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Hodge decomposition of Alexander invariants

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Abstract. Multivariable Alexander invariants of algebraic links calculated in terms of algebro-geometric invariants (polytopes and ideals of quasiadjunction). The relations with log-canonical divisors, the multiplier ideals and a semicontinuity property of polytopes of quasiadjunction are discussed.

1. Introduction

This paper is concerned with two interrelated issues. The first is an algebraic calculation of multivariable Alexander invariants of the fundamental groups of links of plane curves singularities $f_1(x, y) \cdots f_r(x, y)$ with several branches or more precisely the corresponding *characteristic varieties*. The second is the effect of Hodge theory on the structure of these invariants.

Characteristic varieties are attached to a topological space X whose fundamental group admits a surjection on a free abelian group $\mathbf{Z}^r$. They can be defined as follows (cf. [12]). To a surjection onto $\mathbf{Z}^r$ corresponds the abelian cover $\tilde{X}$ for which $\mathbf{Z}^r$ is the group of deck transformations. Consider $H_1(\tilde{X}, \mathbf{C})$ as a module over the group ring of $\mathbf{Z}^r$ and let Φ be the map in a presentation $\mathbf{C}[\mathbf{Z}^r]^m \xrightarrow{\Phi} \mathbf{C}[\mathbf{Z}^r]^n \longrightarrow H_1(\tilde{X}, \mathbf{C}) \longrightarrow 0$ of $\mathbf{C}[\mathbf{Z}^r]$-module $H_1(\tilde{X}, \mathbf{C})$ by generators and relations. The i-th Fitting ideal of $H_1(\tilde{X}, \mathbf{C})$ is the ideal generated by the minors of order $n - i + 1$ in the matrix Φ. The ring $\mathbf{C}[\mathbf{Z}^r]$ can be viewed as the ring of regular functions on the torus $\mathbf{C}^{*r}$ and the i-th characteristic variety of X is defined as the zero set in this torus of the i-th Fitting ideals of the module $H_1(\tilde{X}, \mathbf{C})$.

In the case when X is the complement to a link L with r-components the group $H_1(S^3 - L, \mathbf{Z})$ is isomorphic to $\mathbf{Z}^r$. The first Fitting ideal is a product of a power of the maximal ideal $\mathcal{M}$ of the point in $\mathbf{C}^{*r}$ corresponding to the identity element of the torus and a principal ideal generated by a polynomial $\Delta(t_1, \ldots, t_r)$ (cf. [3]). The latter is called the Alexander polynomial of L. In particular, the first characteristic variety has codimension 1 in $\mathbf{C}^r$. For $i > 1$ the characteristic varieties may have a higher codimension. For any positive $r' < r$ and surjection $H_1(S^3 - L, \mathbf{Z}) \rightarrow \mathbf{Z}^{r'}$, one has the Alexander polynomial of r' variables and characteristic

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varieties in $\mathbf{C}^{*r'}$ which can be found from those of r variables (cf. [27]). The one variable Alexander polynomial can be obtained from this construction applied to the surjection $H_1(S^3 - L, \mathbf{Z}) \to \mathbf{Z}$ given by evaluating the total linking number of loops with L.

If $r = 1$, *all* Fitting ideals are principal and their generators yield a sequence of polynomials $\Delta_i \in \mathbf{C}[t, t^{-1}]$ (defined up to a unit of the latter) such that $\Delta_{i+1}|\Delta_i$. In the case when L is the link of a singularity $f = 0$, both the algebraic calculation of Alexander invariants and their relation to the Hodge theory are well known. Indeed, the corresponding infinite cover of $S^3 - L$ is cyclic, it can be identified with the Milnor fiber of the singularity, and in this identification the deck transformation is the monodromy operator of the singularity. The group H^1 of the Milnor fiber supports a mixed Hodge structure with weights 0,1 and 2, with the identification $N : W_2/W_1 \to W_0$ given by the logarithm of an appropriate power of the monodromy (cf. [24]). All Hodge groups are invariant under the action of the semisimple part of the latter. Let $h^{p,q}_{\zeta}$ (cf. [24]) be the dimension of the eigenspace of this semisimple part acting on the space $H^{p,q}$. These numbers determine the Jordan form of the monodromy as follows. The size of the Jordan blocks of the monodromy does not exceed 2 and the number of blocks corresponding to an eigenvalue ζ of size 1×1 (resp. 2×2) is equal to $h^{1,0}_{\zeta} + h^{0,1}_{\zeta}$ (resp. $h^{0,0}_{\zeta}$). As a consequence:

$$\Delta_i = \prod_{(\zeta)} (t - \zeta)^{a_{\zeta,i}},$$

where

$$a_{\zeta,i} = \begin{cases} h^{1,0}_{\zeta} + h^{0,1}_{\zeta} + 2h^{0,0}_{\zeta} - 2(i-1) & \text{if } 1 \le i \le h^{0,0}_{\zeta} \\ h^{1,0}_{\zeta} + h^{0,1}_{\zeta} - (i - 1 - h^{0,0}_{\zeta}) & \text{if } h^{0,0}_{\zeta} < i \le h^{0,0}_{\zeta} + h^{1,0}_{\zeta} + h^{0,1}_{\zeta} \\ 0 & \text{if } i > h^{0,0}_{\zeta} + h^{1,0}_{\zeta} + h^{0,1}_{\zeta} \end{cases}.$$

All Δ_i can be calculated algebraically in terms of a resolution of the singularity. For Δ_1 this follows from A'Campo's formula for the ζ-function of the monodromy (cf. [2]) and for Δ_i and $i \ge 2$ from Steenbrink's calculation of the Hodge numbers of Mixed Hodge structure on the cohomology of the Milnor fiber (cf. [24]).

Multivariable Alexander polynomials were studied extensively from a topological point of view. They can be found either from a presentation of the fundamental group $\pi_1(S^3 - L)$ using Fox calculus or using an iterative procedure based on the fact that algebraic links are iterated torus links (cf. [26,5], cf. also [20] where an upper bound for the set of zeros of the multivariable Alexander polynomial was obtained algebraically).

In this paper we describe an algebraic procedure for calculating the characteristic varieties. In fact we study a finer than characteristic variety invariant of a singularity having a given algebraic link. This invariant is a collection of polytopes in $\mathbf{R}^r$. These polytopes are equivalent to the local polytopes of quasiadjunction introduced in [15] but are more convenient in the local case. We show that the characteristic varieties are algebraic closures of the images of the faces of these polytopes of quasiadjunction under exponential map: $exp : \mathbf{R}^r \to \mathbf{C}^{*r}$. In the case $r = 1$, polytopes of quasiadjunction are segments having 1 as the right end and faces of quasiadjunction are points in [0,1] which are the left ends of these segments. These points are elements of the Arnold-Steenbrink spectrum of the singularity. (The analogy is going further: we prove in 4.1 a semicontinuity property of faces of quasiadjunction extending one of well known semicontinuity properties of spectra ([25], [28]). Description of characteristic varieties via faces of quasiadjunction is obtained by expressing the mixed Hodge structure on the cohomology of the abelian covers of S^3 branched over the link of singularity in terms of certain ideals in the local ring of singularity (ideals of quasiadjunction). These ideals are the key ingredient in the description of the characteristic varieties of the fundamental groups of the complements to curves in $\mathbf{P}^2$ (cf. [13,15]) and are generalizations of the ideals of quasiadjunction in $r = 1$ case (cf. [11,17]). The ideals of quasiadjunction (in both $r = 1$ and $r > 1$) cases are closely related to more recently introduced multiplier ideals (cf. [19] and Remark 2.6). On the other hand, following an idea of J. Kollar (cf. [10]), we show how log-canonical thresholds of certain divisors can be found in terms of polytopes of quasiadjunction studied here and in [15] (cf. Sect. (4.2)).

In section 4 we point out the effect of the Hodge theory on the homology of the infinite abelian covers of the complements to links. Note that these homology groups typically are infinite dimensional. We show that the intersections of the torus of unitary characters of $\mathbf{Z}^r$ with irreducible components of the space of characters appearing in the representation of $\mathbf{Z}^r$ on $H_1(\tilde{X}, \mathbf{C})$ have natural decomposition into a union of two connected subsets compatible with the Hodge decomposition of the finite abelian covers and illustrate such decomposition by explicit examples. I thank J. Cogolludo for sharing with me his results on Fox calculus calculations in example 2.

Finally note, that polytopes and ideals of quasiadjunction considered here have a natural generalization for arbitrary hypersurface singularities with applications generalizing [14]. We shall return to this elsewhere.

