Eigenvalues for the monodromy of the Milnor fibers of arrangements

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November 6, 2018

Abstract

We decribe upper bounds for the orders of the eigenvalues of the monodromy of Milnor fibers of arrangements given in terms of combinatorics.

1 Introduction.

The central object is the study of the topology of isolated hypersurface singularities is the Milnor fiber. If $f(x_0, ..., x_n)$ has an isolated singularity, say at the origin $\mathcal{O}$, then the Milnor F_f of f is the intersection of $f = t$ with a ball B_ϵ of a small radius ϵ about the origin ($|t| \ll \epsilon$). The Milnor fiber F_f is an n-connected smooth manifold with boundary having the natural action of the monodromy obtained by letting t vary around a small circle around the origin of $\mathbf{C}$. This is a diffeomorphism of F_f constant on the boundary which isotopy class modulo boundary depends only on f. In particular, one has a well defined operator T_f on $H_n(F, \mathbf{Z})$. This operator carries much of information about the topology of f (cf. [17]) and its calculation can be done in several ways. Particularly effective is the method due to A'Campo based on a resolution of the singularity of f (cf. [1]).

If the singularity is not isolated the Milnor fiber may "lose" connectivity. The monodromy diffeomorphism is still well defined up to isotopy and induces a well defined operator $T_{f,i}$ on each homology group $H_i(F_f, \mathbf{Z})$. Though the zeta function: $\prod_i \det(T_{f,i} - I)^{(-1)^{i-1}}$ still can be easily calculated by A'Campo method, the eigenvalues of $T_{f,i}$ for each i reflect more subtle properties of the singularity not necessarily encoded into the topological data about resolutions (cf. [20] [13]).

It is well known that the Milnor fiber of a cone over a projective hypersurface of degree d can be identified with the d-fold cyclic cover of the complement to the hypersurface (cf. for example [11] or section 2 below). In particular information about the homology of cyclic covers is equivalent to the information about the homology of Milnor fibers. The former were studied extensively over the last 20 years (cf. [9], the survey [12] or [4]). In particular it was shown that the eigenvalues of the deck transformations acting on the homology of cyclic covers and hence the eigenvalues of the monodromy of related Milnor fibers depend on position of singularities and bounded by the local type of hypersurfaces and their behavior at infinity.

In this note we shall specialize the results surveyed in [10] and [12] to the case of arrangements. These results fall into two groups. The first contains restrictions on the orders of the eigenvalues (note that eigenvalues of the monodromy are roots of unity as a consequence of the monodromy theorem, cf. [17]). These restrictions imposed entirely by the combinatorics of arrangement (cf. sect. 2.4 and theorem 3.1. In particular one obtains vanishing of the cohomology of certain local systems (i.e. those which cohomology are the components of the cohomology of cyclic covers). Our approach does not depend on Deligne's ([3]) results and gives vanishing of cohomology in specific dimensions i.e. we obtain conditions for "non resonance" in certain range.

The second group of results deals with a calculation of characteristic polynomial of the monodromy in the case of line arrangements in terms of dimensions of system of curves determined by collection of vertices of the arrangement. Methods used below are all contained in our previous work but they give more precise results in the case when the hypersurface is an arrangement. This case of arrangements was considered recently in (cf. [6], [2]).

This work was supported by NSF grants DMS-9872025 and DMS 9803623.

2 Preliminaries.

2.1. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbf{P}^n$ and $l_i(x_1,...,x_{n+1}) = 0, i = 1,..,d$ be the defining equations for the hyperplanes of $\mathcal{A}$. To $\mathcal{A}$ corresponds the cone A given by the equation $\prod l_i = 0$ in $\mathbf{C}^{n+1}$. Since the defining equation of A is homogeneous it follows that the Milnor fiber F_A can be identified with the affine hypersurface $\prod l_i = 1$.

