ELLIPTIC GENERA OF SINGULAR VARIETIES

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Abstract

The notions of orbifold elliptic genus and elliptic genus of singular varieties are introduced, and the relation between them is studied. The elliptic genus of singular varieties is given in terms of a resolution of singularities and extends the elliptic genus of Calabi-Yau hypersurfaces in Fano Gorenstein toric varieties introduced earlier. The orbifold elliptic genus is given in terms of the fixed-point sets of the action. We show that the generating function for the orbifold elliptic genus $\sum \mathrm{Ell}_{\mathrm{orb}}(X^n, \Sigma_n) p^n$ *for symmetric groups* Σ_n *acting on* n*-fold products coincides with the one proposed by R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde. The two notions of elliptic genera are conjectured to coincide.*

1. Introduction

This work started as an attempt to understand the beautiful formula for the generating function for the orbifold elliptic genera of symmetric products due to Dijkgraaf, Moore, Verlinde, and Verlinde which follows (see [19]):

$$\sum_{n\geq 0} p^n \,\mathrm{Ell}_{\mathrm{orb}}(X^n/\Sigma_n; y, q) = \prod_{i=1}^{\infty}\prod_{l,m} \frac{1}{(1-p^i y^l q^m)^{c(mi,l)}}. \tag{1.1}$$

Here X is a Kähler manifold, Σ_n is the symmetric group acting on the n-fold product, and $c(m,l)$ are the coefficients of the elliptic genus $\sum_{m,l} c(m,l) y^l q^m$ of X. The problem is that the orbifold elliptic genus is defined in physical terms, and the arguments given in [19] do not lend themselves to a translation into a mathematical proof.

The two-variable elliptic genus is a very compelling invariant, for the discussion of which we refer to [12]. Here we just note that it is a holomorphic function on the product of **C** and the upper half-plane which is attached to an (almost) complex manifold and that it is a weak Jacobi form if the manifold is Calabi-Yau. For Calabi-Yau manifolds of a dimension smaller than 12 or equal to 13, the elliptic genus can be expressed in terms of the Hirzebruch χ_y genus, but in general the former contains more information than the latter. In all dimensions the elliptic genus specializes into the Hirzebruch χ_y genus, and in particular into the topological Euler characteristic,

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the holomorphic Euler characteristic, the signature, and so on. Special cases of formula (1.1) for these invariants have been proved mathematically for some time. For example, it was shown in [28], using the Macdonald formula (see [31]), that if a finite group G acts on a manifold X and

$$e_{\mathrm{orb}}(X, G) := \frac{1}{|G|} \sum_{fg=gf} e(X^{f,g}) \tag{1.2}$$

(summation is over all pairs of commuting elements; $X^{f,g}$ is the set of fixed points of both f and g), then

$$\sum_{n=0}^{n=\infty} e_{\mathrm{orb}}(X^n, \Sigma_n) = \prod_i \frac{1}{(1-t^i)^{e(X)}}. \tag{1.3}$$

On the other hand, in [24] (see also [21]) it was found that the generating series for the χ_y genera of Hilbert schemes of a surface X is given by

$$\sum_{n=0}^{n=\infty} \chi_{-y}(X^{[n]}) p^n = \exp\Big(\sum_{m=1}^{\infty} \frac{\chi_{-y^m}(X)}{(1-(yp)^m)} \frac{p^m}{m} \Big). \tag{1.4}$$

It was observed in [28] that in the cases when a crepant resolution for X/G does exist, the McKay correspondence (see [33]) can be used to prove that the Euler characteristic of such a resolution coincides with the orbifold Euler characteristic. In [7] this idea was used in a more general case of the χ_y genus, with an appropriately defined orbifold χ_y genus. In the case when X is a surface, the Hilbert scheme provides such a resolution (see [22]), and hence the left-hand side of (1.4) coincides with the generating function for the orbifold χ_y genus of symmetric products of X. Therefore, (1.4) can be viewed as a specialization of (1.1).