2. Invariants of singularities

2.1. Characteristic varieties of algebraic links

Let X, as in Introduction (cf. also [12,15,20]), be a finite CW complex such that $H_1(X, \mathbf{Z}) = \mathbf{Z}^r$. Let $t_1, \dots, t_r$ be a system of generators of the latter. The homology $H_1(\tilde{X}, \mathbf{C})$ of the universal abelian cover $\tilde{X}$ has a structure of $\Lambda = \mathbf{C}[H_1(X, \mathbf{Z})] = \mathbf{C}[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]$-module. Let $F_i(X)$ be the i-th Fitting ideal of $H_1(\tilde{X}, \mathbf{C})$ considered as a Λ-module, i.e. the ideal generated by $(n - i + 1) \times (n - i + 1)$ minors of the matrix of a map $\Phi : \Lambda^m \to \Lambda^n$ such that $\mathrm{Coker}\Phi = H_1(\tilde{X}, \mathbf{C})$.

We shall view $\mathbf{C}[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]$ as a ring of regular functions on a torus $\mathbf{C}^{*r}$ so that the set of zeros of an ideal $F_i(X)$ is a subvariety of $\mathbf{C}^{*r}$ denoted $V_i(X)$. The maximal possible i is called the *depth* of V_i. A translated subgroup of $\mathbf{C}^{*r}$ is an irreducible component of an intersection of codimension one submanifolds given by $t_1^{l_1} \cdots t_r^{l_r} = \lambda (l_i \in \mathbf{Z})$. The following is a local analog of results of [1].

Proposition 2.1. *The characteristic varieties of algebraic links are unions of translated subgroups.*

Proof. Recall that an algebraic link can be obtained from a trivial knot by iteration of cablings $X_1 \to \cdots \to X_N$ where X_1 is a complement to a small tube about the unknot, and the complement X_n to a link L_n is obtained from the complement X_{n-1} to the link L_{n-1} via replacing X_{n-1} by a space $X_{n-1} \cup_{\partial T_n} Y_{n-1}$. Here Y_{n-1} is one of two standard model spaces: the complement in a torus T_n either to a torus knot or to the union of the axis of the torus and the torus knot (cf. [26]). The union is taken by identifying the boundary of a tube about L_{n-1} with ∂T_n. It follows from [26] that for the homology of the universal abelian cover $\tilde{X}_n$ we have:

$$H_1(\tilde{X}_n) = H_1(\tilde{X}_{n-1}) \oplus H_1(\tilde{Y}_{n-1})$$

(cf. p. 118 and p. 119 in [26] for two possible cases of cablings). We can assume by induction that the characteristic varieties of $\tilde{X}_{n-1}$ and $\tilde{Y}_{n-1}$ are the unions of translated subgroups. We have $V_k(X_n) = \mathrm{Supp}(\Lambda^k H_1(\tilde{X}_n))$ (cf. [15]). $\Lambda^k H_1(\tilde{X}_n)$ has a filtration with successive factors $\Lambda^i H_1(\tilde{X}_{n-1}) \otimes \Lambda^{k-i} H_1(\tilde{Y}_{n-1})$. Hence, by [22], $V_k(\tilde{X}_n) = \cap_i V_i(\tilde{X}_{n-1}) \cup V_{k-i}(\tilde{Y}_{n-1})$. In particular, if $V_i(\tilde{X}_{n-1})$ and $V_{k-i}(\tilde{Y}_{n-1})$ are unions of translated subgroups, then so is $V_k(\tilde{X}_n)$.

2.2. *Ideals of log-quasiadjunction*

Let B be a small ball about the origin O in $\mathbf{C}^2$ and let C be a germ of a plane curve having at O singularity with r branches. Let $f_1(x, y) \cdots f_r(x, y) = 0$ be a local equation of this curve (each f_i is assumed irreducible). An abelian cover of type $(m_1, \dots, m_r)$ of ∂B (resp. B) is the branched cover of ∂B (resp. B) corresponding to a homomorphism $\pi_1(\partial B - \partial B \cap C) \to \mathbf{Z}/m_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/m_r\mathbf{Z}$ (resp. the cone over the abelian cover of ∂B). Such cover of ∂B is the link of complete intersection surface singularity:

$$V_{m_1,\dots,m_r}: \quad z_1^{m_1} = f_1(x, y), \dots, z_r^{m_r} = f_r(x, y) \tag{1}$$

The covering map is given by $p : (z_1, \dots, z_r, x, y) \to (x, y)$.

An ideal of quasiadjunction of type $(j_1, \dots, j_r|m_1, \dots, m_r)$ (cf. [13,15]) is the ideal in the local ring of the singularity of C (i.e. $O \in \mathbf{C}^2$) consisting of germs ϕ such that the 2-form:

$$\omega_\phi = \frac{\phi z_1^{j_1} \cdots z_r^{j_r} dx \wedge dy}{z_1^{m_1-1} \cdots z_r^{m_r-1}} \tag{2}$$

extends to a holomorphic form on a resolution of the singularity of the abelian cover of a ball B of type $(m_1, \dots, m_r)$, i.e. a resolution of (1) (we suppress dependence of ω_ϕ on $j_1, \dots, j_r, m_1, \dots, m_r$). In other words, $\phi z_1^{j_1} \cdots z_r^{j_r}$ belongs to the adjoint ideal of (1) (cf. [18]). In particular the condition on ϕ is independent of resolution. We always shall assume that $0 \le j_1 < m_1, \dots, 0 \le j_r < m_r$.

An ideal of log-quasiadjunction (resp. *an ideal of weight one log-quasiadjunction*) of type $(j_1, \dots, j_r|m_1, \dots, m_r)$ is the ideal in the same local ring consisting of germs ϕ such that ω_ϕ extends to a log-form (resp. weight one log-form) on a resolution of the singularity of the same abelian cover. Recall (cf. [4]) that a holomorphic

2-form is weight one log-form if it is a combination of forms having poles of order at most one on each component of the exceptional divisor and not having poles of order one on a pair of intersecting components. These ideals are also independent of a resolution. This follows from the following. Let ω be a holomorphic n-form on a complex space $X \subset \mathbf{C}^N$, $\dim X = n$ with isolated singularity at $x \in X$ and let B_r be a ball about x in $\mathbf{C}^N$ having a radius r. ω extends to a form of weight k on a resolution of X, for which the exceptional divisor has at worst normal crossings, if and only if for sufficiently small $R \gg r > 0$ one has

$$\int_{B_R - B_r} \omega \wedge \bar{\omega} < C|\log r|^k$$

In particular for $k = 0$ one obtains Lemma 1.3 (ii) from [18]. The general case follows, for example, by interpreting the above integral as an integral over the neighborhood of the exceptional locus in a resolution of $x \in X$ and reducing it to the integral over the boundary of $B_R - B_r$. Local calculations near intersection of k components show that the contribution $\int_{r \leq z_1 \leq 1, \ldots, r \leq z_k \leq 1, 0 \leq z_{k+1} \leq 1, 0 \leq z_n \leq 1} \frac{dz_1 \cdots dz_n \wedge d\bar{z}_1 \cdots d\bar{z}_n}{z_1 \bar{z}_1 \cdots z_n \bar{z}_n} < C|\log r|^k$ which yields the estimate as above (similarly to [18]). This characterization gives independence of a particular resolution for both the ideals of log-quasiadjunction and the ideals of weight one log-quasiadjunction.