2.2. Recall that $H_1(\mathbf{P}^n - \mathcal{A}, \mathbf{Z}) = \mathbf{Z}^{d-1}$ with generators given by the loops λ_i each being the boundary of a small 2-disk in $\mathbf{P}^n$ transversal to $l_i = 0$ (i.e. a meridian) at a non-singular point of $\mathcal{A}$. These loops satisfy single relation $\sum_{i=1}^{i=d} \lambda_i = 0$. In particular $\lambda_i \to 1 \bmod d$ defined the homomorphism $H_1(\mathbf{P}^n - \mathcal{A}, \mathbf{Z}) \to \mathbf{Z}/d\mathbf{Z}$. The corresponding branched covering can be realized as the hypersurface $\bar{V}_{\mathcal{A}}$ in $\mathbf{P}^{n+1}$ given by

$$x_0^d = \prod l_i(x_1,...,x_{n+1}) \tag{1}$$

The covering map is induced by the projection

$$\pi : \quad (x_0, x_1, ..., x_{n+1}) \to (x_1, ..., x_{n+1}). \tag{2}$$

The hypersurface $\bar{V}_{\mathcal{A}}$ is singular with the singular locus $Sing$ consisting of the points taken by this projection into points in $\mathbf{P}^n$ belonging to at least two hyperplanes of the arrangement. The unbranched covering $V_{\mathcal{A}}$ of $\mathbf{P}^n - \mathcal{A}$ corresponding to $H_1(\mathbf{P}^n - \mathcal{A}, \mathbf{Z}) \to \mathbf{Z}/d\mathbf{Z}$ is the restriction of this projection to $\bar{V}_{\mathcal{A}} - \{x_0 = 0\}$. The natural identification of $\mathbf{C}^{n+1}$ with the complement to $x_0 = 0$ in $\mathbf{P}^{n+1}$ yields the identification of the Milnor fiber F_A with unbranched covering $V_{\mathcal{A}}$. Moreover the description of the monodromy of Milnor fiber for weighted homogeneous hypersurfaces in [17] yields that a monodromy diffeomorphism of F_A can be identified with the deck transformation $x_0 \to \mu_d \cdot x_0$ (μ_d is a primitive root of unity of degree d) of $V_{\mathcal{A}}$. The operator induced on the homology of either of these spaces we denote as T_A.

An immediate consequence of this description of the monodromy is that for an arrangement of d hyperplanes each eigenvalue of $T_{A,i}$ acting on $H_i(F_A, \mathbf{Z}) = H_i(V_{\mathcal{A}}, \mathbf{Z})$ has an order dividing d. The multiplicity of the eigenvalue 1 for $T_{A,i}$ acting on $H_i(F_A, \mathbf{Z})$ is equal to $\mathrm{k}H_i(\mathbf{P}^n - \mathcal{A}, \mathbf{Z})$. Indeed, in the cohomology spectral sequence $H^q(\mathbf{Z}/d\mathbf{Z}, H^p(V_{\mathcal{A}})) \Rightarrow H^{p+q}(\mathbf{P} - \mathcal{A})$ associated with action of $\mathbf{Z}/d\mathbf{Z}$ all terms with $p \geq 1$ are zeros and the rank of invariant part of $H^p(V_{\mathcal{A}})$ coincides with the multiplicity of the eigenvalue 1 of the monodromy.

