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First order deformations for rank one local systems with a non-vanishing cohomology

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Abstract

We study the tangent cone to the variety V_m^k of rank one local systems on a finite CW-complex for which the dimension of k-dimensional cohomology is at least m. This cone is related to the space of certain complexes of abelian groups with differential induced by the cup product. In particular, for an arrangement of hyperplanes, the set of 1-forms for which the corresponding Aomoto complex has k-dimensional Betti number greater or equal to m is a union of linear spaces. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Local systems on topological spaces contain much information about their homotopy types. We are interested exclusively in the case of *rank one* local systems. Such a local system is just a character of the fundamental group and hence the moduli space of such local systems by itself is very simple: it is the torus $H^1(X, \boldsymbol{C}^*)$ where X is the topological space in question. This torus has, however, an interesting system of subvarieties $V_m^k(X)$, consisting of the local systems L with the property that $\operatorname{rk} H^k(X, L) \geqslant m$.

The case $k = 1$ has attracted much attention in a somewhat different context: V_m^1 depends only on the fundamental group $\pi_1(X)$ and the amount of information contained in V_m^1 is very close to that carried by the Alexander invariants of $\pi_1(X)$ (cf. [12] for discussion of this comparison where these varieties V_m^1 were called the characteristic varieties of X, cf. also [10]). In special cases, e.g., when X is a complement to a link in a sphere (cf. [9]) or

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a complement to an algebraic curve in a projective plane (cf. [12]), particularly detailed results were obtained.

In the case when X is a quasiprojective algebraic variety, subvarieties $V_m^k \subset H^1(X, \boldsymbol{C}^*)$ have a very simple structure: each V_m^k is a union of translated subtori [2] of $H^1(X, \boldsymbol{C}^*)$ (cf. [6,16,1]).

This simplifies substantially the problem of finding the varieties V_m^k explicitly. In special situations mentioned above, i.e., links in spheres or plane algebraic curves, indeed, V_m^k can be related to other interesting information about X. E.g., in the case of curves, V_m^1 can be found in terms of local types of singularities of the curve and their position in the plane. A similar description in terms of position of singularities and the local type, as well as relation to the homotopy groups, also was given for $k > 1$ in the case when X is a hypersurface in projective space with isolated singularities, cf. [11].

In the case when X is a complement to an arrangement $\mathcal{A}$ of hyperplanes in $\boldsymbol{P}^n$, there is a combinatorial method for finding components of V_m^1 having a positive dimension and containing the identity. It is based on the study of the space of Aomoto complexes. The latter are the complexes of the form (X, in fact, can be any finite CW-complex):

$$\begin{aligned} A^\bullet(X, v) : H^0(X, \boldsymbol{C}) &\xrightarrow{v\cup} H^1(X, \boldsymbol{C}) \to \cdots \to H^k(X, \boldsymbol{C}) \\ &\xrightarrow{v\cup} H^{k+1}(X, \boldsymbol{C}) \to \cdots, \end{aligned} \tag{1}$$

where $v \in H^1(X, \boldsymbol{C})$. One can consider a subspace $\mathcal{V}_m^k \subset H^1(X, \boldsymbol{C})$ consisting of v such that $\operatorname{rk} H^1(A^\bullet(X, v)) \geqslant m$. In the case when X is a complement to an arrangement, it was shown that $\mathcal{V}_m^1$ is a union of linear subspaces of $H^1(X, \boldsymbol{C})$ (as was suggested in [5]) which are in one to one correspondence with the components of V_m^1 having a positive dimension and containing the identity of the torus $H^1(X, \boldsymbol{C}^*)$ cf. [4,12]. [3]

The purpose of this note is to present a proof for a similar relation between the varieties $\mathcal{V}_m^k$ and V_m^k for $k > 1$. It is based on the arguments in [6], where the authors study the cohomology of holomorphic line bundles on non singular projective varieties and [7] where, among other things, the restrictions on Kähler groups were obtained (cf. also [8]). In fact one can relate arbitrary components of V_m^k having positive dimension to the space of complexes which are twisted version of (1) i.e., complexes of the form:

$$\to H^k(L) \xrightarrow{v\cup} H^{k+1}(L) \to . \tag{2}$$

More precisely this note contains a proof of the following:

Theorem 1.1. *Let L_η ($\eta \in \operatorname{Char}(\pi_1(X))$) be a rank one local system on a finite CW-complex X which belongs to a component of the variety*

$$V_m^k = \bigl\{\eta \mid H^k(L_\eta) \geqslant m\bigr\} \subset \operatorname{Char}\bigl(\pi_1(X)\bigr)$$

[2] In particular V_m^k has singularities at the intersection points of components.