It is shown in [15] that an ideal of quasiadjunction $\mathcal{A}(j_1, \ldots, j_r | m_1, \ldots, m_r)$ is determined by the vector:

$$\left(\frac{j_1 + 1}{m_1}, \ldots, \frac{j_r + 1}{m_r}\right). \tag{3}$$

This is also the case for the ideals of log-quasiadjunction and weight one log-quasiadjunction. Indeed, these ideals can be described as follows. For a given embedded resolution $\pi : V \to \mathbf{C}^2$ of the germ $f_1 \cdots f_r = 0$ with the exceptional curves $E_1, \ldots, E_k, \ldots, E_s$ let $a_{k,i}$ (resp. c_k, resp. $e_k(\phi)$) be the multiplicity of the pull back on V of f_i ($i = 1, \ldots, r$) (resp. $dx \wedge dy$, resp. ϕ) along E_k. Then ϕ belongs to the ideal of quasiadjunction of type $(j_1, \ldots, j_r | m_1, \ldots, m_r)$ if and only if for any k

$$a_{k,1} \frac{j_1 + 1}{m_1} + \cdots + a_{k,r} \frac{j_r + 1}{m_r} > a_{k,1} + \cdots + a_{k,r} - e_k(\phi) - c_k - 1 \tag{4}$$

(cf. [15]). Similar calculation shows that a germ ϕ belongs to the ideal of log-quasiadjunction corresponding to $(j_1, \ldots, j_r | m_1, \ldots, m_r)$ if and only if the inequality

$$a_{k,1} \frac{j_1 + 1}{m_1} + \cdots + a_{k,r} \frac{j_r + 1}{m_r} \geq a_{k,1} + \cdots + a_{k,r} - e_k(\phi) - c_k - 1 \tag{5}$$

is satisfied for any k. Moreover, a germ ϕ belongs to the ideal of weight one log-quasiadjunction if and only if this germ is a linear combination of germs ϕ satisfying inequality (5) for any collection of k's such that corresponding components do not intersect and satisfying the inequality (4) for k outside of this collection. We shall denote the ideal of quasiadjunction (resp. log-quasiadjunction, resp. weight one log-quasiadjunction) corresponding to $(j_1, \dots, j_r|m_1, \dots, m_r)$ as $\mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$ (resp. $\mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)$, resp. $\mathcal{A}'(j_1, \dots, j_r|m_1, \dots, m_r)$). We have:

$$\mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r) \subseteq \mathcal{A}'(j_1, \dots, j_r|m_1, \dots, m_r) \subseteq \mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)$$

Recall that both (4) and (5) follow from the following calculation (cf. [15], Sect. 2 for complete details). One can use the normalization of the fiber product $\widetilde{V}_{m_1,\dots,m_r} = V \times_{\mathbf{C}^2} V_{m_1,\dots,m_r}$ as a resolution of singularity (1) in the category of manifolds with quotient singularities (cf. [16]). We have:

$$\begin{array}{ccc} \tilde{V}_{m_1,\dots,m_r} & \stackrel{\tilde{p}}{\to} & V \\ \tilde{\pi}\downarrow & & \pi\downarrow \\ V_{m_1,\dots,m_r} & \stackrel{p}{\to} & \mathbf{C}^2. \end{array} \tag{6}$$

The preimage of the exceptional divisor of $V \to \mathbf{C}^2$ in $\widetilde{V}_{m_1,\dots,m_r}$ forms a divisor with normal crossings (cf. [24]), though the preimage of each component is reducible in general (in which case irreducible components above each exceptional curve do not intersect[1]). The order of the vanishing of ω_ϕ on $\widetilde{V}_{m_1,\dots,m_r}$ along E_k is equal to:

$$\Sigma_{i=1}^{i=r}(j_i - m_i + 1)\frac{m_1\cdots\hat{m}_i\cdots m_r\cdot a_{k,i}}{g_{k,1}\cdots g_{k,r}s_k} + \frac{m_1\cdots m_r\cdot ord_{E_k}(\pi^*(\phi))}{g_{k,1}\cdots g_{k,r}\cdot s_k} + \frac{c_k\cdot m_1\cdots m_r}{g_{k,1}\cdots g_{k,r}\cdot s_k} + \frac{m_1\cdots m_r}{g_{k,1}\cdots g_{k,r}\cdot s_k} - 1, \tag{7}$$

where $g_{k,i} = \text{g.c.d.}\ (m_i, a_{k,i})$ and $s_k = \text{g.c.d.}\ (\dots, \frac{m_i}{g_{k,i}}, \dots)$.

A consequence of (7) is that ω_ϕ has an order of pole equal to one (resp. zero) along the component E_k of the above resolution if and only if for such ϕ one has equality in (5) (resp. (4) is satisfied).

Proposition 2.2. 1. *Let $\mathcal{A}''$ be an ideal of log-quasiadjunction. There is a unique polytope $\mathcal{P}(\mathcal{A}'')$ such that a vector $(\frac{j_1+1}{m_1}, \dots, \frac{j_r+1}{m_r}) \in \mathcal{P}(\mathcal{A}'')$ if and only if $\mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)$ contains $\mathcal{A}''$.*[2]

[1] If the Galois group G of $\tilde{p}$ is abelian (as we always assume here) and, in particular, is the quotient of $H_1(B - C \cap B, \mathbf{Z})$, then the Galois group of $\tilde{p}^{-1}(E_i) \to E_i$ is $G/(\gamma_i)$ where for an exceptional curve E_k, γ_k is the image in the Galois group of the homology class of the boundary of a small disk transversal to E_k in V. The components of $\tilde{p}^{-1}(E_i)$ correspond to the elements of $G/(\gamma_i, \dots, \gamma_l, \dots)$ where l runs through indices of all exceptional curves intersecting E_i, while $\tilde{p}_i$ restricted on each component has $(\gamma_i, \dots, \gamma_l, \dots)/(\gamma_i)$ as the Galois group. The points $\tilde{p}^{-1}(E_i \cap E_j)$ correspond to the elements of $G/(\gamma_i, \gamma_j)$ and the points of $\tilde{p}^{-1}(E_i \cap E_j)$ belonging to a fixed component correspond to cosets in $(\gamma_i, \dots, \gamma_l, \dots)/(\gamma_i, \gamma_j)$.

[2] i.e. a subset in $\mathbf{R}^r$ given by a set of linear inequalities $L_s \geq k_s$. We say that an affine hyperplane in $\mathbf{R}^r$ supports a codimension one face of a polytope if the intersection of this

2. *The set of vectors (3) for which* $\mathcal{A}(j_1,\dots,j_r|m_1,\dots,m_r) \neq \mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$ *is a dense subset in the boundary of the polytope having as its closure a union of faces of such a polytope. The closure of the set of vectors (3) for which* $\mathcal{A}'(j_1,\dots,j_r|m_1,\dots,m_r) \neq \mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$ *is also a union of certain faces of such a polytope.*
3. *The ideal* $\mathcal{A}(j_1,\dots,j_r|m_1,\dots,m_r)$ *(resp.* $\mathcal{A}'(j_1,\dots,j_r|m_1,\dots,m_r)$ *and* $\mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$*) is independent of the array* $(j_1,\dots,j_r|m_1,\dots,m_r)$ *as long as the vector (3) varies within the interior of the same face of quasiadjunction.*

We shall call the above faces *the faces of quasiadjunction* (resp. *weight one faces of quasiadjunction*). $\mathcal{A}_\Sigma$ will denote $\mathcal{A}(j_1,\dots,j_r|m_1,\dots,m_r)$ with corresponding vector (3) belonging to the interior of a face of quasiadjunction Σ (similarly for $\mathcal{A}'_\Sigma$ and $\mathcal{A}''_\Sigma$).

Proof. First let us describe the inequalities defining the polytope corresponding to an ideal of log-quasiadjunction $\mathcal{A}''$ with the property described in *1*. For any ideal $\mathcal{B}$ in the local ring of a singular point let $e_k(\mathcal{B}) = \min_{\phi\in\mathcal{B}} \operatorname{ord}_{E_k}(\pi^*(\phi))$. Note that $\phi \in \mathcal{A}''$ (resp. $\phi \in \mathcal{A}$) if and only if

$$\operatorname{ord}_{E_k}(\pi^*(\phi)) \geq e_k(\mathcal{A}') \tag{8}$$

(resp. $\operatorname{ord}_{E_k}(\pi^*(\phi)) \geq e_k(\mathcal{A})$) for all k since if $\phi \in \mathcal{A}''$ it certainly satisfies (8) for each k and vice versa if ϕ satisfies (8) for all k it also satisfies (5) for any k and hence ϕ belongs to ideal $\mathcal{A}''$. The same works for $\mathcal{A}$. We claim that $\mathcal{P}(\mathcal{A}'')$ with property *1* is the subset of the unit cube which consists of solutions of the system of inequalities in $(x_1,\dots,x_r)$:

$$a_{k,1}x_1 + \cdots + a_{k,r}x_r \geq a_{k,1} + \cdots + a_{k,r} - e_k(\mathcal{A}'') - c_k - 1. \tag{9}$$

In order to derive $\mathcal{A}'' \subset \mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$ for (3) satisfying (9) note that if the array $(j_1,\dots,j_r|m_1,\dots,m_r)$ satisfies (9) for all k and if $\phi \in \mathcal{A}''$, i.e. we have $e_k(\phi) \geq e_k(\mathcal{A}'')$ then we also have (5) for all k and hence $\phi \in \mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$. Vice versa, if $\mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$ contains $\mathcal{A}''$ then for any $\phi \in \mathcal{A}''$ and k we have (5) and hence $min_{\phi\in\mathcal{A}''} e_k(\phi)$ satisfies the same inequality. This proves the first part of the proposition. □