2.3. Milnor fiber of a central arrangement is homotopy equivalent to the infinite cyclic cover

$$(\widetilde{\mathbf{C}^{n+1} - A})_\infty \to \mathbf{C}^{n+1} - A \tag{3}$$

corresponding to the homomorphism $\mathbf{Z}^d = H_1(\mathbf{C}^{n+1} - A) \to \mathbf{Z}$ sending a generator corresponding to a hyperplane to the positive generator of $\mathbf{Z}$. Indeed, as a loop around the hyperplane $l_i = 0$ one can take the intersection of n hypersurfaces $l_1 = r_1, .., l_{i-1} = r_{i-1}, l_{i+1} = r_{i+1}, .., l_n = r_n, \prod_{i \geq n} l_i = r_{n+1}$ (equation l_i omitted) and $|l_i| = \epsilon$. This loop is taken by the map $f_A : (x_1, .., x_{n+1}) \to \prod l_i(x_1, ..., x_{n+1})$ into a small circle about the origin of $\mathbf{C}^*$. Hence $(\widetilde{\mathbf{C}^{n+1} - A})_\infty$ is homotopic to the fiber product $\mathbf{C}^{n+1} - A \times_{\mathbf{C}^*} \mathbf{C}$ with respect to f_A and $exp : \mathbf{C} \to \mathbf{C}^*$. This fiber product clearly is homotopy equivalent to $f_A^{-1}(1)$. Alternatively, preimage of $\mathbf{C}_a$ ($a \in \mathbf{P}^n - \mathcal{A}$) in $(\widetilde{\mathbf{C}^{n+1} - A})_\infty$ of $\mathbf{C}^*_a$ which a fiber of $\mathbf{C}^{n+1} - A \to \mathbf{P}^n - \mathcal{A}$ consists of d contractible component since the image of $\pi_1(\mathbf{C}^*)$ in the Galois group of the cover (3) is an infinite subgroup of index d.

If $(\widetilde{\mathbf{C}^{n+1} - A})_e$ is a cyclic cover corresponding to the map $H_1(\mathbf{C}^{n+1} - A) \to \mathbf{Z}/e\mathbf{Z}$ then the eigenvalues of the deck transformation on $(\widetilde{\mathbf{C}^{n+1} - A})_e$ have order which divides the least common multiple of d and e. Indeed, denoting for a CW-complex X and its cyclic cover $\tilde{X}$ by $C_i(X)$ and $C_i(\tilde{X})$ the group of i-chains we have the exact sequence: $0 \to C_i(\tilde{X}) \to C_i(\tilde{X}) \to C_i(X) \to 0$. Here the left homomorphism is the deck transformation minus identity. This yields homology sequence (cf. [16]):

$$... \to H_i((\widetilde{\mathbf{C}^{n+1} - A})_\infty) \to H_i((\widetilde{\mathbf{C}^{n+1} - A})_\infty) \to H_i((\widetilde{\mathbf{C}^{n+1} - A})_e) \to$$

$$H_{i-1}((\widetilde{\mathbf{C}^{n+1} - A})_\infty) \to ... \tag{4}$$

The $\mathbf{C}[t, t^{-1}]$-module $H_i((\widetilde{\mathbf{C}^{n+1} - A})_\infty)$ with t acting as the deck transformation is annihilated by $t^d - 1$ as a consequence of interpretation of $(\widetilde{\mathbf{C}^{n+1} - A})_\infty$ as the Milnor fiber. On the other hand the left map in (4) is multiplication by $t^e - 1$. Hence the claim follows.

As a corollary we obtain the following:

Proposition 2.1 *Let $(\widetilde{\mathbf{C}^{n+1}-A})_e$ be a e-fold cover of $\mathbf{C}^{n+1}-A$ and* d *the number of hyperplanes in A. Then an eigenvalue of a deck transformation acting on $H_i((\widetilde{\mathbf{C}^{n+1}-A})_e)$ has an order dividing both e and d.*

2.4. In the case $n=2$ one can relate the homology of $V_{\mathcal{A}}$ to the homology of a non-singular compactification $\tilde{V}_{\mathcal{A}}$ of $V_{\mathcal{A}}$ (cf. [9],[15]):

$$\mathrm{rk} H_1(\tilde{V}_{\mathcal{A}},\mathbf{Q}) = \mathrm{rk} H_1(V_{\mathcal{A}},\mathbf{Q}) - (d-1) \tag{5}$$