[3] A generalization of to the case when X is a complement to a union of rational curves in $\boldsymbol{P}^2$ was given in [2] where the author constructs a complex detecting the components of V_m^1 starting from equations of irreducible components of these curves.

having positive dimension. For each $v \in H^1(X, \boldsymbol{C})$ let us consider the complex $(H^\bullet(L_\eta), v)$:

$$\to H^\bullet(L_\eta) \xrightarrow{v\cup} H^{\bullet+1}(L_\eta) \to$$

with differential $x \to v \cup x$ given by the cup product corresponding to $L_1 \otimes L_\eta \to L_\eta$ where L_1 is the trivial local system on X. Then the tangent cone to V^k_m at L_η can be identified with a subspace[4] *of $\mathcal{V}^k_m(\eta)$ consisting of v's for which the cohomology of the corresponding complex satisfy*:

$$\operatorname{rk} H^k\big(H^\bullet(L_\eta), v\big) \geqslant m. \tag{$*$}$$

If $\eta \in V^k_m$ is such that $\operatorname{rk} H^k(L_\eta) = m$, then the Zariski tangent space to V^k_m at η coincides with the Zariski tangent space to $\mathcal{V}^k_m(\eta)$.

In the case when X is a quasiprojective algebraic variety the tangent cone to V^k_m at L_η and $\mathcal{V}^k_m(\eta)$ coincide. Complete arguments will be presented elsewhere. Here we shall state this only in a special case when X is a complement to an arrangement of hyperplanes and L_η is the trivial local system.

Corollary 1.2. *Let X be the complement to an arrangement of hyperplanes in $\boldsymbol{C}^n$. Then the set of $v \in H^1(X, \boldsymbol{C})$ with the property that the rank of the kth cohomology group of the complex*

$$\cdots \to H^\bullet(X, \boldsymbol{C}) \xrightarrow{v\cup} H^{\bullet+1}(X, \boldsymbol{C}) \to \cdots$$

is greater than m is a union of linear spaces.

Indeed, recall that if $v \in \mathcal{V}^k_m$ then $\operatorname{rk} H^k(L_{\exp(tv)}) \geqslant \operatorname{rk} H^k(H^\bullet(L), v)$ (cf. [13]). Therefore, if v belongs to $\mathcal{V}^k_m$ then $L_{\exp(tv)} \in V^k_m$ and hence $v \in TC_e(V^k_m)$.[5]

In the case $k = 1$ it yields the results of [4] and theorem (5.4) of [12]. Also, a result similar to Corollary 1.2 was obtained in [3].

2. Preliminaries

To fix notations let us recall first the basic definitions related to local systems (for more details cf. [14]). A local system of rank one on a topological space X is a character $\eta : \pi_1(X) \to \boldsymbol{C}^*$ of the fundamental group. The cohomology of a local system can be calculated as the cohomology of the complex:

$$\to \operatorname{Hom}_{\boldsymbol{C}[H_1(X,\boldsymbol{Z})]}\big(C_{k-1}\big(\widetilde{X}\big), \boldsymbol{C}_\eta\big) \xrightarrow{d^k(\eta)} \operatorname{Hom}_{\boldsymbol{C}[H_1(X,\boldsymbol{Z})]}\big(C_k\big(\widetilde{X}\big), \boldsymbol{C}_\eta\big) \to, \tag{3}$$

where $\widetilde{X}$ is the universal abelian cover of X,

$$\to C_{k+1}\big(\widetilde{X}\big) \xrightarrow{d_{k+1}} C_k\big(\widetilde{X}\big) \xrightarrow{d_k} C_{k-1}\big(\widetilde{X}\big) \to \tag{4}$$

[4] A. Suciu pointed out that an example when the subspace is proper is contained in the thesis of Matei [15] (cf. also Proposition 2.4 of this thesis).

