THE TOPOLOGY OF COMPLEMENTS TO HYPERSURFACES AND NONVANISHING OF A TWISTED DERHAM COHOMOLOGY.

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Abstract. In this note we show how the cohomology of a twisted deRham complex on a complement to certain hypersurfaces in $\mathbf{C}^n$ is related to the invariant of hypersurfaces in $\mathbf{C}^{n+1}$ which we studied in [L5],[L6]. This invariant is the homotopy group of the complement having a dimension depending on the dimension of singular locus of the hypersurface. The first part contains the necessary definitions, discussion and examples from these papers.

1.Introduction

Let $P(z_1, ..., z_{n+1})$ be a polynomial and the hypersurface $D \subset \mathbf{C}^{n+1}$ be its set of zeros. Let $\mathcal{O}(*D)$ be the ring of rational functions holomorphic on $\mathbf{C}^{n+1} - D$ and let $\Omega^i(*D)$ be the $\mathcal{O}(*D)$-module of rational i-forms holomorphic in $\mathbf{C}^{n+1} - D$. The corresponding twisted deRham complex is the complex $(\Omega^*(*D), \nabla_\kappa)$ $(\kappa \in \mathbf{R})$ with the differential given by:

$$\nabla_\kappa(\omega) = d\omega + \kappa \frac{dP}{P} \wedge \omega \tag{1.1}$$

$(\Omega^*(*D), \nabla_\kappa)$ $(\kappa \in \mathbf{R})$ was an object of intense scrutiny (as well as in a more general setting: in analytic spaces and with coefficients in bundles of an arbitrary rank, cf. [D1],[K],[KN] selecting only references relevant to what follows). For example in [KN] the conditions when the cohomologies of this complex vanish were worked out.

Here we shall see that in the case when $P(z_1, ..., z_{n+1})$ is irreducible and $n \geq 2$, certain cohomology group of the twisted deRham complex can be expressed in terms of the homotopy group of $\pi_n(\mathbf{C}^{n+1} - D)$. The latter in turn depends on the local type of singularities of D and their *position* in the sense which will be described below. In the remaining case $n = 1$, the same arguments reestablish the relationship between the cohomology of the twisted deRham complex and the Alexander module of D which was described already in [K]. The properties of the Alexander polynomials of plane curves (cf. [L1],[L2],[L3],[Di2]) are closely related to the classical work of Zariski (cf. [Z]). Actually in the case when D has non-isolated singularities a simple argument, using Lefschetz's hyperplane section theorem (cf. [GM]), shows that one still can express, in a certain sense the first, non-trivial deRham cohomology group in terms of appropriate homotopy group. This homotopy group can be viewed as a high dimensional generalization of the Alexander polynomial of plane algebraic curves (cf. [L1],[L2],[D2]).

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2. Homotopy groups of the complements

Let us look at the homotopy structure of the complement to hypersurfaces in $\mathbf{C}^{n+1}$. There are two general points one should make right away:

1.The complement is an n-dimensional affine algebraic manifold and hence has the homotopy type of CW-complex of (real) dimension n (cf. [Mi1]).

2. The homology is given as follows:

$$H_i(\mathbf{C}^{n+1}-D,\mathbf{Z}) = \begin{cases} \mathbf{Z} & (i=1) \\ 0 & (i \ne 0,1,n,n+1) \\ H^{n+1}(\bar{D},\bar{D}\cap H,\mathbf{Z}) & (i=n) \\ \mathbf{Z}^{N(P)} & i=n+1 \end{cases} \tag{2.1}$$

where $\bar{D}$ is the projective closure of D in $\mathbf{C}P^{n+1}$, A relation between the integer $N(P)$ and $rkH^{n+1}(\bar{D},\bar{D}\cap H)$ can be worked out easily by finding the Euler characteristic of the hypersurface. The latter in turn can be found using expressions for the Euler characteristic of non-singular hypersurfaces (cf. [H]) and the Milnor numbers of the singularities using additivity of the Euler characteristic. The proof of (2.1) can be obtained (cf. [L5]) using exact homology sequences, excision and Poincare duality, combined with Lefschetz theorem on hyperplane sections (cf. [GM]).

One sees from (2.1) that further simplification of the homology of the complement with rational coefficients (i.e. $H_n(\mathbf{C}^{n+1}-D,\mathbf{Q})$) occurs if the following condition takes place:

$$H^{n+1}(\bar{D},\bar{D}\cap H,\mathbf{Q}) = 0 \tag{*}$$

The condition $(*)$ is satisfied in a special case when both $\bar{D}$ and $\bar{D}\cap H$ are $\mathbf{Q}$-manifolds i.e. satisfy the Poincare duality with rational coefficients. This in turn will take place if links of all singularities of D are $\mathbf{Q}$-manifolds. The latter property of a singularity occurs if and only if the characteristic polynomial of the monodromy of a singularity does not vanish at 1. For example, the singularity (2.8) below has as its link a $\mathbf{Q}$-sphere if the exponents p_i are relatively prime to each other.

