. A similar use of the Cayley–Bacharach theorem as in the previous example (the singularities again positioned at the complete intersection of hypersurfaces $f_{p_i\cdot l}^{q_i}$) shows the nonvanishing of $\pi_n\otimes\mathbf{C}$ in this case.

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