For the second part, let us notice that the boundary of the set of solutions of the system (9) is the set of vectors (3) satisfying (9) for a proper subset $\mathcal{S}'$ of the set of exceptional curves $\mathcal{S}$ and the inequalities

$$a_{k,1}x_1 + \cdots + a_{k,r}x_r > a_{k,1} + \cdots + a_{k,r} - e_k(\mathcal{A}'') - c_k - 1 \tag{10}$$

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hyperplane with the boundary of the polytope has dimension $r-1$. A face of a polytope is the intersection of a supporting face of the polytope with the boundary. A codimension one face of a polytope in $\mathbf{R}^r$ is a polytope of dimension $r-1$. By induction one obtains faces of arbitrary codimension for original polytope (for $r=3$ those are called edges and vertices). The boundary of the polytope is the union of its faces.

for $k \in \mathcal{S} - \mathcal{S}'$. For an array $(j_1, \dots, j_r|m_1, \dots, m_r)$, with the corresponding vector (3) in the boundary of $\mathcal{P}(\mathcal{A}'')$, let E_k be a component with $k \in \mathcal{S} - \mathcal{S}'$. If $\phi \in \mathcal{A}''$ is a germ such that $e_k(\phi)$ yields equality in (5), then the form ω_ϕ has pole of order exactly one along E_k. Hence $\phi \notin \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$. Vice versa, if $\mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r) \neq \mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)$ then there exists a form ω_ϕ having a pole of order exactly one along a component E_k. Hence, since $e_k(\phi) = \min\{ord_{E_k}\psi | \psi \in \mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)\}$, the corresponding vector (3) belongs to the boundary of the polytope of $\mathcal{A}''(j_1, \dots, j_r|m_1, \dots, m_r)$.

The vectors (3) corresponding to arrays having distinct ideals $\mathcal{A}$ and $\mathcal{A}'$ belong to faces which are not in the intersection of a pair of codimension one faces corresponding to intersecting exceptional curves (i.e. vectors (3) for which one has equality in (9) for a pair of indices such that $E_k \cap E_l \neq \emptyset$).
Finally, 3 follows since the inequalities imposed by (3) on $ord_{E_k}\phi$ and defining the ideals of quasiadjunction are the same for vectors in the interior of each face of quasiadjunction.

Proposition 2.3. *Any ideal of quasiadjunction is an ideal of log-quasiadjunction (but for a different array $(j_1, \dots, j_r|m_1, \dots, m_r)$) and vice versa.*

Proof. Let $\mathcal{A} = \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$ be an ideal of quasiadjunction and let $(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$ be an array such that the corresponding vector (3) belongs to the boundary of the set of solutions of the system of inequalities (for any $k \in \mathcal{S}$):

$$a_{k,1}x_1 + ... + a_{k,r}x_r \geq a_{k,1} + ... + a_{k,r} - e_k(\mathcal{A}) - c_k - 1 \tag{11}$$

We claim that $\mathcal{A} = \mathcal{A}''(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$. Indeed if $\phi \in \mathcal{A}$ then $ord_{E_k}(\phi) \geq e_k(\mathcal{A})$ together with (11) yields $\phi \in \mathcal{A}'(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$. To get the opposite inclusion $\mathcal{A}'(j'_1, \dots, j'_r|m'_1, \dots, m'_r) \subseteq \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$ notice that the vector (3) corresponding to $(j_1, \dots, j_r|m_1, \dots, m_r)$ is in the interior of the set of solutions of (11) since this vector cannot satisfy equality in the system (11) because otherwise for ϕ such that $e_k(\phi) = e_k(\mathcal{A})$ we shall have $a_{k,1}\frac{j_1+1}{m_1} + \cdots + a_{k,r}\frac{j_r+1}{m_r} = a_{k,1} + \cdots + a_{k,r} - e_k(\phi) - c_k - 1$ contradicting to $\phi \in \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$. Therefore for $\phi \in \mathcal{A}''(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$ we shall have: $e_k(\phi) \geq -a_{k,1}\frac{j'_1+1}{m'_1} - \cdots - a_{k,r}\frac{j'_r+1}{m'_r} + a_{k,1} + \cdots + a_{k,r} - c_k - 1 > -a_{k,1}\frac{j_1+1}{m_1} - \cdots - a_{k,r}\frac{j_r+1}{m_r} + a_{k,1} + \cdots + a_{k,r} - c_k - 1$ i.e. $\phi \in \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$. $\square$

Now let us show that any ideal of log-quasiadjunction, say $\mathcal{A}''(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$, is an ideal of quasiadjunction. Let us choose the array $(j_1, \dots, j_r|m_1, \dots, m_r)$ so that for the corresponding vector (3) none of intervals $-a_{k,1}\frac{j'_1+1}{m'_1} - \cdots - a_{k,r}\frac{j'_r+1}{m'_r} + a_{k,1} + \cdots + a_{k,r} - c_k - 1 > x > -a_{k,1}\frac{j_1+1}{m_1} - \cdots - a_{k,r}\frac{j_r+1}{m_r} + a_{k,1} + \cdots + a_{k,r} - c_k - 1$ contains an integer for all k. Then $\phi \in \mathcal{A}''(j'_1, \dots, j'_r|m'_1, \dots, m'_r)$ is equivalent to $\phi \in \mathcal{A}(j_1, \dots, j_r|m_1, \dots, m_r)$.

The following description of the ideals of quasiadjunction is useful for explicit calculations of the polytopes introduced above and their faces.

Proposition 2.4. *Let $\pi : V \to \mathbf{C}^2$ be a composition of blow ups with the exceptional set $E_1 \cup \cdots \cup E_k$ such that $\pi(\bigcup E_i)$ is the origin O. For a sequence of positive integers $\alpha_1, \dots, \alpha_k$ let $I(\alpha_1, \dots, \alpha_k) = \{\phi \in \mathcal{O}_O | \mathrm{ord}_{E_i} \pi^*(\phi) \geq \alpha_i, i = 1, \dots, k\}$.*

1. *There are germs $\psi_i \in \mathcal{O}_O$ such that $I(\alpha_1, \dots, \alpha_k)$ consists of ϕ's such that the intersection index of $\phi = 0$ and $\psi = 0$ is not less than α_k.*
2. *For $e_i(\mathcal{A}'')$ in (8) we have the identity $\mathcal{A}'' = I(e_1(\mathcal{A}''), \dots, e_k(\mathcal{A}''))$. Let for such $\mathcal{A}''$ we have $\mathcal{A}'' = \mathcal{A}''(j_1, \dots, j_r | m_1, \dots, m_r)$ and the faces of the polytope of quasiadjunction containing corresponding vector (3) are the faces corresponding to all exceptional curves E_k where $k \in \mathcal{S}'$. Then $\mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r) = I(\dots, e_k(\mathcal{A}'') + \epsilon_k, \dots)$ where $\epsilon_k = 1$ for $k \in \mathcal{S}'$ and $\epsilon_k = 0$ otherwise.*
3. *$\mathcal{A}'(j_1, \dots, j_r | m_1, \dots, m_r)$ is the ideal generated by the ideals $I(e_1(\mathcal{A}'') + \epsilon_1, \dots, e_k(\mathcal{A}'') + \epsilon_k)$ corresponding all possible arrays $(\epsilon_1, \dots, \epsilon_k)$ where $\epsilon_i = 1$ if $i \in \mathcal{S}'$ is such that there exists $j \in \mathcal{S}'$ with $E_i \cap E_j \neq 0$ and $\epsilon_i = 0$ otherwise.*

Proof. As ψ_i one can take the local equation of the image of a transversal to E_i in its generic point. The rest follows from (8) and the definitions. □

Remark 2.5. The polytopes of quasiadjunction in this paper are somewhat different than the local polytope of quasiadjunction of [15]. The latter polytopes are defined as the equivalence classes of vectors (3) when two vectors are considered equivalent if and only if the corresponding ideals of quasiadjunction are the same. The interior of a polytope $\mathcal{P}(\mathcal{A})$ is the union of the local polytopes of quasiadjunction from [15]. Vice versa, the convex polytopes in this section determine the local polytopes of quasiadjunction in [15].

