Mathematical Research Letters **6**, 141–149 (1999)

CHERN CLASSES AND THE PERIODS OF MIRRORS

Anatoly Libgober

Abstract. We show how Chern classes of a Calabi Yau hypersurface in a toric Fano manifold can be expressed in terms of the holomorphic at a maximal degeneracy point period of its mirror. We also consider the relation between the Chern classes and the periods of mirrors for complete intersections in Grassmanian Gr(2,5).

0. Introduction

A relation between Chern classes of 3-dimensional Calabi Yau complete intersections in a product of weighted projective spaces and periods of holomorphic 3-forms on their mirrors was found by S.Hosono, A.Klemm, S.Theisen and S-T.Yau. It is expressed by the following "remarkable identities" (cf. [HKTY]). Let X be a 3-dimensional Calabi-Yau complete intersection in a product of k weighted projective spaces. For such Calabi Yau manifolds the standard constructions can be applied to obtain a mirror for X (cf. [B1],[Bo]). Denote by $w_0(z_1,\dots,z_k) = \Sigma_{n_i\geq 0} c(n_1,\dots,n_k) z_1^{n_1}\dots z_k^{n_k}$ the period of a holomorphic 3-form on a mirror of X admitting a holomorphic extension into a point with a maximally unipotent monodromy and normalize this period so that it there takes the value 1. Then, according to [HKTY] (cf. also [HLY96]), the function $c(n_1,\dots,n_r)$ on $\mathbb{Z}^r_{\geq 0}$ is a product of $(l_j(n_1,\dots,n_r)!)^{\pm 1}$ where l_j are linear forms with integer coefficients. Define a holomorphic near the origin function $c(\rho_1,\dots,\rho_r)$ replacing in $c(n_1,\dots,n_r)$ each $(l_j)!$ by $\Gamma(l_j(\dots,\rho_i,\dots)+1)$. For $i=1,\dots,r$ let $J_i \in H^2(X,\mathbb{Z})$ be the class of the pull back on X of the Kähler form of i-th projective space which product contains X. Let, finally, K_{ijk} be the Yukawa coupling (3-point function, cf. [BvS],[HLY] for the references or see below), $\partial_{\rho_i} = \frac{1}{2\pi i}\frac{\partial}{\partial\rho_i}, D_i^{(2)} = \frac{1}{2}K_{ijk}\partial_{\rho_j}\partial_{\rho_k}, D^{(3)} = -\frac{1}{6}K_{ijk}\partial_{\rho_i}\partial_{\rho_j}\partial_{\rho_k}$. Then

$$\int_X c_2 \wedge J_i = -24 D_i^{(2)} c(\rho_1,\dots,\rho_k)|_{(0,\dots,0)}, \tag{0.1}$$

$$\int_X c_3 = i\frac{2\pi^3}{\zeta(3)} D^{(3)} c(\rho_1,\dots,\rho_k)|_{(0,\dots,0)} \tag{0.2}$$

Received August 11, 1998.
Supported in part by NSF.

(cf. (4.21) in [HKTY]).

The purpose of this note is to extend this relationship to arbitrary dimensions. We shall consider the case of hypersurfaces in toric Fano manifolds (the case of complete intersections can be treated similarly). For these Calabi Yau manifolds we show that certain linear combinations of Chern classes can be expressed in terms of the holomorphic at the maximal degeneracy point period of the mirror. These combinations form a certain Hirzebruch's multiplicative sequence, i.e., obtained by applying Hirzebruch construction [Hi] to the series $\frac{1}{\Gamma(1+z)}$. The cohomology classes of multiplicative Γ-sequence, in turn, determine the Chern classes of the manifold (cf. lemma in Sect.1) yielding that the Chern classes also can be expressed via the coefficients of the period of the mirror.