A consequence from (2.1) is that, at least when (*) is satisfied, the homologies are insensitive to any data about singularities besides their local type.

On the other hand, in order to describe the homotopy type of $\mathbf{C}^{n+1}-D$, as we shall see below, one does need additional global geometric information about singularities of D .

Let us first consider two basic examples.

1. Let $P(z_1,..,z_{n+1})$ be such that the corresponding hypersurface is non-singular and transversal to the hyperplane at infinity. Then the complement has the homotopy type of the wedge of spheres:

$$\mathbf{C}^{n+1}-D = S^1 \vee S^{n+1} \vee ... \vee S^{n+1} \tag{2.2}$$

In particular the universal cover is a CW-complex homotopy equivalent to an infinite wedge of spheres S^{n+1} and the hopmotopy groups $\pi_i(\mathbf{C}^{n+1}-D)$ are zeros for $2 \le i < n$.

2. Let $P(z_1,..,z_{n+1})$ be a weighted homogeneous polynomial which has an isolated singularity at the origin. Then $(z_1,..,z_{n+1}) \to P(z_1,..,z_{n+1})$ is a locally trivial fibration

$\mathbf{C}^{n+1} - D \to \mathbf{C}^*$ which has an affine hypersurface homotopy equivalent to the Milnor fibre of the singularity of P as the fibre. The latter has the homotopy type of a wedge of S^n (whose number is equal to the Milnor number of the singularity). Hence the universal cover of $\mathbf{C}^{n+1} - D$ is homotopy equivalent to a finite wedge of S^n's. Note in passing that the last example can be used to get geometric picture of decomposition (2.2). Let V_t^2 be the hypersurface given by $z_1^2 + .. + z_{n+1}^2 = t$ and let us compare the complements to V_0^2 and V_t^2. The Milnor number of the singularity V_0^2 is equal to 1 and the complement to V_0 is $S^1 \times S^n$. On the other hand while one degenerates V_t^2 into V_0^2 the vanishing cycles on V_t^2 (i.e. S^n) get collapsed into a point (singularity of V_0^2) as does a disk in $\mathbf{C}^{n+1}$ which bounds this vanishing cycle (i.e. a relative vanishing cycle). Hence the complement to V_t^2 is the complement to V_0^2 i.e. $S^1 \times S^n$ with attached disk D^{n+1} along a fibre of the projection: $S^1 \times S^n \to S^1$. This complex is clearly equivalent to $S^1 \vee S^{n+1}$.

One of the consequences of calculations of example 1 is the following:

Proposition 2.1 (cf. [L5] lemma 1.5). Let us assume that

1. D is an irreducible hypersurface with the dimension of singular locus equal to s.
2. The hyperplane at infinity $z_0 = 0$ is transversal to all strata of a stratification Δ of D in which all strata of Δ are non singular.

Then

$$\pi_i(\mathbf{C}^{n+1} - D) = \begin{cases} \mathbf{Z} & (i = 1) \\ 0 & (1 < i < s - 1) \end{cases} \tag{2.3}$$

This follows from the Lefschetz theorem on hyperplane sections (cf. [GM]) applied to a section of D by a generic subspace of codimension s. The Lefshetz theorem also reduces the calculation of the homotopy group $\pi_{n-s}(\mathbf{C}^{n+1} - D)$ to the case when D has only isolated singularities. In any case in the situation described in the lemma we call $\pi_{n-s}(\mathbf{C}^{n+1} - D)$ the first non-trivial homotopy group of the complement (despite the fact that it can be trivial as in the case of nonsingular it is 0 according to example 1 and despite that $\pi_1(\mathbf{C}^{n+1} - D) = \mathbf{Z}$).

Another interpretation of this group comes from the isomorphism:

$$H_n(\widetilde{\mathbf{C}^{n+1} - D}, \mathbf{Z}) = \pi_n(\mathbf{C}^{n+1} - D) \tag{2.4}$$

where $\widetilde{\mathbf{C}^{n+1} - D}$ is the universal cover of $\mathbf{C}^{n+1} - D$. The isomorphism (2.4) is a consequence of (2.3) and Hurewicz's theorem ([W]).