[5] For a discussion of relation between V^k_m and $\mathcal{V}^k_m$ and information about the latter cf. [13].

is a chain complex of $\widetilde{X}$ (with coefficients in $\boldsymbol{C}$), $\boldsymbol{C}_\eta$ is a 1-dimensional over $\boldsymbol{C}$ space considered as a $H_1(X,\boldsymbol{Z})$-module using the character η and $d^k(\eta)$ is the map of $\mathrm{Hom}_{\boldsymbol{C}[H_1(X,\boldsymbol{Z})]}(C_{k-1}(\widetilde{X}),\boldsymbol{C}_\eta)$ induced by d_k.

Note that, if $H_1(X,\boldsymbol{Z})$ is torsion free (which we shall assume for simplicity unless otherwise specified) having rank r, and $c_k = \operatorname{rk} C_k(X)$ then $C_k(\widetilde{X})$ can be chosen as a free module of rank c_k over $\Lambda = \boldsymbol{Z}[H_1(X,\boldsymbol{Z})]$. After a choice of generators for $H_1(X,\boldsymbol{Z})$ the ring Λ can be identified with the ring of Laurent polynomials $\boldsymbol{C}[t_1,t_1^{-1},\ldots,t_r,t_r^{-1}]$. In this case, the matrix of the differential d_k can be viewed as a rectangular $c_{k-1}\times c_k$ matrix with entries in $\boldsymbol{C}[t_1,t_1^{-1},\ldots,t_r,t_r^{-1}]$. A matrix of $d^k(\eta)$ can be obtained from the matrix of d_k by transposition and evaluating its entries at η.

The local systems on X are parametrized by the torus $H^1(X,\boldsymbol{C}^*) = \operatorname{Spec}\boldsymbol{C}[H_1(X,\boldsymbol{Z})]$ and the Lie algebra of $H^1(X,\boldsymbol{C}^*)$ can be identified with $H^1(X,\boldsymbol{C})$. This yields the identification of the tangent space T_η at $\eta\in H^1(X,\boldsymbol{C}^*)$ with $H^1(X,\boldsymbol{C})$.

Definition 2.1. The derivative complex (cf. [6]) of the complex (4) at a point η of the torus $\operatorname{Spec}\boldsymbol{C}[H_1(X,\boldsymbol{Z})]$ in the direction of the tangent vector $v\in H^1(X,\boldsymbol{C})$ is the complex:

$$\left(D^\bullet_{\eta,v}(\widetilde{X}), d^\bullet_{D_{\eta,v}}\right),$$

where $D^k_{\eta,v}(\widetilde{X}) = H^k(L_\eta) = \operatorname{Ker}(d^{k+1}(\eta))/\operatorname{Im}(d^k(\eta))$ and the differential

$$d^k_{D_{\eta,v}}:\operatorname{Ker}\left(d^{k+1}(\eta)\right)/\operatorname{Im}\left(d^k(\eta)\right)\to\operatorname{Ker}\left(d^{k+2}(\eta)\right)/\operatorname{Im}\left(d^{k+1}(\eta)\right) \tag{5}$$

induced by

$$\frac{\partial d^{k+1}}{\partial v}(\eta):\operatorname*{Hom}_{\Lambda}\left(C_k(\widetilde{X}),\boldsymbol{C}_\eta\right)\to\operatorname*{Hom}_{\Lambda}\left(C_{k+1}(\widetilde{X}),\boldsymbol{C}_\eta\right).$$

The latter operator is obtained by the differentiation of entries of the transposition of d_{k+1} in the direction of v at the point η.

Definition 2.2. A family of local systems on a space X parametrized by a topological space D is a homomorphism

$$\pi_1(X)\to C(D)^*, \tag{6}$$

where $C(D)$ is a ring of functions on D.