More precisely we prove the following. Let Δ be a reflexive Fano polytope (cf. [B1],[B2]) of dimension d. In other words, Δ is a simplicial polytope with the following properties: (a) its vertices belong to an integral lattice M of rank d; (b) the origin is the only intersection point of the interior of Δ and the lattice M; (c) for each $(d-1)$-dimensional face, its vertices form a basis of M; and (d) the equation of such face is $l = -1$ where l is a linear form on M. The corresponding toric variety $\mathbf{P}_\Delta$ (cf. [F] regarding toric geometry used here) is a Fano variety, i.e., its first Chern class is ample. Moreover, the hypersurface, which is the zero locus for a generic section of the line bundle corresponding to the polytope Δ on $\mathbf{P}_\Delta$ is a Calabi Yau manifold X_Δ. Let $\Delta^* \subset M^* \otimes \mathbb{R}$ ($M^* = \mathrm{Hom}(M, \mathbb{Z})$) be the polar polytope and f_{Δ^*} be a generic linear combination with complex coefficients of characters of $M \otimes \mathbb{C}^*$ belonging to Δ^*. A period of the affine hypersurface $f_{\Delta^*} = 0$ in $\mathrm{Hom}(M^*, \mathbb{C}^*)$ i.e. $\int_\gamma \frac{1}{f_{\Delta^*}} \prod \frac{dx_i}{x_i}$, where x_i's are coordinates in $\mathrm{Hom}(M^*, \mathbb{C}^*)$ corresponding to a choice of a basis in M and γ is a $d{+}1$-cycle in the complement to $f_{\Delta^*} = 0$ in $\mathrm{Hom}(M^*, \mathbb{C}^*)$, satisfies a system of Picard Fuchs PDE. In an appropriate partial compactification of $\mathrm{Hom}(M^*, \mathbb{C}^*)$, constructed in [HLY], there is a maximal degeneracy point such that this Picard-Fuchs system has only one, up to a constant factor, solution which admits a holomorphic extension into this point. Near the maximal degeneracy point this solution has the form $\Sigma_{n_1,\ldots,n_r \in \mathbb{Z}, n_i \geq 0} \prod_i l_i(n_1, \ldots, n_r)^{\pm 1} z_1^{n_1} \cdots z_r^{n_r}$ where l_i are linear forms with integer coefficients. Let $c(\rho_1, \ldots, \rho_r) = \Pi_i \Gamma(l_i(\rho_1, \ldots, \rho_r) + 1)^{\pm 1}$. We shall identify the tangent space to the compactification of the moduli space of hypersurface X_{Δ^*} at a maximal degeneracy point with $H^2(X_\Delta, \mathbb{C})$ and coordinates of $H^2(X_\Delta, \mathbb{C})$ in a basis $J_1, \ldots, J_r \in H^2(X_\Delta, \mathbb{Z})$ with the coordinates $(x_1, \ldots, x_r)$ in the partial compactification of the moduli space of hypersurfaces X_{Δ^*} near a maximal degeneracy point. Let $K_{i_1,\ldots,i_d} = \int_{X_{\Delta^*}} \Omega_{\Delta^*} \wedge \frac{\partial^d \Omega_{\Delta^*}}{\partial x_{i_1} \ldots \partial x_{i_d}}$ be the d-point function corresponding to X_{Δ^*} (cf. [GMP],[BvS]). $K_{i_1,\ldots,i_d}$ depends on a normalization of Ω_{Δ^*} and selected so that the value of d-point function at the maximal degeneracy point is equal to $\int_{X_\Delta} J_{i_1} \wedge \cdots \wedge J_{i_d}$ With such normalization it yields the data of enumerative geometry of rational curves on X_Δ (cf. [GPM]). Then we have:

Theorem. *If the assumption* $(**)$ *below is fulfilled, then the degree* k *polyno-*

mial $Q_k(c_1,\dots,c_k)$ in Chern classes in Hirzebruch's multiplicative sequence corresponding to the series $\frac{1}{\Gamma(1+z)}$ of X_Δ satisfies:

$$(0.3)\quad \int_{X_\Delta} Q_k(c_1,\dots,c_k)\wedge J_{i_1}\wedge\cdots\wedge J_{i_{d-k}} = \sum_{(j_1,\dots,j_k)\subset(1,\dots,r),1\le k\le d,} \frac{1}{k!}\frac{\partial^k c(\rho_1,\dots,\rho_r)}{\partial\rho_{j_1}\cdots\partial\rho_{j_k}}|_{\rho_1=\cdots=\rho_r=0} K_{j_1,\dots,j_k,i_1,\dots,i_{d-k}}|_{x_1=\cdots=x_r=0}$$

for all $i_1,\dots,i_{d-k},(1\le i\le r)$.

As a corollary we obtain an expression for the Chern classes of a Calabi Yau manifold as above in terms of the period and the d-point function of the mirror since (0.3) specifies $Q_k(c_1,\dots,c_k)$ by its cup products with cohomology classes in $H^{2d-2k}(X_\Delta)$.

The starting point for the above theorem was a formal resemblance between the Chern polynomial $\frac{(1+h)^5}{(1+5h)}$ of a quintic in $\mathbf{P}^4$ and the coefficients of the expansion of a period $\Sigma_n\frac{(5n)!}{(n!)^5}z^n$ of its mirror. It can be interpreted as equality of the degree k $(1\le k\le 3)$ terms of the expansion at zero of an (ill defined) "analytic continuation" $\frac{\Gamma(5n+1)}{\Gamma(n+1)^5}$ of coefficients of the period considered as a function of n and the terms of degree k in the Γ-sequence $\frac{\Gamma(5h+1)}{\Gamma(h+1)^5}$ of the quintic (h is the generator of its 2-dimensional cohomology).

The multiplicative Γ-sequence is discussed in section 1 and the proof of the theorem is given in section 2 where also its special cases are made explicit. In section 3 we discuss the simplest example of the theorem and the relation between the periods on Chern classes for manifolds which mirrors were constructed recently in [BCKS] via conifold transitions.

1. Multiplicative Γ-sequence

Let $Q(z)=\frac{1}{\Gamma(1+z)}$. Since $Q(0)=1$, there is a well defined multiplicative sequence corresponding to this power series (cf. [Hi]). The latter associates with a complex manifold M having Chern classes $c_i, i=1,\dots\dim M$ weighted homogeneous polynomials $Q_i(c_1,\dots,c_i)$ of degree $2i$ where $\deg c_i=2i$.

Lemma. *The coefficient of c_i in $Q_i(c_1,\dots,c_i)$ is equal to the value $\zeta(i)$ of Riemann zeta-function for $i\ne 1$ and to the Euler constant $\gamma=\lim_{n\to\infty}\Sigma_{m=1}^{m=n}\frac{1}{m}-\ln n$ for $i=1$. In particular it is non zero and hence c_i is a the polynomial in $Q_j(c_1,\dots,c_j),(j\le i)$.*

Proof. The coefficient of c_i in $Q_i(c_1,\dots,c_i)$ is s_i where

$$(1.1)\qquad 1-z\frac{d}{dz}logQ(z)=\Sigma_{i=0}^{\infty}(-1)^i s_i z^i$$

(cf. [H] sect. 1.4). In the case of Γ-sequence we have (cf. [E] p. 45)

$$\log \Gamma(1+z) = -\gamma z + \sum_{i=2}^{\infty} (-1)^i \zeta(i) z^i / i. \tag{1.2}$$