Indeed, the argument in [9] shows that $\mathrm{rk} H_1(\tilde{V}_{\mathcal{A}}) = rkH_1(\bar{V}_{\mathcal{A}} - \mathrm{Sing})$. The rest follows from the exact sequence of the pair $(\bar{V}_{\mathcal{A}} - \mathrm{Sing}, V_{\mathcal{A}})$. Indeed, from the excision and Lefschetz duality one obtains the isomorphism of $H_i(\bar{V}_{\mathcal{A}} - \mathrm{Sing}, V_{\mathcal{A}})$ and H^{4-i} of the disjoint union of lines of $\mathcal{A}$ and hence $H_1(\bar{V}_{\mathcal{A}} - \mathrm{Sing}, V_{\mathcal{A}}) = 0, H_2(\bar{V}_{\mathcal{A}} - \mathrm{Sing}, V_{\mathcal{A}}, \mathbf{Z}) = \mathbf{Z}^d$. Moreover, the deck transformations act trivially on the latter group and $\mathbf{Z}/d\mathbf{Z}$-invariant subgroup of $H_2(\bar{V}_{\mathcal{A}} - Sing)$ is cyclic which injects into $H_2(\bar{V}_{\mathcal{A}} - \mathrm{Sing}, V_{\mathcal{A}})$. This yields (5).

For a space V acted upon by T let V_ξ be the subspace spanned by the eigenvectors with eigenvalue ξ. The above arguments, since $H_i(F_A) = H_i(V_A)$, also show that

$$\oplus_{\xi\neq 1} H_1(F_A)_\xi = \oplus_{\xi\neq 1} H_1(\tilde{V}_{\mathcal{A}})_\xi, \quad rkH_1(F_A)_1 = d-1 \tag{6}$$

2.5 Finally, recall two results which will be used below. The first is the Lefschetz hyperplane section theorem (cf. [8]). Let X be a stratified complex algebraic variety having the dimension n and let H be a hyperplane transversal to all strata of X. Then the homomorphism $H_i(X\cap H,\mathbf{Z}) \to H_i(X,\mathbf{Z})$ induced by injection $X\cap H \to X$ is an isomorphism for $i \le n-2$ and is surjective for $i = n-1$.

Secondly, we shall use the Leray spectral sequence corresponding to a covering. More precisely, let $X = \bigcup_{i=1}^{i=N} U_i$ be a union of locally closed subsets. Then there is the Mayer-Vietoris spectral sequence:

$$E_2^{p,q} = \oplus_{i_1<\ldots<i_q} H^p(U_{i_1}\cap \ldots \cap U_{i_q}) \Rightarrow H^{p+q}(X)$$

Moreover, if a group G acts on X leaving each U_i invariant then all the maps in this spectral sequence are equivariant (cf. [7]).

3 Bounds on the orders of the eigenvalues.

Let $\mathcal{A}$ be an arrangement in $\mathbf{P}^n$. Let us call two points P and P' in $\mathbf{P}^n$ equivalent if collections of hyperplanes in the arrangement containing P and P' coincide. Each equivalence class is a smooth submanifold of $\mathbf{P}^n$ and equivalence classes form a stratification of $\mathbf{P}^n$ with the union of strata of dimension $n-1$ coinciding with the union of hyperplanes in $\mathcal{A}$. Let $S_1^k, \ldots, S_{s_k}^k$ be the collection of strata of codimension k. For each stratum we define the multiplicity $m(S_i^k)$ as the number of hyperplanes in arrangement containing a point from this stratum. Any hyperplane $H \in \mathcal{A}$ acquires the induced stratification.

Theorem 3.1 *Let H be a hyperplane of the arrangement $\mathcal{A}$ and let $m_1^k(H) = m(S_1^k), ..., m_{s(k)}^k = m(S_{s(k)}^k)$ be the collection of multiplicities of strata of the above stratification of $\mathbf{P}^n$ which belong to H and have codimension k . Let ξ be an eigenvalue of the monodromy of the Milnor fiber F_A of $\mathcal{A}$ acting on $H_{k-1}(F_A, \mathbf{C})$. Then for any $H \in \mathcal{A}$ one has $\xi^{m_i^j(H)} = 1$ for at least one of the multiplicities $m_i^j(H)$ with $j \leq k$.*

Proof. The idea is to bound the orders (of the eigenvalues of the deck transformations acting on the homology of the cyclic covering induced by π (cf (2)) on *a tubular neighborhood of H* and then use a Lefschtez type argument to derive the theorem.