For any $p\in D$ we have a local system on X obtained by composition of homomorphism (6) with evaluation $C(D)^*\to\boldsymbol{C}^*$ at p.

From this point of view, the local system $\pi_1(X)\to(\operatorname{Spec}\boldsymbol{C}[H_1(X,\boldsymbol{Z})])^*$, given by exponentiation of the additive character of abelianization: $\pi_1(X)\to H_1(X,\boldsymbol{Z})$, can be considered as the (universal) family of local systems on $\operatorname{Spec}\boldsymbol{C}[H_1(X)] = \boldsymbol{C}^{*r}$. The latter identification of the universal family with $\boldsymbol{C}^{*r}$ depends on a choice of generators in $H_1(X,\boldsymbol{Z})$. (If $H_1(X,\boldsymbol{Z})$ has torsion, then $\operatorname{Spec}\boldsymbol{C}[H_1(X)] = \coprod\boldsymbol{C}^{*r}$ with the number of components in the disjoint union being equal to the order of $\operatorname{Tor}(H_1(X,\boldsymbol{Z}))$.)

Let us consider the family of local systems parametrized by $\operatorname{Spec}\boldsymbol{C}[\varepsilon]/\varepsilon^2$ such that the induced map $\operatorname{Spec}\boldsymbol{C} = \operatorname{Spec}\boldsymbol{C}[\varepsilon]/\varepsilon\to\operatorname{Spec}\boldsymbol{C}[\boldsymbol{Z}^r]$ corresponds to a fixed point $\eta\in\boldsymbol{C}^{*r}$

belonging to V_m^k. Recall that each such family can be identified with a tangent vector to $\boldsymbol{C}^{*r}$ at η. Indeed, $\operatorname{Spec} \boldsymbol{C}[\varepsilon]/\varepsilon^2 \to \boldsymbol{C}^{*r}$ is equivalent to the homomorphism $\boldsymbol{C}[\boldsymbol{Z}^r] \to \boldsymbol{C}[\varepsilon]/\varepsilon^2$ which (after fixing η) is specified by the map $\boldsymbol{C}[\boldsymbol{Z}^r] \to \boldsymbol{C}$ given by the coefficient of ε. This map is a differentiation. In particular, for a vector v in the tangent space at η, this homomorphism is given by

$$f \to f(\eta) + \varepsilon \frac{\partial f}{\partial v}(\eta). \tag{7}$$

The homomorphism $\boldsymbol{C}[\boldsymbol{Z}^r] \to \boldsymbol{C}[\varepsilon]/\varepsilon^2$ allows to view $\boldsymbol{C}[\varepsilon]/\varepsilon^2$ as a $\boldsymbol{C}[\boldsymbol{Z}^r]$-module with the multiplication by f given by the right hand side of (7).

On the other hand, the group of units in $\boldsymbol{C}[\varepsilon]/\varepsilon^2$ is isomorphic to the group of upper triangular 2×2 matrices with equal eigenvalues via $a + b\varepsilon \to \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. Therefore a family over $\operatorname{Spec} \boldsymbol{C}[\varepsilon]/\varepsilon^2$ corresponding to (η, v), where v is a tangent to $\operatorname{Spec} \boldsymbol{C}[H_1(X, \boldsymbol{Z})]$ at η vector, is equivalent to a 2-dimensional representation of $H_1(X, \boldsymbol{Z})$ (denoted below $(\boldsymbol{C}[\varepsilon]/\varepsilon^2)_{\eta,v}$). In other words, such a family is equivalent to a rank two local system on X (denoted below $L_{(\eta,v)}$). This rank two local system fits into the exact sequence:

$$0 \to L_\eta \to L_{(\eta,v)} \to L_\eta \to 0. \tag{8}$$

Lemma 2.3.
(1) *The coboundary operator of the cohomology sequence corresponding to* (8) *coincides with the differential* $d^k_{D_{\eta,v}} : H^k(L_\eta) \to H^{k+1}(L_\eta)$ *of the derived complex.*
(2) *The coboundary operator of the cohomology sequence corresponding to* (8) *is the cup product* $H^k(L_\eta) \xrightarrow{v\cup} H^{k+1}(L_\eta)$ *with* $v \in H^1(L_1) = H^1(X, \boldsymbol{C})$ *induced by the identification* $L_1 \otimes L_\eta \to L_\eta$ *where* L_1 *is the trivial local system.*