This yields the lemma. □

It is not hard to work out explicitly the polynomials Q_j (using for example the recurrence formulas from [LW]). For small i we have the following (γ is the Euler constant $\lim_{m\to\infty} \sum_1^m \frac{1}{n} - ln\ m$):

$$Q_1(c_1) = \gamma c_1, \tag{1.3}$$

$$Q_2(c_1, c_2) = -\frac{1}{2}\zeta(2)c_1^2 + \zeta(2)c_2 + \gamma^2 c_1^2, \tag{1.4}$$

$$Q_3(c_1, c_2, c_3) = \zeta(3)c_3 - (\zeta(3) - \zeta(2)\gamma)c_1c_2 + \left(\frac{1}{3}\zeta(3) + \frac{1}{6}\gamma^3\right) c_1^3, \tag{1.5}$$

$$\begin{aligned} Q_4(c_1, c_2, c_3, c_4) = &\\ \zeta(4)c_4 + \frac{1}{2}\left(\zeta(2)^2 - \zeta(4)\right) c_2^2 + (-\zeta(4) + \zeta(3)\gamma)c_1c_3 + &\\ \left(\zeta(4) - \zeta(3)\gamma + \frac{1}{2}\zeta(2)\gamma^2 - \frac{1}{2}\zeta(2)^2\right) c_1^2c_2 + &\\ \left(-\frac{1}{4}\zeta(4) - \frac{1}{4}\zeta(2)\gamma^2 + \frac{1}{3}\zeta(3)\gamma + \frac{1}{8}\zeta(2)^2 + \frac{1}{24}\gamma^4\right) c_1^4. & \end{aligned} \tag{1.6}$$

In particular for a Calabi Yau manifold we have the following polynomials:

$$\begin{aligned} Q_2(c_2) &= \zeta(2)c_2, \\ Q_3(c_2, c_3) &= \zeta(3)c_3, \\ Q_4(c_2, c_3, c_4) &= \frac{1}{2}\left(\zeta(2)^2 - \zeta(4)\right) c_2^2 + \zeta(4)c_4. \end{aligned} \tag{1.7}$$

2. Periods for the mirrors of hypersurfaces in toric varieties

We shall use the construction of the maximal degeneracy point and notations from [HLY]. Let M be a lattice $\mathbb{Z}^{d+1}$ and N denotes its dual. A cone in $N \otimes \mathbb{R}$ is called large if it is generated by finitely many vectors from N and its dimension is equal to rkN. A large cone is called regular if generators of its one dimensional faces (edges) form a basis of N. A fan Σ in $N \otimes \mathbb{R}$ is called regular if all its large cones are regular. We consider the fan Σ corresponding to a Fano polytope

$\Delta \subset M\otimes\mathbb{R}$. The intersections of the cones from Σ and the polytope Δ^* polar to Δ form a (maximal, cf. [HLY]) triangulation of Δ^* which we shall denote $\mathcal{T}_{\Delta^*}$. As was mentioned above, corresponding to Δ toric variety $\mathbf{P}_\Delta$ is a non singular Fano manifold.

Let $\Sigma(1)$ be the set of primitive vectors on the edges of the fan Σ. Since Δ is a Fano polytope, the following property (assumed in [HLY], cf. 4.7) is satisfied:

$$(*) \qquad \Sigma(1) \subset \partial\mathrm{conv}(\Sigma(1)).$$

We shall denote the vectors in $\Sigma(1)$ as $\mu_1,\dots,\mu_p$ and put $\mu_0=(0,\dots,0)$. Let $\mathcal{N}$ be a lattice endowed with the basis $n_0,\dots,n_p$, elements of which are in one to one correspondence with the set $\mathcal{A}=\{\mu_0,\dots,\mu_p\}\subset N$. Let $\bar{\mathcal{A}}=\{\bar\mu_0,\dots,\bar\mu_p\}$ be the collection of points in $\bar N_{\mathbb{R}} = N\otimes\mathbb{R}\oplus\mathbb{R}$, all belonging to the affine hyperplane $N\otimes\mathbb{R}\times 1$, and such that the image of $\bar\mu_i$ in $N\otimes\mathbb{R}$ is μ_i. Denote by $R:\mathcal{N}\to\bar N_{\mathbb{R}}$ the map given by $R(n_i)=\bar\mu_i$ and let $L_{\mathcal{A}}=\mathrm{Ker}R$ be the lattice of relations among $\bar\mu_i$'s (i.e. $L_{\mathcal{A}}=\{(l_0,\dots,l_p)\in\mathcal{N}|\Sigma_i l_i\bar\mu_i=0\}$). Let us put $\bar M_{\mathbb{R}}=\bar N^*_{\mathbb{R}}$. We shall view it as a subspace of the space of functions on $\bar{\mathcal{A}}$ and hence as a subspace of $\mathcal{N}^*\otimes\mathbb{R}$. The quotient $\mathcal{N}^*\otimes\mathbb{R}/\bar M_{\mathbb{R}}$ has the canonical identification with $L^*_{\mathcal{A}}\otimes\mathbb{R}$ (i.e. the dual of $L_{\mathcal{A}}\otimes\mathbb{R}$). The space $\mathcal{N}^*\otimes\mathbb{R}$ supports a natural fan edges of which correspond to triangulations of $\mathrm{conv}(\mathcal{A})$ with all simplices having vertices belonging to $\mathcal{A}$. The image of this fan in $L^*_{\mathcal{A}}\otimes\mathbb{R}$ is called the secondary fan $S\Sigma$ (cf. [GKZ]). Moreover the fan $S\Sigma$ admits a natural refinement called the Gröbner fan (cf. [HLY], section 4). We shall assume that:

(**) the cone $C(\mathcal{A})$ in $L^*_{\mathcal{A}}$ corresponding to triangulation $\mathcal{T}_{\Delta^*}$ is a cone of both the secondary and Gröbner fans and that the generators of the edges of this cone form a basis of the integral lattice of $L^*_{\mathcal{A}}$.

The space $PL(\Sigma)$ of piecewise linear functions on $N\otimes\mathbb{R}$, which are linear on each cone of Σ, can be identified with a subspace of $\mathcal{N}^*\otimes\mathbb{R}$. Moreover $L^*_{\mathcal{A}}\otimes\mathbb{R}=\mathcal{N}^*\otimes\mathbb{R}/\bar M\otimes\mathbb{R}=PL(\Sigma)/\bar M\otimes\mathbb{R}$ and the latter is canonically isomorphic to $H^2(\mathbf{P}_\Delta,\mathbb{R})$. The Lefschetz theorem identifies this group with $H^2(X_\Delta,\mathbb{R})$ if $d\geq 3$. In this identification the Kähler cones of X_Δ and $\mathbf{P}_\Delta$ are identified with $C(\mathcal{A})$.

According to [HLY], a consequence of (**) is that in the partial compactification of $\mathbb{C}^{*\mathcal{A}}/\mathbb{C}^*\times\mathrm{Hom}(N,\mathbb{C}^*)$, given by the fan consisting of the cones in the closure of the Kähler cone, the point corresponding to the cone $C(\mathcal{A})$ is a maximal degeneracy point for the GKZ system:

$$(2.1) \quad \left(\prod_{l_\mu>0}\left(\frac{\partial}{\partial a_\mu}\right)^{l_\mu}-\prod_{l_\mu<0}\left(\frac{\partial}{\partial a_\mu}\right)\right)^{-l_\mu}\Pi(a)=$$
$$0(l\in L_{\mathcal{A}}),\left(\Sigma_\mu<u,\bar\mu>a_\mu\frac{\partial}{\partial a_\mu}-<u,\beta>\right)\Pi(a)=0,$$

where $\beta = (-1, 0, \dots, 0)$ and $u \in \bar{M} \otimes \mathbb{R}$. In other words there is only one solution for (2.1) admitting a holomorphic extension in a neighborhood of the point of compactification corresponding to the cone $C(\mathcal{A})$. Moreover the period:

$$\Pi_\gamma(a) = \int_\gamma \frac{1}{f_\mathcal{A}} \prod \frac{dX_i}{X_i}, \quad f_\mathcal{A} = \Sigma a_\mu X^\mu \tag{2.2}$$

(γ, as above, is cycle in the complement in $\mathrm{Hom}(N, \mathbb{C}^*)$ to the hypersurface $f_\mathcal{A}(X, a) = 0$ having dimension $d + 1 = rkN$) is a solution for the system (2.1). Since we assume that Δ is Fano, in fact any solution of (2.1) is a period (cf. [H]).