The group $\pi_n(\mathbf{C}^{n+1} - D)$ (if the projective closure of D is transversal to the hyperplane at infinity) also is isomorphic to $H_n(M_{\bar{P}}, \mathbf{Z})$ where $M_{\bar{P}}$ is the Milnor fibre of "homogenization" $\bar{P}$ of P (cf. [L5], Cor. 4.9). The polynomial $\bar{P}$ has a non-isolated singularity at the origin and the dependence on the position of singularities discussed here is the one mentioned in [St1] p.164.

It will be convenient to use $\mathbf{Q}$ as the field of coefficients i.e. concentrate on $\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{Q}$. The homology group $H_n(\widetilde{\mathbf{C}^{n+1} - D}, \mathbf{Q})$ has the structure of a module over the group ring of the group of deck transformations of the universal cover i.e. over the group ring of $\pi_1(\mathbf{C}^{n+1} - D) = \mathbf{Z}$. This group ring is just the ring of finite Laurent polynomials: $\Lambda = \mathbf{Q}[t, t^{-1}]$. This structure of Λ-module on $\pi_n(\mathbf{C}^{n+1} - D)$ can also be described as the one which comes from the Whitehead product: $\pi_i \times \pi_j \to \pi_{i+j-1}$.

The interpretation (2.4) of the homotopy groups shows that it is a high-dimensional analog of the Alexander module, which also can be defined as the homology of an infinite cyclic cover of the complement to a plane curve. The relation of the latter and the fundamental group of the complement to a curve is the following:

$$H_1(\widetilde{\mathbf{C}^2 - D}, \mathbf{Z}) = \pi_1'/\pi_1'' \tag{2.5}$$

where $'$ and $''$ denotes the first and the second commutator of a group.

(2.4) also suggests that the vanishing of $\pi_n(\mathbf{C}^{n+1} - D)$ is a sort of an analog of commutativity of the fundamental group of the complement to a plane curve or at least the vanishing of its Alexander polynomial. There is a number of results in the theory of plane singular curves which assure commutativity of the fundamental group (cf. [D2],[F],[N]). For example, if a curve has nodes as the only singularities (i.e. near each singular point the curve looks like as a transversal intersection of two smooth transversal branches) then the fundamental group of the complement is commutative. A node of course is a singularity of generic projections of curves into planes. In the case of surfaces, generic projections in $\mathbf{CP}^3$ have non-isolated singularities along a double curve near all but finitely many points of which the image of the generic projection looks like an intersection of two transversal hypersurfaces. These finitely many exceptions are either triple points or pinch points. The above mentioned theorem on commutativity implies that if $\pi : V \to \mathbf{CP}^3$ is a generic projection then $\pi_1(\mathbf{C}P^3 - V) = \mathbf{Z}/degV \cdot \mathbf{Z}$ (the latter is the fundamental group of the complement to any irreducible nodal curve of degree $degV$). The next group, though one could expect it to be non trivial, actually is always trivial at least if one disregards the torsion.

Theorem 2.2 (cf. [L7]) Let $\pi : V \to \mathbf{C}P^3$ be a generic projection, $D = \pi(V)$ and $\mathbf{C}^{n+1} \subset \mathbf{C}P^3$ be the complement to a generic plane . Then

$$\pi_2(\mathbf{C}^{n+1} - D \cap \mathbf{C}^{n+1}) \otimes \mathbf{Q} = 0 \tag{2.6}$$

Another vanishing theorem can be obtained by extending to higher dimensions the ideas in the proof of Nori's theorem (cf. [N]) which gives the strongest condition for the commutativity of the fundamental groups of the complements to divisors on algebraic surfaces.

Theorem 2.3 (cf. [L5]) Let D be a hypersurface in $\mathbf{C}^{n+1} \subset \mathbf{C}P^{n+1}$ with isolated singularities and transversal to the hyperplane at infinity. Let us assume the condition (*) . Suppose that $\phi : W \to \mathbf{C}P^{n+1}$ is an embedded resolution of the singularities of $D \subset \mathbf{C}^{n+1}$ i.e. a birational morphism such that the union of the strict preimage D' of D and the exceptional set of ϕ form a divisor in W with normal crossings. Assume that D' is ample on W. Then $\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{Q} = 0$.

It would be interesting to find some explicit conditions which will assure ampleness of the strict transform which is sufficient to force vanishing of the homotopy group in question.