Proof. (1) The coboundary operator in the cohomology sequence corresponding to (8) is the coboundary operator of the sequence (with $\Lambda = \boldsymbol{C}[H_1(X, \boldsymbol{Z})]$):

$$\begin{aligned} 0 \to\ & \operatorname*{Hom}_{\Lambda}\big(C_\bullet(\widetilde{X}), \boldsymbol{C}_\eta\big) \to \operatorname*{Hom}_{\Lambda}\big(C_\bullet(\widetilde{X}), \big(\boldsymbol{C}[\varepsilon]/\varepsilon^2\big)_{(\eta,v)}\big) \\ & \to \operatorname*{Hom}_{\Lambda}\big(C_\bullet(\widetilde{X}), \boldsymbol{C}_\eta\big) \to 0. \end{aligned} \tag{9}$$

We shall denote this coboundary operator in (9) as $\delta^\bullet$. Let ϕ be a k-cocycle of L_η. We need to check the following:

$$\delta^k(\phi) = \frac{\partial d^{k+1}}{\partial v}(\eta)(\phi) \tag{10}$$

modulo coboundaries:

$$\operatorname{Im} d^{k+1}(\eta)\Big(\operatorname*{Hom}_{\Lambda}\big(C_\bullet(\widetilde{X}), \boldsymbol{C}_\eta\big)\Big).$$

If $\tilde{\phi}$ is a lift of ϕ into $\operatorname{Hom}_\Lambda(C_k(\widetilde{X}), (\boldsymbol{C}[\varepsilon]/\varepsilon^2)_{(\eta,v)})$ then $\delta^k(\phi) = d^{k+1}(\tilde{\phi})$ considered as the element of $\operatorname{Hom}_\Lambda(C_{k+1}(\widetilde{X}), \boldsymbol{C}_\eta)$. Let $\tilde{\phi} = \phi + \varepsilon\psi,\ \phi, \psi \in \operatorname{Hom}_\Lambda(C_k(\widetilde{X}), \boldsymbol{C}_\eta)$. Then

$$d^{k+1}(\phi + \varepsilon\psi) = d^{k+1}(\phi) + \varepsilon\left(d^{k+1}(\psi) + \frac{\partial d^{k+1}}{\partial v}(\eta)(\phi)\right).$$

In the case $\operatorname{rk} C_k(\widetilde{X}) = \operatorname{rk} C_{k+1}(\widetilde{X}) = 1$ this follows since the Λ-module structure on $\boldsymbol{C}[\varepsilon]/\varepsilon^2$ is given by (7) and the general case follows from the rank one case since the modules of chains are Λ-free. Since ϕ is a cocycle $d^{k+1}\phi = 0$ and the ε-component of $d^{k+1}(\phi + \varepsilon\psi)$ is $\delta^k(\phi)$ this yields the claim.

(2) The coboundary operator of the long exact sequence corresponding to (8) is the cup product with the extension class defining (8). Under the identification $\operatorname{Ext}^1(L_\eta, L_\eta) = H^1(X, \boldsymbol{C})$ this extension class is equal to v. This yields the claim. □

3. Tangent cones

Recall that the Zariski tangent space to an algebraic variety V at a point η is the subset of $\operatorname{Hom}(\operatorname{Spec} \boldsymbol{C}[\varepsilon]/\varepsilon^2, V)$ consisting of morphisms taking the closed point of $\operatorname{Spec} \boldsymbol{C}[\varepsilon]/\varepsilon^2$ to η. Recall also, that the tangent cone to V at a point η is a subset of the Zariski tangent space to V given by the initial forms of the defining equation of V at η or $TC_\eta(V) = \operatorname{Spec} \oplus \mathcal{I}_V \cap \mathcal{M}_\eta^k / \mathcal{I}_V \cap \mathcal{M}_\eta^{k+1}$ where $\mathcal{M}_\eta$ is the maximal ideal of η and $\mathcal{I}_V$ is the ideal of V.