This brings in the basic question: How are the orbifold Euler characteristic and the orbifold χ_y genus (or more generally, the orbifold elliptic genus of an action on a variety) related to the corresponding invariants of an arbitrary, not necessarily crepant, resolution of the singularities of the orbifold? This question has been addressed in several papers (see, e.g., [5], [7], [18]). Paper [5] contains mathematical definitions of the orbifold E-function and of an E-function of singular varieties calculated via resolutions, which is called a stringy E-function. The E-function of a smooth manifold is equivalent to the data given by the Hodge numbers of the manifold, and it specializes to the χ_y genus. The stringy E-function is defined for singular varieties with log-terminal singularities and more generally for log-terminal pairs. Works [5] and [18] show that the orbifold E-function for a pair (X, G) coincides with the stringy E-function for the pair $(X/G,$ image of ramification divisor$)$. The published version of [5] has a gap in its canonical abelianization algorithm, but it has now been corrected by Batyrev [6].

In this paper, two notions of the elliptic genus for singular varieties are proposed. The first is called the *singular elliptic genus* and is defined for pairs (variety, divisor). The singular elliptic genus specializes to the χ_y genus derived from the stringy E-function of [5]. The second notion of elliptic genus, called the *orbifold elliptic genus*, is defined for any pair (X, G) of a manifold and a finite group of its automorphisms. The orbifold elliptic genus specializes to the χ_y genus derived from the orbifold E-function. We conjecture that the two elliptic genera coincide for $(X/G,$ image of ramification divisor) and (X, G), up to an explicit normalization factor. The advantage of the orbifold elliptic genus is that it is well suited for the mathematical proof of formula (1.1). On the other hand, the singular elliptic genus provides an interesting new invariant of singular varieties. Instead of the non-Archimedian integrals over spaces of arcs techniques of [5] and [18], we use the recent result in factorization of birational maps into a sequence of smooth blow-ups and blow-downs (see [1]).

The content of the paper is as follows. In Section 2 we collect some standard definitions and results that are relevant to the subject but may not be familiar to the reader. In Section 3 we define the singular elliptic genus of a $\mathbf{Q}$-Gorenstein complex projective variety Z as follows. If $\pi : Y \to Z$ is a resolution of singularities of Z and $\alpha_k \in \mathbf{Q}$ are defined from the relation $K_Y = \pi^* K_Z + \sum \alpha_k E_k$, then

$$\widehat{\mathrm{Ell}}_Y(Z; z, \tau) := \int_Y \Big(\prod_l \frac{(y_l/(2\pi \mathrm{i}))\theta(y_l/(2\pi \mathrm{i}) - z)\theta'(0)}{\theta(-z)\theta(y_l/(2\pi \mathrm{i}))} \Big) \times \Big(\prod_k \frac{\theta(e_k/(2\pi \mathrm{i}) - (\alpha_k + 1)z)\theta(-z)}{\theta(e_k/(2\pi \mathrm{i}) - z)\theta(-(\alpha_k + 1)z)} \Big),$$

where $\theta(z, \tau)$ is the Jacobi theta function and y_l are Chern roots of Y and $e_k = c_1(E_k)$. It is shown that $\widehat{\mathrm{Ell}}_Y(Z; z, \tau)$ depends only on Z (rather than on the desingularization Y). Moreover, this definition is extended to pairs (variety, divisor), and the singular elliptic genus has the transformation properties of a Jacobi form if the pair satisfies a natural Calabi-Yau condition. Some difficulties arise only when some α_k equal (-1) and we assume that the pairs are Kawamata log-terminal. One application of the singular elliptic genus is to the problem raised by M. Goreski and R. McPherson (see [9]). They were trying to determine which Chern numbers can be defined for singular spaces so that they are invariant under small resolutions. B. Totaro found a remarkable connection between this problem and the elliptic genus. In [35] he showed that such Chern numbers must be among the combinations of the coefficients of the two variable elliptic genus by showing that these are the only Chern numbers invariant under the classical flops. As a corollary of our definition of singular elliptic genus, we show that the elliptic genera of any two IH-small resolutions (or, more generally, two crepant resolutions) of a singular variety coincide, which in a sense completes the paper of Totaro. Unfortunately, most varieties do not admit such resolutions, and it appears that Chern numbers may not be a good invariant to look for because singular elliptic genera generally do not lie in the span of the elliptic genera of smooth varieties. However, coefficients of Taylor expansions of elliptic genera do provide an analog of Chern numbers for singular varieties.