Remark 2.6. Recall that for a $\mathbf{Q}$-divisor D on a non singular manifold X its multiplier ideal $\mathcal{J}(D)$ (cf. ([19]) can be defined as follows. Let $f : Y \to X$ be an embedded resolution of D and $f^*(D) = -E$. Then $\mathcal{J}(D) = f_*(\mathcal{O}_Y(K_Y - f^*(K_X) - \lfloor E \rfloor))$ where $\lfloor E \rfloor$ is round-down of a $\mathbf{Q}$-divisor. In this terminology one can define the ideals of quasiadjunction as follows. For an array $(\gamma_1, \dots, \gamma_r)$, $(\gamma_i \in \mathbf{Q})$ let $D_{\gamma_1, \dots, \gamma_r}$ be given by equation $f_1^{\gamma_1} \cdots f_r^{\gamma_r}$. Then $\mathcal{J}(D_{\gamma_1 \dots \gamma_r}) = \mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r)$ where $\gamma_i = 1 - \frac{j_i+1}{m_i}$ for $i = 1, \dots, r$. This follows immediately from (4).

2.3. Mixed Hodge structure of the cohomology of links of singularities and on the local cohomology

Here we shall summarize several well known facts used in the next section. Cohomology of the link L of an isolated singularity x of a complex space X ($\dim X = n$) can be given a Mixed Hodge structure, for example using canonical identification $H^k(L) = H^*_{\{x\}}(X)$ with the local cohomology and using the construction of mixed Hodge structure on the latter due to Steenbrink [23]. The Hodge numbers:

$h^{kpq}(L) = \dim Gr_F^p Gr_{p+q}^W H^k(L)$ have the following symmetry properties (cf. [9, 23]):

$$h^{kpq} = h^{2n-k-1,n-p,n-q} \tag{12}$$

If E is the exceptional divisor for a resolution, then for $k < n$ one has (cf. [9])

$$h^{kpq}(L) = h^{kpq}(E) \quad \text{if} \quad p+q < k$$

$$h^{kpq}(L) = h^{kpq}(E) - h^{2n-k,n-p,n-q}(E) \quad \text{if} \quad p+q = k \tag{13}$$

$$h^{kpq}(L) = 0 \quad \text{if} \quad p+q > k$$

The mixed Hodge structure on cohomology of a link is related to the mixed Hodge structure on vanishing cohomology of the Milnor fiber B via the exact sequence (corresponding to the exact sequence of a pair):

$$0 \to H^{n-1}(L) \to H^n_c(B) \to H^n(B) \to H^n(L) \to 0 \tag{14}$$

which is an exact sequence of mixed Hodge structures (cf. [24] and (2.3) in [23]).

Steenbrink also put Mixed Hodge structure on the local cohomology $H^*_E(\tilde{X})$ ([23]) where $\tilde{X}$ is a resolution of X. In the case $\dim_{\mathbf{C}} \tilde{X} = 2$ we have

$$H^*_E(\tilde{X}) = \mathrm{Hom}(H^{4-*}(E), \mathbf{Q}(-2)), \tag{15}$$

where $\mathbf{Q}(-2)$ is Tate Hodge of type $(2,2)$. Since the Hodge and weight filtrations on $H^1(E)$ have form:

$$H^1(E) = W_1 \supset W_0 \supset 0, \quad H^1(E) = F^0 \supset F^1 \supset F^2 = 0$$

we have on $H^3_E(\tilde{X})$

$$H^3_E(\tilde{X}) = W_4 \supset W_3 \supset W_2 = 0, H^3_E(\tilde{X}) = F^1 \supset F^2 \supset F^3 = 0.$$

Moreover

$$F^1 H^1(L) = F^1 H^1(E) = F^2 H^3_E(\tilde{X}). \tag{16}$$

One can use the following complex for description of this mixed Hodge structure:

$$0 \to A^2_E(\tilde{X}) \to A^3_E(\tilde{X}) \to 0, \tag{17}$$

where

$$A^2_E(\tilde{X}) = \Omega^1_{\tilde{X}}(log\ E)/\Omega^1_{\tilde{X}}, A^3_E(\tilde{X}) = \Omega^2_{\tilde{X}}(log\ E)/\Omega^2_{\tilde{X}}$$

with filtrations given by

$$F^2 A^p_E(\tilde{X}) = 0\ for\ p < 3, F^2 A^p_E(\tilde{X}) = A^p_E(\tilde{X})\ for\ p \geq 3,$$

$$W_3 A^3_E(\tilde{X}) = W_1 \Omega^2_{\tilde{X}}(log\ E)/\Omega^2_{\tilde{X}}.$$

Since $H^3(E) = 0$, the relations (13) and (15) yield that the complex (17) completely determines h^{1pq} (and hence all Hodge numbers h^{kpq} by (12)).

3. Characteristic varieties and polytopes of quasiadjunction

3.1. Main theorem

We shall view the unit cube $\mathcal{U}$, considered in the last section and containing the polytopes of quasiadjunction, as the fundamental domain for the Galois group $H^1(S^3-L,\mathbf{Z})$ of the universal abelian cover $H^1(S^3-L,\mathbf{R})$ of the group $H^1(S^3-L,\mathbf{R}/\mathbf{Z})$ of the unitary characters of $H_1(S^3-L,\mathbf{Z})$ (i.e. the maximal compact subgroup of $\mathrm{Char}(H_1(S^3-L,\mathbf{Z}))=H^1(S^3-L,\mathbf{C}^*)$). $\exp:\mathcal{U}\to \mathrm{Char}(H_1(S^3-L,\mathbf{Z}))$ will denote the restriction of $H^1(S^3-L,\mathbf{R})\to H^1(S^3-L,\mathbf{R}/\mathbf{Z})$ on $\mathcal{U}$.

For any sub-link $\tilde{L}$ of L, i.e. a link formed by components of L, we have surjection $\pi_1(S^3-L)\to\pi_1(S^3-\tilde{L})$ induced by inclusion. Hence $\mathrm{Char}\,H_1(S^3-\tilde{L},\mathbf{Z})$ is a sub-torus of $\mathrm{Char}\,H_1(S^3-L,\mathbf{Z}))$ (in coordinates in the latter torus corresponding to the components of L it is given by equations of the form $t_\alpha=1$ where subscripts correspond to components of L absent in $\tilde{L}$). Moreover, since the homology of the universal abelian cover $H_1(\widetilde{S^3-L})$ surjects onto $H_1(\widetilde{S^3-\tilde{L}})$, it follows that $V_i(S^3-\tilde{L})$ belongs to a component of $V_i(S^3-L)$ (cf. 1.2.1 in [15]). We shall call a character of $\pi_1(S^3-L)$ (or a connected component of $V_i(S^3-L)$) *essential* if it does not belong to a subtorus $\mathrm{Char}\,H^1(S^3-\tilde{L})$ for any sublink $\tilde{L}$ of L.

Let $L_{m_1,\dots,m_r}$ be the link of singularity (1) or equivalently the cover of S^3 branched over the link L and having a quotient $H_{m_1,\dots,m_r}=\mathbf{Z}/m_1\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}/m_1\mathbf{Z}$ of $H_1(S^3-L,\mathbf{Z})$ as its Galois group. We shall view $\mathrm{Char}\,H_{m_1,\dots,m_r}$ as a subgroup of $\mathrm{Char}\,H_1(S^3-L,\mathbf{Z})$. The group $H_{m_1,\dots,m_r}$ acting on $H^1(L_{m_1,\dots,m_r})$ preserves both Hodge and weight filtrations.

Theorem 3.1. *An essential character $\chi\in Char(H_1(S^3-L,\mathbf{Z}))$ is a character of the representation of $H_{m_1,\dots,m_r}$ acting on $F^1(H^1(L_{m_1,\dots,m_r}))$ if and only if it factors through the Galois group $H_{m_1,\dots,m_r}$ and belongs to the image of a face of quasiadjunction under the exponential map.*

The multiplicity of χ in this representation of the Galois group is equal to $\dim \mathcal{A}''_\Sigma/\mathcal{A}_\Sigma$ *where $\mathcal{A}''_\Sigma$ (resp. $\mathcal{A}_\Sigma$) is the ideal of log-quasiadjunction (resp. ideal of quasiadjunction) corresponding to a vector (3) belonging to the face of quasiadjunction Σ.*

A character χ is a character of the representation of the Galois group of the cover on $W_0(H^1(L_{m_1,\dots,m_r}))$ if and only if it factors through the Galois group $H_{m_1,\dots,m_r}$ and it belongs to the image under the exponential map of a weight one face of quasiadjunction.