The result of this section is the following:

Theorem 3.1. *The tangent cone to the variety V_m^k at point η is the subset* (*possibly proper*) *of $H^1(X, \boldsymbol{C})$ consisting of vectors v such that* $\dim H^k(D^\bullet_{\eta,v}) \geqslant m$:

$$TC_\eta(V_m^k) \subseteq \{ v \in H^1(X, \boldsymbol{C}) \mid \dim H^k(D^\bullet_{\eta,v}) \geqslant m \}.$$

Proof. The proof, which we give for completeness, follows [6] closely. First, note that the condition $\theta \in V_m^k$ is equivalent to

$$\operatorname{rk} d^k(\theta) + \operatorname{rk} d^{k+1}(\theta) \leqslant b_k - m.$$

Hence if

$$V_{a,d^k} = \{ \theta \in \operatorname{Spec} \boldsymbol{C}[H_1(X, \boldsymbol{Z})] \mid \operatorname{rk} d^k(\theta) \leqslant a - 1 \},$$
$$V_{b,d^{k+1}} = \{ \theta \in \operatorname{Spec} \boldsymbol{C}[H_1(X, \boldsymbol{Z})] \mid \operatorname{rk} d^{k+1}(\theta) \leqslant b - 1 \},$$

then

$$V_m^k = \bigcap_{\substack{a+b=b_k-m+1,\\ a,b\geqslant 0}} V_{a,d^k} \cup V_{b,d^{k+1}} \tag{11}$$

Therefore V_m^k is the set of zeros of the ideal:

$$\mathcal{I} = \mathcal{I}(V_m^k) = \sum_{a+b=b_k-m+1} \mathcal{I}_a(d^k) \cdot \mathcal{I}_b(d^{k+1}), \tag{12}$$

where $\mathcal{I}_a(d^k)$ is the ideal in $\boldsymbol{C}[H_1(X, \boldsymbol{Z})]$ generated by the $a \times a$ minors in the matrix of d^k.

The tangent cone to V_m^k at η hence is the homogeneous ideal in $\bigoplus_{l=0}^{\infty} \mathcal{M}_\eta^l / \mathcal{M}_\eta^{l+1} = \mathrm{Sym}(H^1(X, \boldsymbol{C}))$ given by:

$$\bigoplus_l \big(\mathcal{I}\big(V_m^k\big) \cap \mathcal{M}_\eta^l / \mathcal{I}\big(V_m^k\big) \cap \mathcal{M}_\eta^{l+1}\big). \tag{13}$$

Similarly, if $h = \dim H^k(L_\eta)$, then the ideal I

$$I = \sum_{a+b=h-m+1} I_a\big(\delta^k\big) \cdot I_b\big(\delta^{k+1}\big),$$

where $\delta^\bullet = d^\bullet_{\eta,v}$ are the differentials of the complex $D^\bullet_{\eta,v}$, has as its set of zeros the cone $\{v \in H^1(X, \boldsymbol{C}) \mid \dim H^k(D^\bullet_{\eta,v}) \geqslant m\}$. In the latter formula $I_a(\delta^k)$ is the ideal defined by the vanishing of the determinants in the matrix of the map $H^k(L_\eta) \to H^{k+1}(L_\eta) \otimes T_\eta^*$, whose entries are elements in T_η^*. The latter map is such that the image of $\alpha \in H^k(L_\eta)$ evaluted on v is equal to $d^k_{\eta,v}(\alpha) = \delta^k(\alpha)$.