In Section 4 we propose a definition of an orbifold elliptic genus which does not use the resolution of singularities but uses only information about the manifold and the fixed-point sets. Let G be a finite group acting on a manifold X. For $h \in G$, let X^h be a connected component of the fixed-point set of h, and let $TX|_{X^h} = \bigoplus V_\lambda$, $\lambda \in \mathbf{Q} \cap [0, 1)$ be a decomposition into direct sum, such that h acts on V_λ as the multiplication by $e^{2\pi \mathrm{i}\lambda}$. Let $F(h, X^h \subset X) = \sum_\lambda \lambda(h)$ be the fermionic shift (see [7], [38]), and let

$$\begin{aligned} V_{h,X^h\subseteq X} := \bigotimes_{k\geq 1} \Big[&(\Lambda^\bullet V_0^* y q^{k-1}) \otimes (\Lambda^\bullet V_0 y^{-1} q^k) \otimes (\mathrm{Sym}^\bullet V_0^* q^k) \otimes (\mathrm{Sym}^\bullet V_0 q^k) \\ &\otimes \Big[\bigotimes_{\lambda\neq 0} (\Lambda^\bullet V_\lambda^* y q^{k-1+\lambda(h)}) \otimes (\Lambda^\bullet V_\lambda y^{-1} q^{k-\lambda(h)}) \\ &\qquad \otimes (\mathrm{Sym}^\bullet V_\lambda^* q^{k-1+\lambda(h)}) \otimes (\mathrm{Sym}^\bullet V_\lambda q^{k-\lambda(h)}) \Big]\Big]. \end{aligned}$$

Then we define (see Section 4)

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) := y^{-\dim X/2} \sum_{\{h\}, X^h} y^{F(h, X^h\subseteq X)} \frac{1}{|C(h)|} \sum_{g\in C(h)} L(g, V_{h,X^h\subseteq X}),$$

where $\{h\}$ is a conjugacy class in G, $C(h)$ is the centralizer of h, and $L(g, V_{h,X^h\subseteq X}) = \sum_i (-1)^i \operatorname{tr}(g, H^i(V_{h,X^h\subseteq X}))$ is the holomorphic Lefschetz number. Using the Atiyah-Singer holomorphic Lefschetz theorem, the orbifold elliptic genus can be rewritten as follows. For a pair $g, h \in G$ of commuting elements, let $X^{g,h}$ be a connected component of the set of points in X fixed by both g and h, let x_λ be the Chern roots of a subbundle V_λ of $TX|_{X^{g,h}}$ on which both g and h act via the multiplication by $\exp(2\pi \mathrm{i}\lambda(g))$ and $\exp(2\pi \mathrm{i}\lambda(h))$, respectively, and let

$$\Phi(g, h, \lambda, z, \tau, x) := \frac{\theta(x/(2\pi \mathrm{i}) + \lambda(g) - \tau\lambda(h) - z)}{\theta(x/(2\pi \mathrm{i}) + \lambda(g) - \tau\lambda(h))} e^{2\pi \mathrm{i} z\lambda(h)z}.$$

Then

$$E_{\mathrm{orb}}(X, G; z, \tau) = \frac{1}{|G|} \sum_{gh=hg} \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}]. \quad (1.5)$$

This formula generalizes (1.2). (As we mentioned earlier, (1.2) has as a consequence (1.3), as was shown in [28].) For a thus-defined orbifold elliptic genus, we prove the formula of Dijkgraaf, Moore, Verlinde, and Verlinde (1.1). We also show that if X is a Calabi-Yau manifold, then $E_{\mathrm{orb}}(X, G; z, \tau)$ is a weak Jacobi form.