The study of the structure of the first non-trivial homotopy group is based on the following observation: if all singularities of D are isolated then $\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{Q}$ is a Λ-torsion module (cf. [L5]). In particular

$$H_n(\widetilde{\mathbf{C}^{n+1} - D}, \mathbf{Q}) = \oplus \Lambda/(\lambda_i(t)) \tag{2.7}$$

where $\lambda_i(t)$ are Laurent polynomials defined up to a unit of Λ. Let $\Delta_D(t) = \Pi \lambda_i$ be the Λ-order of $H_n(\widetilde{\mathbf{C}^{n+1} - D}, \mathbf{Q})$. The dependence of $\Delta_D(t)$ on the local type of singularities is given by the following:

Divisibility Theorem (2.4) Let $Sing(D)$ be the collection of singular points of the hypersurface D. For any $p \in Sing(D)$ let $\Delta_{D,p}$ be the characteristic polynomial of the monodromy of Milnor fibration of this singularity ([Mi2]). Then

1. $\Delta_D(t)$ divides the product $\prod_{p \in Sing(D)} \Delta_{D,p}$.

2.If a projective closure of D transversal to the hyperplane at infinity then all roots of Δ_D are the roots of unity of degree $d = degD$.

A typical consequence of this theorem is a vanishing of $\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{Q}$ if none of the roots of $\Delta_{D,p}$ is a root of unity of degree $d = deg(D)$. For example this is the case when all singularities of D locally look like

$$x_1^{p_1} + ... x_{n+1}^{p_{n+1}} = 0 \tag{2.8}$$

(i.e. near p a coordinate system can be selected so that D is given by this equation) and $g.c.d.(p_i, d) = 1$ for any i (one can easily formulate even stronger conditions which follow from theorem 2.4). Indeed the characteristic polynomial of singularity (2.8) is $\Pi(t - \xi_{j_1} \cdots \xi_{j_{n+1}})$ where $\xi_{j_i} (i = 1, ..., n+1)$ runs through all roots of unity of degree p_i different from 1.

Now let us consider the coverings associated with D. Since $\pi_1(\mathbf{C}^{n+1} - D) = \mathbf{Z}$ by (2.3) for any positive integer k here is a unique unbranched covering of $\mathbf{C}^{n+1} - D$ with Galois group $\mathbf{Z}/k\mathbf{Z}$. Let us denote it $(\widetilde{\mathbf{C}^{n+1} - D})_k$. Its homology can be calculated as follows:

Lemma 2.5 (cf. [L6]). The dimension of the $\mathbf{Q}$-space $H_n((\widetilde{\mathbf{C}^{n+1} - D})_k, \mathbf{Q})$ is equal to the sum over i of the numbers of common root of $t^k - 1$ and the polynomials $\lambda_i(t)$ from the cyclic decomposition (2.7).

3. Mixed Hodge Structures on the Homotopy groups

Projective models of the cyclic covers $(\widetilde{\mathbf{C}^{n+1} - D})_k$ can be considered as coverings of $\mathbf{CP}^{n+1}$ with the branching locus consisting of the projective closure of D and possibly the hyperplane at infinity (this ramification takes place unless k is a divisor of the degree of the hypersurface D). The Hodge number $h^{n,0}(X_k)$ of a smooth model X_k of a projctive closure of $\widetilde{(\mathbf{C}^{n+1}-D)}_k$ is a birational invariant (cf.[KS]) and hence depends only on D and the degree of the covering. Other birationally invariant Hodge numbers $h^{i,0}(X_k)$ of smooth models of cyclic coverings are zeros if $0 < i < n$ and $h^{n+1,0}(X_k)$ can be calculated using only the local data about the singularities and $h^{n,0}$. It turns out that one can relate $h^{n,0}(X_k)$ to $\pi_n(\mathbf{C}^{n+1}-D)$. The relationship can be described in terms of the mixed Hodge structure on $\pi_n(\mathbf{C}^{n+1}-D)\otimes\mathbf{C}$. A mixed Hodge structure was put on homotopy groups of a quasiprojective variety in [Mo] but only in the case when the space in *nilpotent* which is rarely the case for $\mathbf{C}^{n+1}-D$. The idea of the construction of this mixed Hodge structure comes from the fact that the action of the generator t of $\pi_1(\mathbf{C}^{n+1}-D)$ satisfies $t^d=1$ and in particular $\pi_n(\mathbf{C}^{n+1}-D)$ can be identified with $H_n(\widetilde{(\mathbf{C}^{n+1}-D)}_d)$ with the action of t on the latter being induced by the deck transformation. Now the mixed Hodge structure on π_n is defined to be the one corresponding to Deligne's canonical mixed Hodge structure on the homology of an open algebraic manifold (cf. [D2]). Note that if one does not assume transversality of the projective closure of D to the hyperplane at infinity the situation gets a bit more involved (i.e. one constructs the Mixed Hodge structure only on the part of π_n cf. [L6] sect. 2).