First, however, let us notice that it is enough to show the theorem in the case $k = n$. Indeed if $k < n$ then for a generic linear subspace $L \in \mathbf{P}^n$ having dimension k and transversal to all strata of $\mathcal{A}$ we have the induced arrangement $\mathcal{L} = \mathcal{A} \cap L$ which has, due to transversality, as the multiplicities of its strata the integers $m_i^j, j \leq k, 1 \leq i \leq s_j$. Moreover, one has an equivariant with respect to the group of deck transformations map: $H_j(V_{L\cap\mathcal{A}}) \rightarrow H_j(V_{\mathcal{A}})$, which, by Lefschetz theorem, is an isomorphism for $j \leq k-2$ and surjective for $j = k-1$. Hence the above theorem for the arrangement $L \cap \mathcal{A}$ yields the result in general and we shall assume from now on that $k = n$.

Let $T(H) \subset \mathbf{P}^n$ be a small tubular neighborhood of H, $B = H - H \cap \mathcal{A}$ and $T(B) = T(H) - \mathcal{A}$. The above stratification of $\mathbf{P}^n$ yields a partition of $T(B)$ into union of subsets of $\mathbf{P}^n$ corresponding to the strata of the above stratification of H so that each subset is a locally trivial fibration over the corresponding stratum in H. The fiber of this fibration over a stratum of dimension k is a central arrangement of hyperplanes in $\mathbf{C}^{n-k}$ and the number of hyperplanes in the latter is equal to the multiplicity of the stratum. Let $\mathcal{S}_l^k$ be a collection of subsets of $T(B)$ each of which fibers over the stratum S_l^k and chosen so that their union in $T(B)$.

Intersection of subsets $\mathcal{S}_{l_1}^{k_1}$ and $\mathcal{S}_{l_2}^{k_2}$ such that $k_1 \geq k_2$ is non-empty if and only if the stratum $S_{l_2}^{k_2}$ is in the closure of $S_{l_1}^{k_1}$. In this case this the intersection is the fibration with the same fiber as $\mathcal{S}_{l_1}^{k_1} \rightarrow S_{l_1}^{k_1}$ and the base being a subset in the stratum $S_{l_1}^{k_1}$ (more precisely the base is the complement in a small neighborhood of the closure of $S_{l_2}^{k_2}$ in the closure of $S_{l_1}^{k_1}$ to the union of the hyperplanes of the arrangement induced by $\mathcal{A}$ on the closure of $S_{l_1}^{k_1}$) .

We shall denote the π-preimage of each of the sets $\mathcal{S}_i^k$ (resp. $T(B)$) as $\tilde{\mathcal{S}}_i^k$ (resp. $\widetilde{T(B)}$).

To illustrate this, let us consider $\mathcal{A}$ which is the set of zeros of $x \cdot y \cdot z \cdot$ in $\mathbf{P}^3$. We have one stratum of dimension 3, four strata of dimension 2 corresponding to four planes of the arrangement, six one dimension strata and corresponding to lines and strata of dimension zero. Sets $\mathcal{S}_i^2$ are fibrations over $\mathbf{P}^2$ minus three lines and having as a fiber the circle. Sets $\mathcal{S}_i^1$ are fibered with the fiber homeomorphic to $\mathbf{C}^2$ minus a pair of intersecting lines. The base is $\mathbf{C}$ minus a point, etc. Intersection of strata having as their closure a plane and a line in this plane is the fibration with the fiber being a circle over a regular neighborhood of $\mathbf{P}^1$ in $\mathbf{P}^2$ minus two of its fibers.