Proof. 1. log-2-forms on $V_{m_1,\dots,m_r}-p$. Let $\tilde{p}:\tilde{V}_{m_1,\dots,m_r}\to V_{m_1,\dots,m_r}$ be a resolution such that the exceptional locus is a divisor $\tilde{E}=\bigcup\tilde{E}_i$ on $\tilde{V}_{m_1,\dots,m_r}$ with normal crossings (e.g. (6)). The group $H_{m_1,\dots,m_r}$ acts on both sheaves: $\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log\ E)$ and $\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}$.

We are going to identify the eigenspace of $\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log\ E)/\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}$ corresponding to the character χ, which is the exponent of a vector (3) belonging to a face of quasiadjunction Σ, with the quotient of ideals $\mathcal{A}''_\Sigma/\mathcal{A}_\Sigma$.

First notice that any 2-form $\frac{z_1^{j_1},\dots,z_r^{j_r}\phi(x,y)dx\wedge dy}{z_1^{m_1-1}\cdots z_r^{m_r-1}}$ is holomorphic on $\tilde{V}_{m_1,\dots,m_r} - E = V_{m_1,\dots,m_r} - p$ since it is a residue of a holomorphic $(r+2)$-form on $\mathbf{C}^{r+2} - V_{m_1,\dots,m_r}$. It is an eigenform corresponding to the character χ such that $\chi(\gamma_i) = \exp(2\pi\sqrt{-1}\frac{m_i-j_i-1}{m_i})$ (here $\gamma_i \in H^1(S^3 - L, \mathbf{Z})$ are the generators corresponding to the components of the link L).

Vice versa, any 2-form, holomorphic on $V_{m_1,\dots,m_r} - p$, is a residue of $r+2$-form given by $\frac{\phi(z_1,\dots,z_r,x,y)dz_1\wedge\cdots\wedge dz_r\wedge dx\wedge dy}{(z_1^{m_1}-f_1(x,y))\cdots(z_r^{m_r}-f_r(x,y))}$ where ϕ is a polynomial. Decomposition of ϕ into a sum of monomials corresponds to the decomposition of the form into sum over characters.

Secondly a form $\frac{z_1^{j_1}\cdots z_r^{j_r}\phi dx\wedge dy}{z_1^{m_1-1}\cdots z_r^{m_r-1}}$ is log-2-form (resp. holomorphic 2-form) if and only if $\phi(x,y)$ is in the ideal of log-quasiadjunction (resp. quasiadjunction) corresponding to $(\frac{j_1}{m_1},\dots,\frac{j_r}{m_r})$. These ideals do not coincide if and only if (3) belongs to a face of quasiadjunction.

2. Hodge and weight filtration on cohomology of link. Now we want to identify $\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log\ E)/\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}$ with $F^1H^1(L_{m_1,\dots,m_r})$. (16) yields that the latter is isomorphic to $F^2H^3_E(\tilde{V}_{m_1,\dots,m_r})$. Using description of the Hodge filtration on $H^3_E(\tilde{V}_{m_1,\dots,m_r})$ from (2.3) it can be identified with the hypercohomology of the complex $0 \to A^3_E(\tilde{V}_{m_1,\dots,m_r}) \to 0$ i.e. with $H^0(\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log\ E)/\Omega^2_{\tilde{V}_{m_1,\dots,m_r}})$. Since by the Grauert-Riemenschneider theorem $H^1(\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}) = 0$ we see that the latter space is isomorphic to $H^0(\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log\ E))/H^0(\Omega^2_{\tilde{V}_{m_1,\dots,m_r}})$ and the claim follows.

3. Conclusion of the proof. Similarly to the above, it follows that a character χ is a character of the representation of $H_{m_1,\dots,m_r}$ on $W_0H^1(L_{m_1,\dots,m_r})$ if it is a character of this group acting on $W_2/W_1(\Omega^2_{\tilde{V}_{m_1,\dots,m_r}}(\log \tilde{E})/\Omega^2_{\tilde{V}_{m_1,\dots,m_r}})$ (cf. (2.3)). In other words $\chi = \exp(\frac{2\pi i j_1}{m_1},\dots,\frac{2\pi i j_r}{m_r})$ where $\mathcal{A}'(j_1,\dots,j_r|m_1,\dots,m_r) \neq \mathcal{A}''(j_1,\dots,j_r|m_1,\dots,m_r)$ i.e. $(\frac{j_1}{m_1},\dots,\frac{j_r}{m_r})$ belongs to a face of weight one log quasiadjunction. Moreover, the dimension of the χ-eigenspace of the action of the Galois group of the cover on $W_0H^1(L_{m_1,\dots,m_r})$ is equal to $\dim \mathcal{A}''_\Sigma/\mathcal{A}'_\Sigma$ where Σ is the face of weight one log-quasiadjunction to which the χ belongs. $\square$

3.2. Essential components of characteristic varieties

Theorem 3.1 allows one to describe essential components of characteristic varieties. Indeed, each component of $V_i(S^3 - L)$ is a torus translated by a point of a finite order in $\mathrm{Char}\, H_1(S^3 - L, \mathbf{Z})$ (cf. 2.1) and each such sub-torus is Zariski closure of the set of points of finite order in it. It follows from [21] that an essential character of finite order belongs to $V_i(S^3 - L)$ if and only if it is a character of $H_{m_1,\dots,m_r}$ on $H_1(L_{m_1,\dots,m_r})$ for some array $(m_1,\dots,m_r)$. A character χ appears as either a character on $W_0H^1(L_{m_1,\dots,m_r})$, in which case according to 3.1 it is an exponent of a vector in a face of quasiadjunction, or χ is a character on $W_1/W_0H^1(L_{m_1,\dots,m_r})$, in which case either χ or $\bar{\chi}$ is a character of the Galois group acting on $H^{1,0}(W_1/W_0)$. In each of the cases, the multiplicity of the character is $\dim \mathcal{A}''_{\Sigma}/\mathcal{A}_{\Sigma}$ where Σ is the face of quasiadjunction to exponent of which χ belongs. If $\mathcal{L}_{\chi}$ will denote the local system on $S^3 - L$ corresponding to the character χ then this multiplicity is equal to to $\dim H^1(\mathcal{L}_{\chi})$, as follows from arguments in [21]. On the other hand, $\dim H^1(\mathcal{L}_{\chi})$ is the depth of the characteristic variety to which χ belongs. We obtain therefore:

Proposition 3.2. *For $\xi = (x_1, \dots, x_r) \in \mathcal{U}$ let $\bar{\xi} = (1 - x_1, \dots, 1 - x_r)$. For a face of quasiadjunction Σ let $\tilde{\Sigma} = \Sigma$ or $\tilde{\Sigma} = \Sigma \cup \bar{\Sigma}$ depending on whether $\mathcal{A}'_{\Sigma}/\mathcal{A}_{\Sigma} = 0$ or not. Then the Zariski closure of $\exp(\tilde{\Sigma})$ is a component of the characteristic variety of depth $\dim \mathcal{A}'_{\Sigma}/\mathcal{A}_{\Sigma} + \dim \mathcal{A}'_{\bar{\Sigma}}/\mathcal{A}_{\bar{\Sigma}} + \dim \mathcal{A}''_{\Sigma}/\mathcal{A}'_{\Sigma}$. Vice versa, any essential component of $V_i(S^3 - L)$ has such a form.*

3.3. Quasiadjunction and higher Alexander polynomials

We shall show that in the case $r = 1$, information from the faces and ideals of quasiadjunction determines *all* Hodge numbers $h^{p,q}_{\zeta}$ of the Milnor fiber of $f = 0$ considered in the introduction and in particular *all* Alexander polynomials Δ_i (cf. Introduction). The idea of relating the constants of quasiadjunction to the Hodge theory appeared first in [17] where the case of the first Alexander polynomial Δ_1 was studied.

For $r = 1$ each polytope of quasiadjunction is a segment $[\kappa, 1]$. κ is called (cf. [11]) the constant of quasiadjunction (resp. log-quasiadjunction and weight one quasiadjunction).