One has the following:

Lemma 3.1 The length of the weight filtration on $\pi_n(\mathbf{C}^{n+1}-D)\otimes\mathbf{C}$ is at most 2. More precisely if D has only isolated singularities and satisfies condition (*) then: $Gr^W_k(\pi_n(\mathbf{C}^{n+1}-D))=0$ for $k\neq -n,-n-1$.

In the situation of example 2 from section 1 (the complement to the set of zeros of a weighted homogeneous polynomial) the mixed Hodge structure via identification of π_n with the homology of the Milnor fibre gets identified with the Steenbrink's mixed Hodge structure of (co)homology of the Milnor fibre (cf. [St3]). In this case the statement of the lemma is proven in [St2].

This mixed Hodge structure can be used to define:

$$L_{p,q}(D)=Gr_F^{-p}Gr^W_{-p-q} \tag{3.1}$$

$L_{p,q}$ is a $\mathbf{C}[t,t^{-1}]$-submodule of $\pi_n(\mathbf{C}^{n+1}-D)\otimes\mathbf{C}$ and hence has a cyclic decomposition:

$$L_{p,q}=\oplus\mathbf{C}[t,t^{-1}]/(\lambda_{i,(p,q)}) \tag{3.2}$$

The $\mathbf{C}[t,t^{-1}]$-orders of this module $\Delta^{p,q}=\Pi_i\lambda_{i,(p,q)}(t)$ give a factorization of the order of $\pi_n(\mathbf{C}^{n+1}-D)\otimes\mathbf{C}$ and one has $\Delta_D(t)=\Pi_{p+q=n,n+1}\Delta^{p,q}$. In the case of curves this construction of the Mixed Hodge structure leads to the Mixed Hodge structure on $\pi_1'/\pi_1''\otimes\mathbf{C}$ which one can show is pure (cf. [L2] remark 2.5). One obtains the factorization of the Alexander polynomial of a curve: $\Delta(t)=\Delta^{1,0}(t)\cdot\Delta^{0,1}(t)$.

Now one can express $h^{n,0}(X_k)$ in terms of decomposition (3.2) as follows:

Proposition 3.2 The Hodge number $h^{n,0}(X_k)$ of a smooth model of a projective model of cyclic cover $(\widetilde{\mathbf{C}^{n+1}-D})_k$ is equal to the sum of the numbers of common roots of t^k-1 and polynomials $\lambda_{i,(n,0)}$ in decomposition (3.2).

The invariant $h^{n,0}(X_k)$ provides the link between the homotopy group and the projective geometry of the set of singularities of D. Namely this Hodge number can be related to the dimensions of certain linear systems of hypersurfaces defined by the singular points.

To describe how this comes about let us start by description of the ideals defining these linear systems. To this end we shall define the sequence of ideals in the local ring of a points in $\mathbf{C}^{n+1}$ associated with a germ of a function $f(z_1,..,z_{n+1})$ which has an isolated singularity at this point (which we assume is the origin). The construction is based on the theory of adjoints (cf. [BL],[MT]). For a germ G of an isolated hypersurface singularity $g(z_1,..,z_N)=0$ at the origin one can define the adjunction ideal as the stalk at the origin of the sheaf $\pi_*(\Omega_{\tilde{G}})$. Here $\pi:\tilde{G}\rightarrow G$ is a resolution of the singularity of G. This is indeed an ideal in the local ring of the origin and is independent of the chosen resolution. Let us denote it $\mathcal{A}_g$. Now the singularity of f defines the sequence of singularities $g_m(z_1,....z_{n+2})=z_{n+2}^m+f(z_1,..,z_{n+1})$ (cyclic branched covers of $\mathbf{C}^{n+1}$ branched over $f=0$). We have the following:

Lemma 3.3. Let $\phi(z_1,..,z_{n+1})$ be a germ of a holomorphic function at the origin. Then there exist a rational number κ_ϕ such that:

$$[\kappa_m\cdot m]=min\{l|z_{n+2}^l\cdot\phi\in\mathcal{A}_{g_m}\} \tag{3.3}$$

The collection of rational numbers κ_ϕ (we call them the constants of quasiadjunction) is a finite set Θ_f. It can be explicitly determined from the collection of multiplicities of exceptional divisors in a resolution of the germ G (cf.[L6]). The number of constants of quasiadjunction $card\Theta_f$ does not exceed the Milnor number of the singularity g.