In Section 5 we conjecture (see Conjecture 5.1) that the two notions of elliptic genera coincide, which would extend the results of [5] and [18]. We prove this conjecture for the toric case and in dimension one. For Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties, the elliptic genus has already been defined in [12], using the description of the cohomology of chiral de Rham complex $\mathscr{M}SV$ for such hypersurfaces from [10] and borrowing the definition of elliptic genus via chiral de Rham complex in the nonsingular case:

$$\mathrm{Ell}(X) = y^{\dim X/2}\mathrm{Supertrace}_{H^*(\mathscr{M}SV(X))}\, y^{J[0]}q^{L[0]}.$$

Here $\mathscr{M}SV$ is the chiral de Rham complex constructed in [32] and $J[0]$ and $L[0]$ are the operators of the $N = 2$ super-Virasoro algebra acting on $H^*(\mathscr{M}SV(X))$. We use the combinatorial description of this genus, proved in [12], and the calculation of [11] to show that it coincides with the singular elliptic genus, up to an explicit normalization factor.

We continue to discuss Conjecture 5.1 in Section 6. We show that both notions of elliptic genera are invariant under complex cobordisms of G action. By using the known result about cobordism classes of the action of a cyclic group of prime order p, we prove Conjecture 5.1 for involutions.*

2. Preliminaries

2.1. Elliptic genus

Let X be a compact (almost complex) manifold. For a bundle V on X, we consider the following elements in the ring of formal power series over $K(X)$:

$$S_t(V) = \sum_i S^i(V)t^i, \qquad \Lambda_t(V) = \sum_i \Lambda^i(V)t^i,$$

where S^i (resp., Λ^i) is the ith symmetric (resp., exterior) power of V.

Let T_X (resp., $\bar{T}_X$) be the tangent (resp., cotangent) bundle. The elliptic genus of X can be defined as

$$\mathrm{Ell}(X; y, q) = \int_X \mathrm{ch}(\mathscr{E}LL_{y,q})\, \mathrm{td}(X),$$

where

$$\mathscr{E}LL_{y,q} := y^{-d/2} \otimes_{n\geq 1} \left(\Lambda_{-yq^{n-1}}\bar{T}_X \otimes \Lambda_{-y^{-1}q^n}T_X \otimes S_{q^n}\bar{T}_X \otimes S_{q^n}T_X\right).$$

*A proof of Conjecture 5.1 is given in our paper [13].

Clearly, this is an invariant of the cobordism class of X, and, moreover, $\mathscr{E}LL_{y,q}(X \times Y) = \mathscr{E}LL_{y,q}(X) \cdot \mathrm{Ell}_{y,q}(Y)$; that is, the elliptic genus is a genus in the sense of [27]. If x_i are the Chern roots of X, that is, for the total Chern class, we have $c(X) = \prod_l (1 + x_l)$; then

$$\mathrm{Ell}(X; y, q) = \int_X \prod_l x_l \frac{\theta(x_l/(2\pi \mathrm{i}) - z, \tau)}{\theta(x_l/(2\pi \mathrm{i}), \tau)}, \tag{2.1}$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ and $y = \mathrm{e}^{2\pi \mathrm{i}z}$. In (2.1)

$$\theta(z, \tau) = q^{1/8}(2 \sin \pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l \mathrm{e}^{2\pi \mathrm{i}z})(1 - q^l \mathrm{e}^{-2\pi \mathrm{i}z})$$

is the Jacobi theta function (see [14]).

In other words, (2.1) is the genus corresponding to the series $Q(x) = x\big(\theta(x/(2\pi \mathrm{i}) - z, \tau)/\theta(x/(2\pi \mathrm{i}), \tau)\big)$ (see [27]). It is not normalized in the sense that $Q(0) = (1/(2\pi i))(\theta(-z, \tau)/\theta'(0, \tau)) \neq 1$. It is often convinient to use the normalized version of the elliptic genus:

$$\mathrm{Ell}(X; y, q) = \int_X \prod_l \frac{x_l}{2\pi \mathrm{i}} \frac{\theta(x_l/(2\pi \mathrm{i}) - z, \tau)\theta'(0, \tau)}{\theta(x_l/(2\pi \mathrm{i}), \tau)\theta(-z, \tau)}. \tag{2.2}$$