On the other hand we have the following series representation:

$$\text{(2.3)} \quad a_0 \Pi_\gamma(a) =$$
$$(2\pi i)^d \Sigma_{l_1\mu_1 + \dots + l_p\mu_p = 0, l_1, \dots, l_p \geq 0} (-1)^{l_1 + \dots + l_p} \frac{(l_1 + \dots + l_p)!}{(l_1)! \dots (l_p)!} \frac{a_1^{l_1} \cdots a_p^{l_p}}{a_0^{l_1 + \dots + l_p}}.$$

The summation in the latter can be changed into a summation over dual to $C(\mathcal{A})$ cone in $L_\mathcal{A}$ (i.e. the Mori cone) by assigning to $(l_1, \dots, l_p)$ the vector $(l_0, l_1, \dots, l_p) \in L_\mathcal{A}$ where $l_0 = -l_1 - \dots - l_p$. If $l^{(1)}, \dots, l^{(p-d)}$ is a basis of the Mori cone, then in corresponding canonical coordinates (cf. (3.1) [HLY]) $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$ we have (cf. (5.12) in [HLY] and [HLY96])

$$\text{(2.4)} \quad a_0 \Pi_\gamma(x_1, \dots, x_{p-d}) =$$
$$\Sigma_{m_1, \dots, m_{p-d}, \Sigma m_k l_0^{(k)} \leq 0} \frac{\Gamma(-\Sigma m_k l_0^{(k)} + 1)}{\Gamma(\Sigma m_k l_1^{(k)} + 1) \cdots \Gamma(\Sigma m_k l_p^{(k)} + 1)} x_1^{m_1} \cdots x_{p-n}^{m_{p-n}}.$$

Let $r = p - d - 1 = rkH^2(X_\Delta, \mathbb{C})$ and let $J_k (i = 1, \dots, r)$ be the elements of $L_\mathcal{A}^*$ forming a basis dual to $l^{(k)}$. Let $D_i, (i = 1, \dots, p)$ be the cohomology classes in $H^2(\mathbf{P}_\Delta, \mathbb{Z})$ dual to codimension one orbits of $\mathbf{P}_\Delta$. Since D_i's correspond to the generators of one dimensional cones of Σ and, under identification $H^2(V_\Delta, \mathbb{Z})$, correspond to μ_i, we have $D_i = \Sigma_k J_k l_i^{(k)}, (i = 1, \dots, p)$. The total Chern class of the Calabi Yau hypersurface X_Δ, which has as the dual cohomology class $D_1 + \dots + D_p$, is the sum of the terms of degree not exceeding d in the expansion of

$$\frac{(1 + D_1) \cdots (1 + D_p)}{(1 + D_1 + \dots + D_p)} = \frac{(\Sigma_k J_k l_1^{(k)} + 1) \dots (\Sigma_k J_k l_p^k + 1)}{(\Sigma_k - J_k l_0^{(k)} + 1)}. \tag{2.5}$$