If $g(z_1,..,z_{n+1})$ is a weighted homogeneous the description of the adjoint ideals in [MT] allows one to determine the constants of quasiadjunction explicitly. Indeed each monomial $z_1^{i_1}\cdots z_{n+2}^{i_{n+2}}$ defines the point $(i_1,...,i_{n+2})$ in $\mathbf{R}^{n+2}$. A monomial $z_1^{i_1}\cdots z_{n+2}^{i_{n+2}}$ belongs to the adjoint ideal of a weighted homogeneous polynomial if and only if $(i_1+1,...,i_{n+2}+1)$ is strictly above the hyperplane defined by the monomials sum of which is g (cf. [MT]). This description of the adjoint ideal makes the lemma 3.3 obvious in this case. One can readily see that the constant of quasiadjunction of the singularity $z_1^{q_1}+....z_{n+1}^{q_{n+1}}=0$ corresponding to monomial $z_1^{i_1}\cdots z_{n+2}^{i_{n+2}}$ is equal to $max(1-\Sigma_k\frac{i_{k+1}}{q_k},0)$.

The constants of quasiadjunction of $f(z_1,..,z_{n+1})$ now define the filtration of the local ring $\mathcal{O}_{0,\mathbf{C}^{n+1}}$ (*the ideals of quasiadjunction*). We put $\mathcal{A}_{f,\kappa,0}=\{\phi\in\mathcal{O}_{0,\mathbf{C}^{n+1}}|\kappa=min\lambda\in\Theta_f,\kappa_\phi<\lambda\in\Theta\}$. Particularly noteworthy is the ideal of the constant of quasiadjunction corresponding to the monomial 1. In this case the ideal corresponding to this constant quasiadjunction is the maximal ideal i.e. a germ ϕ belongs to this ideal of quasiadjunction iff the hypersurface given by $\phi=0$ contains the singular point of g i.e. the origin. In general the condition on ϕ of belonging to an ideal of quasiadjunction is an interesting geometric condition on the germ of hypersurface $\phi=0$.

Using the ideals of quasiadjunction of singular points one can attach to the hypersurface D the ideals sheafs on $\mathbf{CP}^{n+1}$ (which by an abuse of notation will denoted $\mathcal{A}_\kappa$) consisting of germs of holomorphic functions on $\mathbf{CP}^{n+1}$ which belong to the ideal of quasiadjunction corresponding to the constant quasiadjunction κ at each point of $\mathbf{CP}^{n+1}$ which is a singular point of the hypersurface D. With all this set we have the following:

Theorem 3.4 Let $\tilde{X}_m$ be a $\mathbf{Z}_m$-equivariant resolution of a smooth model of a projective closure of $(\mathbf{C}^{n+1} - D)_m$. Let ζ be a root of unity of degree m and d is the degree of D. Let $H^{n,0}(\tilde{X}_m)_\zeta$ be the eigenspace of the linear map induced by the deck transformation acting on the space of holomorphic n-forms on $\tilde{X}_m$ corresponding to the eigenvalue ζ. Then

$$dimH^{n,0}(\tilde{X}_m)_\zeta = \Sigma_\kappa dimH^1(\mathbf{C}P^{n+1}, \mathcal{A}_\kappa(d-n-2-\kappa\cdot d)) \qquad (3.4)$$

where the summation is over all constants of quasiadjunction of all singularities such that $exp(2\pi i\kappa) = \zeta$ and $\kappa \cdot m \in \mathbf{Z}$.

This theorem in fact gives a complete calculation of the module $L_{n,0}$ which is part of the $\mathbf{C}[t,t^{-1}]$ module $\pi_n(\mathbf{C}^{n+1} - D)$. To make this theorem useful for the study of the latter one needs to calculate $dimH^1(\mathbf{C}P^{n+1}, \mathcal{A}_\kappa(d-n-2-\kappa\cdot d))$. This of course is contingent on exact information on the local ideals and information on the geometry of positions of singularities of D. However in the case of $\mathcal{A}_{\kappa_1}$ where κ_1 is the constant of quasiadjunction of the monomial 1 this sheaf is just the ideal sheaf of the (reduced) subscheme $Sing(D) \subset \mathbf{C}P^{n+1}$. The following classical result is useful in some cases:

Theorem 3.5 (Cayley -Bacharach. cf. [Seg] p.120) Let $V_1, .., V_{n+1}$ be a generic hypersurfaces of degrees $e_1, .., e_{n+1}$ and let S be the set of $(e_1 \cdots e_{n+1})$ points which form a complete intersection these hypersurfaces. Then $H^1(\mathbf{C}P^{n+1}, \mathcal{I}_S(\Sigma_{i=1}^{i=n+1} e_i - n - 2)) = \mathbf{C}$.

The dimension of the cohomology group in the statement of theorem 3.5 classically is called the *superabundance* of the linear system of hypersurfaces of degree $\Sigma e_i - n - 2$ passing through the points of the set S (i.e. the difference between the actual and "expected" dimensions of this linear system).