Let us consider the Mayer-Vietoris spectral sequence (cf. 2.5):

$$E_2^{p,q} : \oplus H^p(\tilde{\mathcal{S}}^{k_1}_{t_1} \cap ... \cap \tilde{\mathcal{S}}^{k_{q+1}}_{t_{q+1}}) \Rightarrow H^{p+q}(\widetilde{T(B)}) \tag{7}$$

The sequence (7) is equivariant with respect to the action of the group of deck transformations of the cover π restricted to $\widetilde{T(B)}$. An eigenvalue of the deck transformation acting on $E_2^{p,q}$ must satisfy: $\xi^{m^i_{t_i}}$ where $i \le k$. To see this notice that each summand in (7) fibers over a subset of S^i_j with the fiber being the cyclic cover of an arrangement of m^i_j hyperplanes. Consider the Leray spectral sequence for such fibration:

$$H^a(S^{k_1}_{t_1} \cap ... \cap S^{k_l}_{t_l}, H^b(\mathcal{F}(\widetilde{\mathcal{S}^{k_1}_{t_1} \cap ... \cap \mathcal{S}^{k_l}_{t_l}}))) \Rightarrow H^{a+b}(\tilde{\mathcal{S}}^{k_1}_{t_1} \cap ... \cap \tilde{\mathcal{S}}^{k_l}_{t_l}) \tag{8}$$

where $\mathcal{F}(\widetilde{S^{k_1}_{t_1} \cap ... \cap \mathcal{S}^{k_l}_{t_l}})$ is the cover of the arrangement $\mathcal{F}(S^{k_1}_{t_1} \cap ... \cap \mathcal{S}^{k_l}_{t_l})$ which is the fiber of the fibration associated with corresponding stratum and hence is an arrangement of $m^i_{t_i}$ hyperplanes where $i \le p$. (8) yields that the degree of an eigenvalue on $H^p(S^{k_1}_{t_1} \cap ... \cap \mathcal{S}^{k_l}_{t_l})$ is $m^i_{t_i}$ and the claim follows.

Now let us consider a generic (i.e. transversal to all strata) hypersurface $\mathcal{V}$ which belongs to $T(B)$: hypersurface $x_0^N = \tilde{B}$ where $\tilde{B}$ is a sufficiently small deformation of B is a possible choice. Since $\mathcal{V}$ is an ample divisor in $\mathbf{P}^{n+1}$ by Lefschetz theorem ([18]) we have surjections:

$$H_k(\mathcal{V} - \mathcal{L}) \to H_k(V_{\mathcal{A}} - \mathcal{L}) \quad (k \le n-1)$$

This yields that in the diagram

$$\begin{array}{ccc} H_k(\mathcal{V} - V_{\mathcal{A}}) & \longrightarrow & H_k(T(B) - V_{\mathcal{A}}) \\ & \searrow & \downarrow \\ & & H_k(V_{\mathcal{A}} - \mathcal{L}) \end{array}$$

the vertical arrow is surjective for $k \le n-1$. Hence the claim about the eigenvalues of the monodromy follows.

Remark 3.2 One can restrict the collection of strata S^j_i in theorem 3.1 by considering only strata which closures are dense (cf. [19]) edges. Recall that a dense edge is an intersection of hyperplanes of arrangement L such that the central arrangement $\mathcal{A}_L = \{H \in \mathcal{A} | L \subseteq H\}$ is indecomposable in the sense that the latter arrangement cannot be split into a union of two subarrangement which can be written in appropriate coordinates as arrangements of disjoint sets of variables.

Indeed, a Thom-Sebastiani (cf. [21]) type argument shows that orders of eigenvalues of decomposable arrangements are least common factor of the orders of monodromy on each factor.

Remark 3.3 One can replace m^j_i by the *least common multiple* of the eigenvalues of the monodromy of Milnor fiber of the arrangement which appear in the transversal to the stratum section. Such l.c.m. is a divisor of the multiplicity m^j_i (cf. section 2) but can be smaller than m^j_i. The simplest example is two lines through a point. Here the multiplicity is 2 but the only eigenvalues of the monodromy is 1.

Remark 3.4 Theorem 3.1 gives also restriction on the local systems corresponding to the homomorphism of the fundamental group sending each meridian to $\frac{1}{e}$ which have non vanishing cohomology in dimension $k-1$: non vanishing occurs only if e divides at least one of the multiplicities $m_k^j, j \le k$.