The term $Q_k(c_1, \dots, c_k)$ of the degree k of Γ-sequence coincides with the image in $H^*(X_\Delta)$ of the term of degree k in the formal power series in variables $\bar{J}_i, (i = 1, \dots, r)$:

$$c(\bar{J}_1, \dots, \bar{J}_{n-p}) = \frac{\Gamma(-\Sigma \bar{J}_k l_0^{(k)} + 1)}{\Gamma(\Sigma \bar{J}_k l_1^{(k)} + 1) \cdots \Gamma(\Sigma \bar{J}_k l_p^{(k)} + 1)}, \tag{2.6}$$

when $\bar{J}_i \to J_i$. The latter is equal to

$$\sum_{j_1,\dots,j_k} \frac{1}{k!} \frac{\partial^k c(\bar{J}_1,\dots,\bar{J}_k)}{\partial \bar{J}_{j_1},\dots,\partial \bar{J}_{j_k}}|_{\bar{J}_1=\dots=\bar{J}_r=0} \bar{J}_{j_1}\cdots \bar{J}_{j_k}. \tag{2.7}$$

The coefficient in (2.7) coincides with the corresponding partial derivative of the coefficient of period (2.4) after replacement $m_i \to \rho_i (i=1,\dots,r)$. The cup product of $Q_k(c_1,\dots,c_k)$ with $J_{i_1} \wedge \dots \wedge J_{i_{n-d}}$ hence is equal to the right hand side of (0.3) since the value of $J_{j_1} \wedge \dots J_{j_k} \wedge J_{i_1} \wedge \dots \wedge J_{i_{d-k}}$ on the fundamental class of X_Δ is the value of d-point function $K_{j_1,\dots,j_k,i_1,\dots,i_{d-k}}$ at the maximal degeneracy point.

Corollary.

1. (*cf.* (0.1), (0.2),[HKTY].) *Let X be a Calabi Yau hypersurface of dimension 3. Then*

$$\int_X c_2 \wedge J_i = \frac{3}{\pi^2} K_{ijk} \frac{\partial^2 c}{\partial \rho_j \partial \rho_k}(0,\dots,0), \tag{2.8}$$

$$\int_X c_3 = \frac{6}{\zeta(3)} K_{ijk} \frac{\partial^3 c}{\partial \rho_i \partial \rho_j \partial \rho_k}(0,\dots,0). \tag{2.9}$$

2. *Let X be a Calabi-Yau hypersurface of dimension 4 in a non-singular toric Fano manifold. Then*

$$\begin{aligned} \int_X c_2 \wedge J_i \wedge J_j &= \frac{3}{\pi^2} K_{ijkl} \frac{\partial c^2(0,\dots,0)}{\partial \rho_k \partial \rho_l}, \\ \int_X c_3 \wedge J_l &= \frac{6}{\zeta(3)} K_{ijkl} \frac{\partial^3 c(0,\dots,0)}{\partial \rho_i \partial \rho_j \partial \rho_k}, \end{aligned} \tag{2.10}$$

$$\int_X \left(\frac{1}{2}\zeta(2)^2 - \zeta(4)\right) c_2^2 + \zeta(4) c_4 = \frac{1}{24} K_{ijkl} \frac{\partial c^4(0,\dots,0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_k}. \tag{2.11}$$

These identities are a consequences of (0.3) and the identities (1.7).

3. Concluding remarks

1. **Example.** For a hypersurface V_d of degree d in $\mathbf{P}^d$ the Chern polynomial is the sum of the terms degree less than $d+1$ in the series $\frac{(1+h)^{d+1}}{(1+dh)}$, i.e., the Γ-sequence for V_d is the sum of the terms of less than $d+1$ in $\frac{\Gamma(1+dh)}{\Gamma(1+h)^{d+1}}$. (0.3) is a consequence of the fact that after replacing h by $m \in \mathbb{Z}^+$ the latter becomes the coefficient of the holomorphic at the maximal degeneracy point period of the d-form on the mirror of V_d.