For $q = 0$ we have $\mathrm{Ell}(X; y, q = 0) = y^{-d/2}\chi_{-y}(X)$, where

$$\chi_y(X) = \sum_{p,q} (-1)^q \dim H^q(X, \Omega_X^p) y^p$$

is Hirzebruch χ_y-genus (see [27]). In particular, $\mathrm{Ell}(X; y = 1, q = 0)$ is the topological Euler characteristic, $(-1)^{d/2} \mathrm{Ell}(X; y = -1, q = 0)$ is the signature, and so on.

If X is a Calabi-Yau, that is, if $K_X \sim 0$, then $\mathrm{Ell}(X; y, q)$ is a weak Jacobi form. Recall (see [20], [25]) that a weak Jacobi form of weight $k \in \mathbf{Z}$ and index $r \in \mathbf{Z}/2$ is a function on $H \times \mathbf{C}$ that satisfies

$$\phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k e^{2\pi i (rcz^2/(c\tau+d))} \phi(\tau, z),$$
$$\phi(\tau, z + m\tau + n) = (-1)^{2r(\lambda+\mu)} e^{-2\pi i r (m^2\tau + 2mz)} \phi(\tau, z)$$

and has a Fourier expansion $\sum_{l,m} c_{m,l} y^l q^m$ with nonnegative m.

2.2. Log-terminal singularities

We recall basic definitions related to singular varieties. Let Z be a normal irreducible projective variety. The $\mathbf{Q}$-Weil (resp., $\mathbf{Q}$-Cartier) divisor is a linear combination with rational coefficients of codimension one subvarieties (resp., Cartier divisors) on Z.

The canonical divisor K_Z of Z is a Weil divisor $\operatorname{div}(s)$, where $s = df_1 \wedge \cdots \wedge df_{\dim Z}$ (f_i are meromorphic functions) is a nonzero rational differential on Z. We call Z Gorenstein (resp., $\mathbf{Q}$-Gorenstein) if K_Z is Cartier (resp., $\mathbf{Q}$-Cartier).

A resolution of singularities of a variety Z is a proper birational morphism $f : Y \to Z$, where Y is smooth.

Definition 2.1
An IH-small resolution of Z is a regular map $Y \to Z$ such that for every $i \geq 1$ the set of points $z \in Z$ such that $\dim(f^{-1}(z)) \geq i$ has codimension greater than $2i$ in Z (see [9]).

Definition 2.2
Z has at worst log-terminal singularities if the following two conditions hold.

(i) Z is $\mathbf{Q}$-Gorenstein.

(ii) For a resolution $f : X \to Z$ whose exceptional set is a divisor with simple normal crossings, one has $\alpha_i > -1$ for all i in the relation $K_X = f^*K_Z + \sum \alpha_i E_i$.

A well-known result of birational geometry (see, e.g., [15]) states that for any resolution of a log-terminal variety Z, the coefficients α_i (called *discrepancies*) are bigger than (-1). A similar definition of log-terminality exists for pairs (Z, D), where D is a $\mathbf{Q}$-Weil divisor on a normal variety Z such that $(K_Z + D)$ is $\mathbf{Q}$-Cartier.

2.3. G-bundles
Let X be a complex manifold, and let G be a finite group of holomorphic transformations acting on X. Let V be a holomorphic G-bundle on X; that is, the action of G on X is extended to the action on V. The holomorphic Lefschetz number of $g \in G$ is

$$L(g, V) = \sum_i (-1)^i \operatorname{tr}\big(g, H^i(X, V)\big).$$

Let V^G be the sheaf whose sections over open sets are the G-invariants of the sections of V. We have (see [26], spectral sequences degenerate due to finiteness of G)

$$\chi(V^G) = \frac{1}{|G|} \sum_{g \in G} L(g, V).$$

The Lefschetz numbers are given by the data around the fixed-point sets (see [2]) as follows. Let N^g be the normal bundle to the fixed-point set X^g of g, and let N^{g*} be its dual. In the case when the action of G on a space Y is trivial, we have $K_G(Y) = K(Y) \otimes R(G)$ (see [2]), and hence one can define $W(g) \in K(Y)$ corresponding to