Example 3.6 Let $f_d(z_0, .., z_{n+1})$ be a generic form of degree d, $d_i \in \mathbf{Z}$ $(i = 1, ..., n+1)$ such that $\Sigma \frac{1}{d_i} < 1$, $D_0 = \Pi_{j=1}^{j=n+1} d_i$, and $D_i = \frac{D_0}{d_i}$. Let

$$P = f_{D_i}^{d_1} + ... f_{D_{n+1}}^{d_{n+1}} \qquad (3.5)$$

Then the singularities of $P(z_0, .., z_{n+1}) = 0$ are the points satisfying $f_{D_1} = ... f_{D_{n+1}} = 0$. If we let $\zeta = exp(-2\pi i(\Sigma \frac{1}{d_i})$ then according to theorem 3.4 we can calculate $dimH^{n,0}(X_{D_0})_\zeta$ as $dimH^1(\mathbf{C}P^{n+1}, \mathcal{I}(D_0 - (1 - \Sigma \frac{1}{d_i})D_0 - n - 2)$. By the Cayley-Bacharach theorem above the latter is equal to 1. In particular $\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{C} \neq 0$. Using different methods one can actually calculate $\pi_n(\mathbf{C}^{n=1} - D) \otimes \mathbf{Q}$ completely. It turns out that this module is isomorphic to the Milnor fibre of a local singularity of P with the module structure given by the action of the monodromy operator. In particular the order is equal to $\Pi(t-\xi_{j_1}\cdots\xi_{j_{n+1}})$ where $\xi_{j_i}(i=1,..,n+1)$ runs through all primitive roots of unity of degree d_i (cf. [L5]).

Example 3.7 Similar use of the Cayley Bacharach theorem shows nonvanishing of $\pi_n(\mathbf{C}^{n+1}-D)$ where D is the affine portion of the hypersurface given by:

$$P_l = f_{D_i\cdot l}^{d_1} + ...f_{D_{n+1}\cdot l}^{d_{n+1}} \tag{3.6}$$

with l a positive integer and other notation as in example 3.6 above. The family of hypersurfaces (3.6) is a high dimensional counterpart of the family of curves studied in [T].

4. Twisted DeRham complex

Now we can show how the first non-vanishing homotopy group can be used to calculate the homology of the complement $\mathbf{C}^{n+1}-D$ with twisted coefficients. This can be done using the following:

Lemma 4.1 Let A be a cyclic group, t one of its generators and Λ the group ring of A over $\mathbf{Q}$. Let X be a CW-complex for which there exists an integer n such that if $n>1$ then the cyclic cover $\tilde{X}_\phi$ corresponding to a homomorphism $\phi:\pi_1(X)\to A$ is acyclic below dimension n (i.e. $H_i(\tilde{X}_\phi,\mathbf{Q})=0$). Let χ be a character of A and $\mathcal{L}_\chi$ be the local system on X corresponding to homomorphism $\chi\circ\phi$. Let $H_n(\tilde{X})=\Lambda^k\oplus\Lambda/(\Delta_i)$ be a cyclic decomposition of $H_n(\tilde{X})$ viewed as a Λ-module. Let $l=\#\{i|\Delta(\chi)=0\}$. Then $rkH_n(X,\mathcal{L}_\chi)=k+l$.

Proof. Let $C_*(\tilde{X},\mathbf{Q})=C_*(\tilde{X},\mathbf{Q})\otimes\mathbf{Q}[\pi_1(X)]$ and $C_*(X,\mathcal{L}_\chi)$ be the chain complexes of $\tilde{X}$ and of the local system $\mathcal{L}_\chi$ respectively. Recall that a chain complex of a local system $\mathcal{L}$ with the fibre V and corresponding to a homomorphsm $\psi:\pi_1(X)\to Aut(V)$ is defined as:

$$\begin{gathered} C_i(X,\mathcal{L}) = C_i(\tilde{X})\otimes_{\mathbf{Q}[\pi_1(X)]} V = \\ = C_i(X)\otimes_{\mathbf{Q}}\mathbf{Q}[\pi_1(X)]\otimes_{\mathbf{Q}} V/\{g\otimes c-\psi(g)(c)|c\in\mathbf{C}[\pi_1(X)]\otimes V\} \end{gathered} \tag{4.1}$$

Here V is given the structure of $\pi_1(X)$ module using the homomorphism ψ. This implies that we have the following exact sequence of chain complexes:

$$0\to C_*(\tilde{X})\to C_*(\tilde{X})\to C_*(X,\mathcal{L}_\chi)\to 0 \tag{4.2}$$

in which the first homomorphism is the multiplication by $t-\chi(t)\in\mathbf{Q}[A]$. The corresponding homology sequence yields:

$$H_n(\tilde{X},\mathbf{Q})\to H_n(\tilde{X},\mathbf{Q})\to H_n(X,\mathcal{L}_\chi)\to 0 \tag{4.3}$$

since the assumption of acyclicity and the claim follows.