2. Let $X_{1,1,3}$ (resp. $X_{1.2.2}$) be a non singular complete intersection of the Grassmanian $Gr(2,5)$ embedded via Plucker embedding in $\mathbf{P}^9$ and hypersurfaces of degrees $(1,1,3)$ (resp. $(1,2,2)$). It is shown in [BKCS] that the holomorphic period of the mirror has the presentation

$$\sum_m \left[(3m)!m!^2 \sum_{r,s} \frac{1}{m!^5} \binom{m}{r} \binom{s}{r} \binom{m}{s}^2\right] z^m, \tag{3.1}$$

resp.

$$\sum_m \left[m!(2m)!^2 \sum_{r,s} \frac{1}{m!^5} \binom{m}{r} \binom{s}{r} \binom{m}{s}^2\right] z^m. \tag{3.2}$$

Though the corresponding differential equations are not of hypergeometric type (since they have more than three singular points) and the ratio of coefficients is not a rational function of m, nevertheless the ratio of the coefficients of the series (3.1) and (3.2) is equal to the value of the ratio of the terms in the Hirzebruch Γ-sequences for $X_{1,1,3}$ and $X_{1,2,2}$. Indeed the Chern polynomials of the these manifolds are the cohomology classes of corresponding manifolds given by $\frac{c(Gr(2,5))}{(1+h)^2(1+3h)}$ and $\frac{c(Gr(2,5))}{(1+h)(1+2h)^2}$ respectively, where $c(Gr(2,5))$ is the Chern polynomial of the Grassmanian and h is the cohomology class of the hyperplane section. That is the ratio of the Γ-sequences is $\frac{\Gamma(3h+1)\Gamma(h+1)}{\Gamma(2h+1)}$, which, after replacing h by $m \in \mathbb{Z}_{\geq 0}$ is equal to the ratio of the coefficients of the periods of mirrors of corresponding Calabi Yau manifolds.

References

[B1] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), 493–535.

[B2] V. Batyrev, *On the classification of toric Fano 4-folds*, math.AG 9801107.

[BKCS] V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D .v. Straten,, *Conifold transitions and mirror symmetry for Calabi Yau complete intersections in Grassmanians*, Nuclear Phys. B. **514** (1998), 640–666.

[BvS] X. Batyrev and D. v. Straten, *Generalized hypergeometric functions and rational curves on Calabi Yau complete intersections in toric varieties*, Comm. Math. Phys. **168** (1995), 493–533.

[Bo] L. Borisov, *Towards the mirror symmetry or Calabi-Yau complete intersections in Gorenstein toric Fano varieties*, alg-geom/9310001.

[E] A. Erdelyi (ed.), *Higher transcendental functions*, vol. I, McGraw Hill, New York, 1953.

[F] W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton, NJ, 1993.

[GKZ] I.M. Gelfand, M. Kapranov, and A. Zelevinski, *Discriminants, Resultants and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.

[GMP] B. Greene, D. Morrison, and R. Plesser, *Mirror manifolds in higher dimensions*, Comm. Math. Phys. **173** (1995), 1699–1702.

[Hi] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag, Berlin-New York, 1966.

[H] S. Hosono, *GKZ systems, Gröbner Fans and moduli spaces of Calabi Yau hypersurfaces*, alg-geom/ 9707003.

[HKTY] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersections Calabi Yau spaces*, Comm. Math. Phys. **167** (1995), 301–350.

[HLY96] S. Hosono, B.H. Lian, and S.-T. Yau,, *GKZ Generalized hypergeometric systems in mirror symmetry of Calabi Yau hypersurfaces*, Comm. Math. Phys. **182** (1996), 535–578.

[HLY] S. Hosono, B.H. Lian, and S.-T. Yau, *Maximal degeneracy points of GKZ systems*, J. Amer. Math. Soc. **10** (1997), 427–443.

[LW] A. Libgober and J. Wood, *Uniqueness of the complex structure on Kähler manifolds of certain homotopy type*, J. Differential Geom. **32** (1990), 139–154.

Department of Mathematics, University of Illinois at Chicago, 851 S.Morgan, Chicago, Ill, 60607

E-mail address: libgober@math.uic.edu