Since

$$\bar{\delta}^k : \operatorname{Ker} d^{k+1}(\eta) \to \operatorname{Coker} d^{k+1}(\eta) \otimes T_\eta^*$$

and

$$\delta^k : \operatorname{Ker} d^{k+1}(\eta) / \operatorname{Im} d^k(\eta) \to \operatorname{Ker} d^{k+2}(\eta) / \operatorname{Im} d^{k+1}(\eta) \otimes T_\eta^*$$

have the same rank we can rewrite the ideal I in a form more suitable for comparison of its zero set with the tangent cone for the zero set of $\mathcal{I}$:

$$I = \sum_{a+b=h-m+1} I_a\big(\bar{\delta}^k\big) \cdot I_b\big(\bar{\delta}^{k+1}\big).$$

Let $r = \operatorname{rk} d^k(\eta)$. In the bases $e_1, \ldots, e_r, \ldots, e_{b_k}$ and $f_1, \ldots, f_r, \ldots, f_{b_{k-1}}$, chosen in $C_k(\widetilde{X})$ and $C_{k-1}(\widetilde{X})$, respectively so that at η one has $d_k(e_i) = f_i$ for $i = 1, \ldots, r$ and $d_k(e_i) = 0$ for $i = r+1, \ldots, b_k$, the matrix of d_k has the form:

$$M(d_k) = \begin{pmatrix} 1 + F_{11} & F_{12} & \ldots & F_{1r} & F_{1(r+1)} & \ldots & F_{1b_{k-1}} \\ F_{21} & 1 + F_{22} & \ldots & F_{2r} & F_{2(r+1)} & \ldots & F_{2b_{k-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_{r1} & F_{r2} & \ldots & 1 + F_{rr} & F_{r(r+1)} & \ldots & F_{rb_{k-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_{b_k1} & F_{b_k2} & \ldots & F_{b_kr} & F_{b_k(r+1)} & \ldots & F_{b_kb_{k-1}} \end{pmatrix}$$

for appropriate polynomials in the local parameters of η. The matrix of $\bar{\delta}_k$ (dual to $\bar{\delta}^k$) is:

$$M(\bar{\delta}_k) = \begin{pmatrix} dF_{(r+1)(r+1)} & \ldots & dF_{(r+1)b_{k-1}} \\ \vdots & \ddots & \vdots \\ dF_{b_k(r+1)} & \ldots & dF_{b_kb_{k-1}} \end{pmatrix}.$$

A minor of size $a \times a$ in $M(d_k)$ can have a leading term of degree $a - r$ only if it contains the first r rows and columns and has a bigger degree otherwise. This term of degree $a - r$ is the leading term of a minor of an order $(a - r) \times (a - r)$ in the matrix $M(\bar{\delta}_k)$. Hence the graded component of degree $a - r$ in the ideal generated by the minors of size $a \times a$ in $M(d_k)$ is the same as the graded component of degree $a - r$ in the ideal generated by minors of order $a - r$ in the matrix of $M(\bar{\delta}_k)$: $\mathrm{gr}_{a-r}\,\mathcal{I}_a(V_m^k) = \mathrm{gr}_{a-r}\,I_{a-r}(\delta^k)$ (the degree and the grading in $\mathcal{I}(V_m^k)$ is taken with respect to the local parameters of η as in (13)).

This implies that the graded component of degree $h - m + 1$ in the ideal of $\mathcal{I}(V_m^k)$ is equal to the graded component of the same degree in the ideal I. The claim follows since I is generated by polynomials of degree $h - m + 1$ (because all entries of $M(\bar{\delta}^k)$ are linear and not all of them are zeros if m is such that $\dim H^k(L_\theta) = m$ at a generic point of V_m^k which we tacitly assume). The inclusion $I \subseteq \mathrm{Gr}_*(\mathcal{I}(V_m^k))$ will be proper if $\mathrm{Gr}_*(\mathcal{I}(V_m^k))$ will require generators of degree exceeding $h - m + 1$.

Remark 3.2. If $h = m$ then the graded component I_1 and $\mathrm{Gr}_1\,\mathcal{I}(V_m^k) \cap \mathcal{M}_\eta$ coincide, i.e., the Zariski tangent spaces involved are the same.

Theorem 3.1 and Remark 3.2 yield Theorem 1.1 in the introduction.