$W \in K_G(Y)$. In this notation,

$$L(g, V) = \frac{\operatorname{ch} V|_{X^g}(g)\,\operatorname{td}(T_{X^g})}{\operatorname{ch} \Lambda_{-1}(N^g)^*(g)}[X^g]. \tag{2.3}$$

For $g \in G$, the normal bundle N_{X^g} to the fixed-point set X^g can be decomposed into the direct sum $N_{X^g} = \bigoplus_i N(\theta_i)$, $\theta_i \in \mathbf{Q}$, where each $N(\theta_i)$ is the subbundle on which g acts as multiplication by $e^{2\pi i \theta_i}$. If $x_{\theta_i, j}$ are the Chern roots of $N(\theta_i)$, that is, if $c(N(\theta_i)) = \prod_j (1 + x_{\theta_i, j})$, then (2.3) can be rewritten as

$$L(g, V) = \frac{\operatorname{ch}(V|_{X^g})}{\prod_{i,j}(1 - e^{-x_j - \theta_{i,j}})} \operatorname{td}(X^g)[X^g].$$

3. Singular elliptic genus

In this section we define *singular elliptic genus* for a large class of singular varieties and more generally for pairs consisting of a variety and a $\mathbf{Q}$-Cartier divisor on it. This is by far the most general definition of the elliptic genus for singular varieties constructed to date. All varieties are assumed to be proper over $\operatorname{Spec}(\mathbf{C})$.

Definition 3.1

Let Z be a $\mathbf{Q}$-Gorenstein variety with log-terminal singularities, and let $\pi : Y \to Z$ be a desingularization of Z whose exceptional divisor $E = \sum_k E_k$ has simple normal crossings. The discrepancies α_k of the components E_k are determined by the formula

$$K_Y = \pi^* K_Z + \sum_k \alpha_k E_k.$$

We introduce Chern roots y_l of Y by $c(TY) = \prod_l (1 + y_l)$ and define cohomology classes $e_k := c_1(E_k)$. The *singular elliptic genus* of Z is then defined as a function of two variables z, τ given by

$$\begin{aligned}\widehat{\operatorname{Ell}}_Y(Z; z, \tau) := \int_Y \Big(\prod_l \frac{(y_l/(2\pi \mathrm{i}))\theta(y_l/(2\pi \mathrm{i}) - z)\theta'(0)}{\theta(-z)\theta(y_l/(2\pi \mathrm{i}))}\Big) \\ \times \Big(\prod_k \frac{\theta(e_k/(2\pi \mathrm{i}) - (\alpha_k + 1)z)\theta(-z)}{\theta(e_k/(2\pi \mathrm{i}) - z)\theta(-(\alpha_k + 1)z)}\Big),\end{aligned}$$

where $\theta(z, \tau)$ is the Jacobi theta function (see [14]). We often suppress the τ-dependence in our formulas.

We usually abuse notation and consider $\widehat{\operatorname{Ell}}$ to be a function of $y = \mathrm{e}^{2\pi \mathrm{i} z}$ and $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$. Strictly speaking, this function is multivalued because rational powers of y may occur.

The key result of this section is the following theorem.

THEOREM 3.2
The above-defined $\widehat{\mathrm{Ell}}_Y(Z; y, q)$ *does not depend on the choice of desingularization* Y *and therefore defines an invariant of* Z*, which we denote simply by* $\widehat{\mathrm{Ell}}(Z; y, q)$.

Proof
Because of the weak factorization theorem of [1], it suffices to show that $\widehat{\mathrm{Ell}}_Y(Z; y, q) = \widehat{\mathrm{Ell}}_{\tilde{Y}}(Z; y, q)$ when $\tilde{Y}$ is obtained from Y by a blow-up along a nonsingular subvariety X. We remark that the algorithm of [1] is compatible with the normal crossing condition (see [1, Theorem 0.3.1]), so we may assume that X has normal crossings with the components of the exceptional divisor of $\pi : Y \to Z$.