Corollary 3.5 Order of the monodromy operator acting on $H_i(F_A), 1 \le i \le n-1$ for an arrangement of d hyperplanes in $\mathbf{P}^n$ divides d and at least one of numbers $m_j^i(H)$ for each hyperplane H of the arrangement. In particular, if for at least one hyperplane each $m_i^j(H)$ is relatively prime to d, then eigenvalues different from 1 appear only in top dimension (i.e. n).

4 Line arrangements

Let $\mathcal{A}$ be an arrangement of d lines in $\mathbf{P}^2$. This is as a curve of degree d having only ordinary singularities. We shall apply the calculation of the Alexander module of plane algebraic curves to this special curve. We refer to [10], [15] or [14] for definition of constants and ideals of quasiadjunction. Since the singularities of the curve in question are (weighted) homogeneous one can use the description of the constants of quasiadjunction from [10] sect. 5 (cf. also [14] sect.3). One obtains that in the case of a point of multiplicity m the constants of quasiadjunction are $\frac{m-2}{m}, \frac{m-3}{m}, ..., \frac{1}{m}$ and the ideal of germs ϕ in the local ring of the singular point satisfying $\kappa_\phi < \alpha$ is $\mathcal{M}^{m-[\alpha \cdot m]-2}$ where $\mathcal{M}$ is the maximal ideal of the singular point.

Theorem 4.1 *Let $d = \mathrm{Card}\mathcal{A}$ and m be a divisor of d. Let $\sigma_k(m)$ be the superabundance of the curves of degree $d-3-\frac{kd}{m}$ such that the local equation at a point of multiplicity m belongs to the ideal $\mathcal{M}^{m-[\frac{k \cdot d}{m}]-1}$. Then the multiplicity of an eigenvalue $exp(\frac{2\pi ik}{m})$ of the monodromy of the Milnor fiber acting on $H_1(F_A, \mathbf{C})$ is equal to the $\sigma_k(m) + \sigma_{d-k}(m)$.*

Proof. We shall use the identification (6). We have $H^{0,1}(\tilde{V}_\mathcal{A})_\xi = H^{1,0}(\tilde{V}_\mathcal{A})_{\bar{\xi}}$ and $H^1(\tilde{V}_\mathcal{A}) = H^{0,1}(\tilde{V}_\mathcal{A}) \oplus H^{0,1}(\tilde{V}_\mathcal{A})$. According to theorem 5.1 in [10] we have:

$$H^{0,1}(\tilde{V}_\mathcal{A})_{exp\frac{2\pi ik}{m}} = H^1(\mathbf{P}^2, \mathcal{I}(d-3-\frac{k \cdot d}{m}))$$

for the ideal sheaf $\mathcal{I}$ defined as follows. The support of $\mathcal{O}_{\mathbf{P}^2}/\mathcal{I}$ coincide with the set of vertices of the arrangement $\mathcal{A}$ and the stalk at a singular point P consists of germs $\phi \in \mathcal{O}_P$ such that $\kappa_\phi < \frac{k}{m}$. Now the theorem follows from the above description of ideals of quasiadjunction of ordinary singularities.

5 Remarks and Examples

Remark 5.1 *ζ-function of monodromy.* One can easily see the the relation:

$$\zeta_\mathcal{A}(t) = \prod det(1 - T_{f,i}t, H_i(V_A))^{(-1)^i} = (1-t^d)^{\chi(\mathbf{P}^n - A)} \qquad (9)$$

Indeed, we have $\zeta_{\mathcal{A}}(t) = \prod det(1 - T_{f,i}t, H_i(V_A))^{(-1)^i} = \prod det(1 - T_{f,i}t, C_i(V_A))^{(-1)^i}$ where $C_i(V_A)$ denote the i-chains.