Proposition 3.3. 1. *ζ is an eigenvalue of the monodromy acting on the Milnor fiber of f if and only if ζ or $\bar{\zeta}$ is equal to $\exp 2\pi\kappa_l$ for some constant of quasiadjunction κ_l.*

2. *If $\zeta = \exp 2\pi\kappa_l$ and κ_l is a weight one constant quasiadjunction then $h^{0,0}_{\zeta} = \dim \mathcal{A}''_{\kappa_l}/\mathcal{A}'_{\kappa_l}$ and $h^{1,0}_{\zeta} = \dim \mathcal{A}'_{\kappa_l}/\mathcal{A}_{\kappa_l}$. In particular $h^{0,0}_{\zeta} = 0$, unless the corresponding κ_l is a weight one constant of quasiadjunction.*

Proof. $\zeta \neq 1$ such that $\zeta^m = 1$ is an eigenvalue of the monodromy of the Milnor fiber of f if an only if it is an eigenvalue of the Galois group of acting on $H^1(L_m) = H^1_{\{\mathrm{Sing} V_m\}}(V_m)$ where V_m is given by $z^m = f$ (cf. (1)). On the other hand, the exact sequence

$$0 \to H^1_{\{\mathrm{Sing} V_m\}}(V_m) \to H^2_c(M(V_m)) \xrightarrow{j} H^2(M(V_m)) \to$$

(cf. (14)) in which $M(V_m)$ is the Milnor fiber of V_m shows that the Hodge structure on $H^1(L_m)$ is isomorphic to the Hodge structure induced from $H^2_c(M(V_m))$ on $Ker(T_c - I)$. Indeed $\ker j = \ker T_c - I$ since $T_c - I = \mathrm{Var} \circ j$ (cf. [23], (2.4); recall also that $\mathrm{Var} : H^2(M(V_m)) \to H^2_c(M(V_m))$ is an isomorphism as a consequence for example of a well known relation between the Seifert form and variation operator).

The result follows since the Hodge structure on $H^2_c(M(V_m))$ is determined by the Hodge structure on the Milnor fiber of $f = 0$ via a Thom–Sebastiani type theorem (cf. [24]). $\square$

4. Properties and applications of polytopes of quasiadjunction

4.1. Semicontinuity

Theorem 4.1. 1. *Let C_t be a family of plane curve singularities in a ball B with r branches such that the limit curve has r branches as well. Then the number of components of $V_1(C_t)$ does not exceed the number of components of $V_1(C_0)$.*
2. *The total volume of all codimension one faces of quasiadjunction is semicontinuous, provided that the volume of each face is calculated with respect to the measure on the hyperplane containing this face in which the measure of the simplex containing no integral points is equal to $\frac{1}{(r-1)!}$.*

Proof. First notice that the intersection form on H_2 of the smoothing of the complete intersection surface singularity which is an abelian cover of B branched over C_t (i.e the singularity (1)) embeds into the intersection form of the singularity which is the abelian cover of the same type branched over C_0. On the other hand, $b_1(L_{m_1,\ldots,m_r})$ is the dimension of the radical of this intersection form. In particular we have $b_1(L_{n,\ldots,n}(C_t)) \le b_1(L_{n,\ldots,n}(C_0))$. On the other hand $b_1(L_{n,\ldots,n}) = kn^{r-1} + \alpha \cdot n^{r-1} + \cdots$ for almost all n as follows from [21] where k is the number of components in $V_1(C)$ since the number of points of order n on a torus of dimension l translated by a point of finite order is n^l for almost all n. Hence $k = \lim_{n\to\infty} \frac{b_1(L_{n,\ldots,n})}{n^{r-1}}$ which yields the first part.

Similarly, asymptotically the number of the points in the lattice $(\frac{k}{n}, \ldots, \frac{k}{n}) \subset \mathcal{U}$ which belong to codimension one faces of quasiadjunction is the total volume of the faces. On the other hand this number of the points is $\dim F^1 H^2_1(M)$ i.e. the dimension of the Hodge filtration F^1 on the subspace of the cohomology of the Milnor fiber consisting of monodromy invariants. Similarly to [25] for the Milnor fibers M_t and M_0 of the smoothings of abelian covers of B having type $(n, \ldots, n)$ and branched over C_t and C_0 respectively we have: $\dim F^1 H^2_1(M_t) - \dim F^1 H^2_1(M_0) = \dim F^1 \mathbf{H}_1(R\Phi)$ which yields the second part. □

4.2. Log canonical divisors

Recall ([10]) that a pair (X, D) where X is normal and D is a $\mathbf{R}$-divisor such that $K_X + D$ is $\mathbf{R}$-Cartier is called log-canonical at $x \in X$ if for any birational morphism $f : Y \to X$, with Y normal, in the decomposition

$$K_Y = f^*(K_X + D) + \sum_E a(E, X, D)E \tag{18}$$

for each irreducible E having center at x one has $a(E, X, D) \ge -1$. This coefficient is called *discrepancy* of divisor D on X along E.

Proposition 4.2. *The local ring $\mathcal{O}_O$ of a singularity $f_1 \cdots f_r = 0$ at the origin O of $\mathbf{C}^2$ considered as the ideal in itself is an ideal of log-quasiadjunction. Let $\mathcal{P}$ be the corresponding polytope of log-quasiadjunction. Let D_i be the divisor in $\mathbf{C}^2$ with the local equation $f_i = 0$ near the origin.*

Then for $\{(\gamma_1, \dots, \gamma_r)\} \in \mathbf{R}^r$ *the divisor* $\gamma_1 D_1 + \cdots + \gamma_r D_r$ *is log-canonical at* $(0,0) \in \mathbf{C}^2$ *if and only if* $(\gamma_1 + 1, \dots, \gamma_r + 1)$ *belongs to the polytope* $\mathcal{P}$.

Proof. Let us consider the polytope given by inequalities (11) in which one puts $e_k(\mathcal{A}'') = 0$, i.e.

$$a_{k,1}x_1 + \cdots + a_{k,r}x_r \geq a_{k,1} + \cdots + a_{k_r} - c_k - 1 \tag{19}$$

Let $(j_1, \dots, j_r | m_1, \dots, m_r)$ be such that the corresponding vector (3) belongs to the boundary of this polytope. Then $1 \in \mathcal{A}''(j_1, \dots, j_r | m_1, \dots, m_r)$ and hence $\mathcal{A}''(j_1, \dots, j_r | m_1 \mathfrak{t}, \dots, m_r)$ is the local ring of the origin.

If $\pi : V \to \mathbf{C}^2$ is an embedded resolution $\pi^*(f_1^{\gamma_1} \cdots f_r^{\gamma_r} dx \wedge dy)$ has as the order of vanishing along E_k:

$$a_{k,1}\gamma_1 + \cdots + a_{k,r}\gamma_r + c_k$$

i.e. the discrepancy along each E_k is not less than -1 if and only if $(\gamma_1 + 1, \dots, \gamma_r + 1)$ satisfies (19).

Remark 4.3. If $r = 1$ the polytope of quasiadjunction $\mathcal{P}$ described above has as its face (i.e. the end) the constant of quasiadjunction (cf. [11] for a definition) κ_1 and one obtains that the log-canonical threshold $c_0(f)$ of $f = 0$ satisfies: $c_0(f) + 1 = \kappa_1$. This result is due to J. Kollar (cf. ([10]) Prop. 9.8).

Remark 4.4. Mixed Hodge structure on cohomology of universal abelian covers. It seems at the moment very little is known about the Hodge theory on cohomology of complex manifolds which are not finite CW-complexes. Theorem 3.1 gives some hints what one may and may not expect. It yields a decomposition of the torus of unitary characters into a union of connected subsets (faces of quasiadjunction or their conjugates) in which the characters appearing on certain Hodge and weight type on a finite level are dense. These subsets are not algebraic subvarieties. In particular one cannot expect filtrations on the cohomology of infinite abelian covers with Galois-invariant Hodge and weight filtrations. Results of [15] yield decomposition of the torus of unitary characters of the fundamental group of the complement to an affine algebraic curve similar to the one given by Theorem 3.1 and reflecting the Hodge theory on H^1 of the infinite abelian cover of this complement.

5. Examples

5.1. Links of Ordinary singularities

Let us consider the link of the singular point of $L_1 \cdots L_r = 0$ where L_i are distinct linear forms on $\mathbf{C}^2$ (the Hopf link with r components). The faces of quasiadjunction are

$$\Sigma_l : \quad x_1 + \cdots + x_r = l, \quad (l = 1, \dots, r-2) \tag{20}$$

This follows directly since a resolution is achieved by single blow up and the multiplicity of L_i on the exceptional curve is equal to 1 (i.e. the coefficients $a_{k,i}$ in (5) are all equal to 1). Moreover, if $\mathcal{M}$ is the maximal ideal in the local ring of the origin, then the ideal of quasiadjunction $\mathcal{A}_{\Sigma_l}$ corresponding to the face (20) is $\mathcal{M}^{r-1-l}$ and $\mathcal{A}'_{\Sigma_l} = \mathcal{A}''_{\Sigma_l} = \mathcal{M}^{r-2-l}$. We have $\dim \mathcal{M}^{l-1}/\mathcal{M}^l = l$. Hence, if χ is a character of the fundamental group having multi-order $(m_1, \dots, m_r)$ and given on the generators $\gamma_i \in H_1(S^3 - L, \mathbf{Z})$ corresponding to the components of L by

$$\chi(\gamma_i) = \exp(2\pi\sqrt{-1}x_i) \quad \text{where } \Sigma x_i = l \tag{21}$$

then the dimension of the eigenspace $H^{1,0}(L_{m_1,\dots,m_r})_\chi$ is $r-1-l$. The conjugate character $\bar{\chi}$ has the same multiplicity $r-1-l$ on $H^{0,1}(L_{m_1,\dots,m_r})$. Therefore the character in (21) has on $H^{0,1}(L_{m_1,\dots,m_r})$ multiplicity $l-1$ and its multiplicity on $H^1(L_{m_1,\dots,m_r})$ is equal to $r-2$. In particular, the characteristic variety has only one essential component V_{r-2}. This is well known from the calculations using the Fox calculus (cf. [12] for $r=3$).