We use the notation of Fulton [23] for the blow-up diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

where $\tilde{X}$ is the exceptional divisor of the blow-down morphism. We also have $\pi : Y \to Z$ and $\pi \circ f : \tilde{Y} \to Z$. The discrepancies of the exceptional divisors of these morphisms are related by

$$K_Y = \pi^* K_Z + \sum_k \alpha_k E_k,$$
$$K_{\tilde{Y}} = f^* \pi^* K_Z + \sum_k \alpha_k E'_k + \Big(\sum_k \alpha_k \beta_k + r - 1\Big)\tilde{X},$$

where β_k is the multiplicity of E_k along X and r is the codimension of X in Y.

We use for a while the following technical assumption:

$$\begin{gathered}\text{The normal bundle } N \text{ to } X \text{ inside } Y \text{ is a pullback under } i\\ \text{of some rank } r \text{ bundle } M \text{ on } Y.\end{gathered} \tag{3.1}$$

We have the following exact sequences of coherent sheaves on $\tilde{Y}$ (see [23, Section 15.4]):

$$0 \to T\tilde{Y} \to f^* TY \to j_* F \to 0,$$
$$0 \to j_* \mathcal{O}_{\tilde{X}}(-1) \to j_* g^* i^* M \to j_* F \to 0,$$
$$0 \to \mathcal{O} \to \mathcal{O}(\tilde{X}) \to j_* \mathcal{O}_{\tilde{X}}(-1) \to 0,$$
$$0 \to f^* M(-\tilde{X}) \to f^*(M) \to j_* g^* i^* M \to 0.$$

Here F is the tautological quotient bundle on $\tilde{X}$. This implies

$$c(T\tilde{Y}) = c(f^*TY)\cdot(1+\tilde{x})\cdot\prod_l \frac{(1+f^*m_l-\tilde{x})}{(1+f^*m_l)},$$

where $c(M)=\prod_l(1+m_i)$ and $\tilde{x}=c_1(\mathscr{O}(\tilde{X}))$. Note also that $c_1(E_k')=f^*e_k-\beta_k\tilde{x}$. Therefore,

$$\begin{aligned}
&\widehat{\mathrm{Ell}}_{\tilde{Y}}(Z;y,q)\\
&\quad=\int_{\tilde{Y}}\Big(\prod_l\frac{(f^*y_l/(2\pi\mathrm{i}))\theta(f^*y_l/(2\pi\mathrm{i})-z)\theta'(0)}{\theta(-z)\theta(f^*y_l/(2\pi\mathrm{i}))}\Big)\\
&\qquad\times\Big(\frac{(\tilde{x}/(2\pi\mathrm{i}))\theta(\tilde{x}/(2\pi\mathrm{i})-z)\theta'(0)}{\theta(-z)\theta(\tilde{x}/(2\pi\mathrm{i}))}\Big)\\
&\qquad\times\Big(\prod_l\frac{\theta((f^*m_l-\tilde{x})/(2\pi\mathrm{i})-z)((f^*m_l-\tilde{x})/(2\pi\mathrm{i}))\theta(f^*m_l/(2\pi\mathrm{i}))}{\theta((f^*m_l-\tilde{x})/(2\pi\mathrm{i}))(f^*m_l/(2\pi\mathrm{i}))\theta(f^*m_l/(2\pi\mathrm{i})-z)}\Big)\\
&\qquad\times\Big(\prod_k\frac{\theta((f^*e_k-\beta_k\tilde{x})/(2\pi\mathrm{i})-(\alpha_k+1)z)\theta(-z)}{\theta((f^*e_k-\beta_k\tilde{x})/(2\pi\mathrm{i})-z)\theta(-(\alpha_k+1)z)}\Big)\\
&\qquad\times\Big(\frac{\theta(\tilde{x}/(2\pi\mathrm{i})-(\alpha_{\tilde{X}}+1)z)\theta(-z)}{\theta(\tilde{x}/(2\pi\mathrm{i})-z)\theta(-(\alpha_{\tilde{X}}+1)z)}\Big),
\end{aligned}$$

where $\alpha_{\tilde{x}}=r-1+\sum_k\alpha_k\beta_k$.