4. Examples

4.1. Tangent space outside of identity

Let X be the complement in $\boldsymbol{C}^2$ to the union of lines $\mathcal{A}$: $x \cdot y \cdot (x - y) = 0$. The characteristic variety V_1^1 is the subtorus in $H^1(X, \boldsymbol{C}^*) = \boldsymbol{C}^3$ given by $t_1 t_2 t_3 = 1$ (cf. [10]). Let $\eta = (\omega_3, \omega_3, \omega_3) \in V_1^1$ where ω_3 is a primitive root of 1 of degree 3. We have $\mathrm{rk}\, H^0(L_\eta) = 0, \mathrm{rk}\, H^1(L_\eta) = 1$ (cf. [10]) and also $\mathrm{rk}\, H^2(L_\eta) = 1$ since for the Euler characteristic we have $\chi(L_\eta) = \chi(L_1) = 0$. The complex in question hence is:

$$0 \to \boldsymbol{C} \xrightarrow{\wedge\omega} \boldsymbol{C} \to 0.$$

To calculate the cup product with the elements $\omega \in H^1(X, \boldsymbol{C})$, let us use the isomorphism:

$$H^i(X_3, \boldsymbol{C}) \cong H^i(L_1) \oplus H^i(L_\eta) \oplus H^i(L_{\eta^2}),$$

where X_3 is the threefold cyclic cover of $\boldsymbol{C}^2 - \mathcal{A}$. X_3 is a $\boldsymbol{C}^*$ fibration over an elliptic curve (3-fold cyclic cover of the line at infinity branched over 3 points) with three branch points removed. The cup product in $H^*(X_3)$ is given by the Künneth formula. One obtains a two-dimensional space of ω's for which the middle map in the above sequence is trivial. This space is the tangent space to V_1^1 at η.

4.2. Calculation of $\mathcal{V}_m^k$ at the trivial local system for arrangements

This calculation is straightforward and is combinatorial (i.e., it depends only on the lattice of the arrangement) since it depends only on the cohomology algebra $H^*(\boldsymbol{P}^n - \mathcal{A})$.

4.3. Examples of arrangements with non-trivial V_m^k for $k > 1$

Let $\mathcal{A}$ be an arrangement in $\boldsymbol{P}^n$ and L a local system such that $L \in V_m^k(\mathcal{A})$ but $L \notin V_1^i$ for $i \leqslant k-1$. The simplest example of such is the generic arrangement with the local system corresponding to the character of the fundamental group taking non trivial values on generators dual to the hyperplanes of the arrangement. Let $\mathcal{B}$ be an arrangement in $\boldsymbol{P}^{n+m}$ such that for some subspace $B \in \mathcal{B}$ the projection π from B onto $\boldsymbol{P}^n$ has as its discriminant the arrangement $\mathcal{A}$ and is a locally trivial fibration (with fiber being the complement to an arrangement in $\boldsymbol{P}^m$). Then $\operatorname{rk} H^k(\boldsymbol{P}^{n+m} - \mathcal{B}, \pi^*(L)) \geqslant m$. In particular $\pi^*(L) \in V_m^k(\pi^*(L))$.

Indeed, let us consider the Leray spectral sequence corresponding to π:

$$E_2^{p,q} = H^p\big(\boldsymbol{P}^n - \mathcal{A}, R_*^q\pi\big(\pi^*(L)\big)\big) \Rightarrow H^{p+q}\big(\boldsymbol{P}^{n+m} - \mathcal{B}, \pi^*(L)\big).$$

Since, by projection formula $R_*^q\pi(\pi^*(L)) = L \otimes R_*^q(\boldsymbol{C})$, $H^i(L) = 0$ for $i < k$ and $R_*^q(\boldsymbol{C}) = H^q(\pi^{-1}(p))$ for a generic $p \in \boldsymbol{P}^n - \mathcal{A}$, we see that $H^k(\pi^*(L)) = E_2^{k,0} = H^k(L)$ and the claim follows.

For example, one can start with a generic arrangement and consider either the cone over it or delete several hyperplanes which intersections belong to the hyperplane of the cone (to keep projection a local trivial fibration). This construction in the case when $\mathcal{B}$ is an arrangement of lines produces non-essential (cf. [12]) components corresponding to points of $\mathcal{B}$ having multiplicity greater than 2. This construction also includes the case of products of affine arrangements.

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