Example 5.2 *Braid arrangement i.e. the arrangement in* $\mathbf{P}^n$ *with hyperplanes given by* $x_i = x_j (i, j = 0, ..., n, i \neq j)$. In the case $n = 2$ (three lines in $\mathbf{P}^2$ passing through a point) section 4.1 yields that the multiplicity of eigenvalue ω_3 is equal to 1. Since $\chi(\mathbf{P}^3 - \mathcal{A}) = -1$ and $\dim H_1(\mathbf{P}^3 - \mathcal{A}) = 2$ we obtain $(1-t), (1-t)(1-t^3), 1$ as the characteristic polynomials of the monodromy acting on H_0, H_1 and H_2 respectively.

In the case $n = 3$ we have the arrangement of 6 planes with 4 lines and one vertex. Each plane contains two strata of codimension 2 in $\mathbf{P}^3$ (each has a line as a closure) having multiplicity 3 and hence the eigenvalues of the monodromy acting on H_1 of the Milnor fiber $\prod(x_i - x_j) = 1$ are the roots of unity of order 3 or 1. Similarly the eigenvalues of the monodromy acting on H_2 are roots of unity of degree either $1, 3$ or 6 (in fact all orders do occur (cf. [5])).

In fact ω_3 *is* an eigenvalue of T_1. Indeed, by Lefschetz type argument (cf 2.5) eigenvalues are the eigenvalues of the monodromy of the arrangement of 6 lines in $\mathbf{P}^2$ formed the lines containing the sides and the medians of a triangle. One can find the multiplicity of the eigenvalue of ω_3 as the $\dim H^1(\mathcal{I}(1))$ where $\mathcal{I}$ is the ideal sheaf of the collection of triples points in this arrangement of lines. Since these 4 triple points form a complete intersection of two quadrics we have:

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-2) \oplus \mathcal{O}(-2) \to \mathcal{I} \to 0$$

and hence $H^1(\mathcal{I}(1)) = H^2(\mathcal{O}(-3)) = \mathbf{C}$. Therefore the multiplicity of the eigenvalue ω_3 is 1.

The divisibility theorem has the following consequence. Since a generic plane section has only triple points the eigenvalues most have the order 3 or 1 and the order must be a divisor of $\frac{n(n-1)}{2}$. Hence if $n = 2$ mod 3 the order of an eigenvalue must be 1.

Similarly, generic section of a braid arrangement by a 3-space has only points of multiplicity 3 and 6. Hence if n is relatively prime to 6 then the eigenvalues of the monodromies T_1 and T_2 (i.e. on H_1 and H_2 respectively) are equal to 1, etc.

Example 5.3 More generally, consider an arrangement such that only three hyperplanes meet in each edge of codimension two (cf. previous example). Then if the number of hyperplanes is not divisible by 3 then the eigenvalues of the monodromy are equal to 1. In particular the homology of the Milnor fiber in all dimensions except top coincide with the cohomology of $\mathbf{P}^n - \mathcal{A}$.

Example 5.4 Consider a line arrangement having only triple points. It follows from (6) that the multiplicities of the eigenvalues $exp\frac{2\pi i}{3}$ and $exp\frac{4\pi i}{3}$ are the same. It follows from the divisibility theorem at infinity [9] that the characteristic polynomial of the monodromy divides $(t^d-1)^{d-2}(t-1)$. In particular the multiplicity of the eigenvalue $exp\frac{2\pi i}{3}$ does not exceed $d-2$ Since the multiplicity of the eigenvalue 1 is $d-1$ we obtain:

$$H_1(F_A, \mathbf{C}) = (\mathbf{C}[t, t^{-1}]/(t^3-1))^s \oplus (\mathbf{C}[t, t^{-1}]/(t-1))^{d-1-s}$$

where s is the superabundance of the curves of degree $d-3-\frac{d}{3}$ passing through the set of vertices of multiplicity 3. (cf. [2])

Example 5.5 Consider an arrangement of p hyperplanes where p is a prime and such that not all hyperplanes are passing through a point. Then eigenvalues different from 1 appear only in the top dimension. Indeed the multiplicity of any stratum is less than p and the claim follows from 3.5.

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