5.2. *Singularity* $(x^2+y^5)(y^2+x^5)$ (cf. also [6])

A standard sequence of blow ups lead to the resolution V for this singularity pictured on Fig. 1 (E_1, E_2 are the exceptional curves in the last blow up and B_1, B_2 are the branches of the singularity).

We have $e_{E_1}(x) = 5$, $e_{E_1}(y) = 2$, $e_{E_2}(x) = 2$, $e_{E_2}(y) = 5$ and hence $e_{E_1}(x^2 + y^5) = 10$, $e_{E_1}(y^2+x^5) = 4$, $e_{E_2}(x^2+y^5) = 4$, $e_{E_2}(y^2+x^5) = 10$. Moreover, $e_{E_1}(dx \wedge dy) = e_{E_2}(dx \wedge dy) = 6$ (the orders of all functions and forms on $\mathbf{C}^2$ meant to be calculated on the chosen resolution V). As in (1), V_{m_1,m_2} is the abelian branched cover: $z_1^{m_1} = x^2+y^5$, $z_2^{m_2} = x^5+y^2$. A form ω_ϕ given by (2) admits a holomorphic extension over the preimages of the curves E_1 and E_2 on the normalization $\widetilde{V}_{m_1,m_2}$ of $V \times_{\mathbf{C}^2} V_{m_1,m_2}$ if and only if

$$10\Big(\frac{j_1+1}{m_1}-1\Big) + 4\Big(\frac{j_2+1}{m_2}-1\Big) + e_{E_1}(\phi) + 7 \geq 0 \tag{22}$$

$$4\Big(\frac{j_1+1}{m_1}-1\Big) + 10\Big(\frac{j_2+1}{m_2}-1\Big) + e_{E_2}(\phi) + 7 \geq 0$$

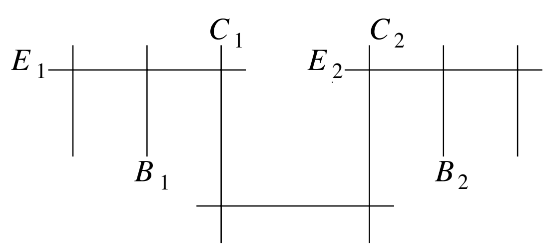

Fig. 1. Resolution of $(x^2+y^5)(x^5+y^2)$

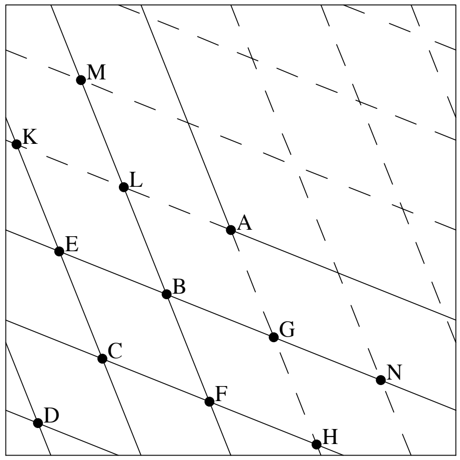

Fig. 2. Faces of quasiadjunction for $(x^2 + y^5)(x^5 + y^2)$

(cf. (5)). Similar inequalities should be written for other exceptional curves but calculation shows that the inequalities expressing the condition that ω_ϕ extends over the preimages of remaining exceptional curves in Fig. 1 follow from (22).

The quotient $\mathcal{A}''(j_1, j_2|m_1, m_2)/\mathcal{A}(j_1, j_2|m_1, m_2) \neq 0$ if and only if there exists a function ϕ for which (22) is satisfied and such that a left hand side in at least one of inequalities (22) is zero. Since the possibilities for $e_{E_i}(\phi)$ are $0, 2, 4, 5, 6, 7, \ldots,$ the faces of quasiadjunction are subsets of the lines:

$$10x_1 + 4x_2 = 7, 5, 3, 2, 1$$

$$4x_1 + 10x_2 = 7, 5, 3, 2, 1$$

The value $e_{E_1}(\phi) = 5$ (i.e. $\phi = x$) for $i = 1$ does not yield a face of quasiadjunction since it implies that $e_{E_2}(\phi) = 2$ and no pair $x_1 = \frac{j_1+1}{m_1}$, $x_2 = \frac{j_2+1}{m_2}$ in the unit square satisfies inequalities (22) with such $e_{E_i}(\phi)$. Similarly, $e_{E_2}(\phi) = 5$ also does not yield a face of quasiadjunction.

Next let us consider pairs $(\frac{j_1+1}{m_1}, \frac{j_2+1}{m_2})$ satisfying $10x_1 + 4x_2 = 7$. It follows from the first inequality in (22) that ϕ is a non zero in $\mathcal{A}''(j_1, j_2|m_1, m_2)/\mathcal{A}(j_1, j_2|m_1, m_2)$ only if $e_{E_1}(\phi) = 0$ and hence the lowest order term of ϕ is a constant (i.e. this is the only case when ω_ϕ has a pole of order 1 along E_1). Hence the second inequality in (22) is $4x_1 + 10x_2 \geq 7$ and hence only "the half" of the segment $10x_1 + 4x_2 = 7$ in the unit square is the face of quasiadjunction. Similarly "the half" of the segment $4x_1 + 10x_2 = 7$ is also the face of quasiadjunction.

On the other hand for a pair $(\frac{j_1+1}{m_1}, \frac{j_2+1}{m_2})$ on the segment $10x_1 + 4x_2 = 5$, assuming that ω_ϕ has a pole of order one along E_1, the first inequality (22) yields that the lowest order term of ϕ is ay, $a \in \mathbf{C}^*$ and the second one in (22) is satisfied for any pair $(\frac{j_1+1}{m_1}, \frac{j_1+1}{m_1})$ on $10x_1 + 4x_2 = 5$ in the unit square. Therefore the segment $10x_1 + 4x_2 = 5$ is a face of quasiadjunction. Similar calculations show that we obtain the diagram of faces of quasiadjunction given on Fig. 2. Moreover, the points B, C, D, E, F are the only ones for which one has $\dim \mathcal{A}''/\mathcal{A} = 2$ (for the remaining points on the faces of quasiadjunction this dimension is 1).

Exponential map takes the union of the faces of quasiadjunction and their conjugates into the union of of translated subgroups $t_1^2t_2^5+1=0$ and $t_1^5t_2^2+1=0$ (and coincides with the union of translated maximal compact subgroups of the latter). V_2 consists of exponents of the points of the following types:

(a) Points where $\dim \mathcal{A}''/\mathcal{A} = 2$ (corresponding characters appear on holomorphic part); these are the points B, C, D, E, F.
(b) Conjugates of the points in (a) (corresponding characters appear on anti-holomorphic part).
(c) Points belonging to faces of quasiadjunction and having conjugates on faces of quasiadjunction as well; corresponding characters appear with multiplicty one on both the holomorphic and anti-holomorphic parts; these are points H, G, K, L, M, N (N is the conjugate of M.)
(d) Conjugates of the points in (c).

We obtain that V_2 consists of all (twenty) points of intersection of two translates $t_1^2t_2^5+1=0$ and $t_1^5t_2^2+1=0$ except the point $(-1,-1)$.

The eigenspace in $H^1(L)$ corresponding to the character $\exp(2\pi i A) = (-1,-1)$ consists of the weight zero classes in $H^1(L)_{(-1,-1)}$ since in this case $\mathcal{A}''/\mathcal{A}'$ is generated by 1 and ω_1 has poles of order 1 along E_1, E_2 and C_1, C_2 i.e. the weight of ω_1 is two.

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