Theorem 4.2 Let P be a polynomial satisfying conditions 1 and 2 of Proposition 2.1. Then the rank of $H^n((\Omega^*(*D), \nabla_\kappa))$ is equal $\#\{i|\lambda_i(exp(2\pi\sqrt{-1}\kappa)) = 0\}$.

In the case $n = 1$ this reduces to the algebraic computation of the Alexander polynomial (cf. [K]).

Proof. It is standard by now (cf. [K],[KN]) to identify the cohomology of a local system with the cohomology of the twisted deRham complex: the local system corresponding to the complex with the differential ∇_κ is defined by the homomorphism of $\pi_1(\mathbf{C}^{n+1} - D)$ into $\mathbf{C}^*$ sending the generator of the fundamental group to $exp(2\pi\sqrt{-1}\kappa)$ (cf. [D1], lemma II.5.6). Hence the theorem follows.

Combining the theorem with the results of [L2] one obtains the following:

Corollary 4.3 Let D be a hypersurface in $\mathbf{C}^{n+1}$ satisfying 1. and 2. such that all singularities of D are locally isomorphic to $x_1^{d_1} + ...x_{n+1}^{d_{n+1}}$ and such that $d = degD$. Then the rank of $H^n(\Omega^*(*D), \nabla_\kappa)$ is zero unless $\kappa \cdot d_1 \cdots d_{n+1} \in \mathbf{Z}$ In the latter case it is bigger than the superabundance of the linear system of hypersurfaces of degree $d(\Sigma\frac{1}{d_i}) - n - 2$ passing through the singularities of D.

Examples 4.4. 1. Let $P(z_0, ..., z_{n+1})$ be a polynomial such that the corresponding hypersurface is non singular and transversal to the hyperplane $z_0 = 0$.
Then $H^n(\Omega^*(*D), \nabla_\kappa) = 0$

2. Let $p(z_1, ..., z_{n+1})$ be a weighted homogeneous polynomial of degree 1 and weights $w_1, ..., w_{n+1}$ which has an isolated singularity at the origin.
Then the rank of $H^n(\Omega^*(*D), \nabla_\kappa)$ is one if $\frac{\kappa}{w_i} \in \mathbf{Z}$ for some i and zero otherwise. For example if $P(z_1, ..., z_{n+1}) = z_1^3 + z_2^2 + ...z_{n+1}^2$ then the twisted deRham cohomology vanishes unless $6\kappa \in \mathbf{Z}$. Indeed the homology of the universal cyclic cover of the complement to $p = 0$ can be identified with the Milnor fibre of the singularity of p The action on homology of the operator given by the deck transformation of the cover coincides with the action of the monodromy operator. Hence the claim here follows from the calculations in [M] ch 9.

3.Let $f_d(z_0, .., z_{n+1})$ be a generic form of degree d, $d_i \in \mathbf{Z}$ $(i = 1, ..., n+1)$, $D_i = \frac{\Pi_{j=1}^{j=n+1} d_j}{d_i}$. Let

$$P_{d_1,...,d_{n+1}} = f_{D_i}^{d_1} + ...f_{D_{n+1}}^{d_{n+1}} \tag{4.4}$$

Then the cohomology $H^s(\Omega_{\mathbf{C}P^{n+1}}(*V), \nabla_\kappa) = 0$ for $s \le n-1$ (any κ) and for $s = n$ unless $d_i \cdot \kappa \in \mathbf{Z}$ for one of d_i. In the latter case $H^s(\Omega_{\mathbf{C}P^{n+1}}(*V), \nabla_\kappa) \neq 0$. Indeed the order of the first non vanishing homology group (which has dimension n) is $\Pi(t - \xi_{j_1} \cdots \xi_{j_{n+1}})$ where ξ_{j_i} runs through all primitive roots of unity of degree d_i, $(i = 1, ..., n+1)$ (cf. [L] Ch.5). The nonvanishing also follows from the results of example 3.6 i.e the use of Caley-Bacharach theorem.

4. The same conclusion is valid for the affine portion of the hypersurface:

$$P_{d_1,..,d_{n+1},l} = f_{D_i \cdot l}^{d_1} + ... f_{D_{n+1} \cdot l}^{d_{n+1}} \tag{4.5}$$

which has degree $d_1 \cdots d_{n+1} \cdot l$. This follows from the discussion in example 3.7.

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