We now use $\int_{\tilde{Y}}a=\int_Y f_*(a)$. We write the Taylor expansion $\sum_n R_n(y,q)\tilde{x}^n$ of the expression under $\int_{\tilde{Y}}$ in the above identity. Observe that $f_*R_0(y,q)$ is exactly the class in $A(Y)$ whose integral is $\widehat{\mathrm{Ell}}_Y(Z;y,q)$; so we need to show that the contribution of the rest of the terms vanishes. Notice that $f_*\tilde{x}^n=0$ for $1\le n\le r-1$ and that $f_*\tilde{x}^{r+n}=i_*(s_n(i^*M))(-1)^{n+r-1}$, where $\sum_{n\ge 0}s_nt^n$ is the Segre polynomial of a vector bundle (see [23]). Hence, one needs to calculate

$$\begin{aligned}
&\int_Y\sum_{n\ge 0}i_*s_n(i^*M)(-1)^{n+r-1}\\
&\quad\times(\text{Coeff. at }t^{r+n})\Big[\Big(\prod_l\frac{(y_l/(2\pi\mathrm{i}))\theta(y_l/(2\pi\mathrm{i})-z)\theta'(0)}{\theta(-z)\theta(y_l/(2\pi\mathrm{i}))}\Big)\\
&\qquad\times\Big(\frac{(t/(2\pi\mathrm{i}))\theta(t/(2\pi\mathrm{i})-(\alpha_{\tilde{X}}+1)z)\theta'(0)}{\theta(t/(2\pi\mathrm{i}))\theta(-(\alpha_{\tilde{X}}+1)z)}\Big)\\
&\qquad\times\Big(\prod_l\frac{\theta((m_l-t)/(2\pi\mathrm{i})-z)((m_l-t)/(2\pi\mathrm{i}))\theta(m_l/(2\pi\mathrm{i}))}{\theta((m_l-t)/(2\pi\mathrm{i}))(m_l/(2\pi\mathrm{i}))\theta(m_l/(2\pi\mathrm{i})-z)}\Big)\\
&\qquad\times\Big(\prod_k\frac{\theta((e_k-\beta_kt)/(2\pi\mathrm{i})-(\alpha_k+1)z)\theta(-z)}{\theta((e_k-\beta_kt)/(2\pi\mathrm{i})-z)\theta(-(\alpha_k+1)z)}\Big)\Big].
\end{aligned}\tag{3.2}$$

We denote $n_l = i^*m_l$, $f_k = i^*e_k$ and use the fact that

$$\sum_{n\geq 0} s_n(i^*M)(-1)^n t^{-n} = \frac{t^r}{\prod_l (t-n_l)}$$

to rewrite (3.2) as

$$\begin{aligned}\text{const.} \int_X (\text{Coeff. at } t^{-1})\Big[\Big(\prod_l \frac{(x_l/(2\pi \mathrm{i}))\theta(x_l/(2\pi \mathrm{i})-z)\theta'(0)}{\theta(-z)\theta(x_l/(2\pi \mathrm{i}))}\Big) \\ \times \Big(\frac{\theta(t/(2\pi \mathrm{i})-(\alpha_{\tilde X}+1)z)\theta'(0)}{\theta(t/(2\pi \mathrm{i}))\theta(-(\alpha_{\tilde X}+1)z)}\Big) \\ \times \Big(\prod_l \frac{\theta((n_l-t)/(2\pi \mathrm{i})-z)\theta(n_l/(2\pi \mathrm{i}))}{\theta((n_l-t)/(2\pi \mathrm{i}))(n_l/(2\pi \mathrm{i}))\theta(n_l/(2\pi \mathrm{i})-z)}\Big) \\ \times \Big(\prod_k \frac{\theta((f_k-\beta_k t)/(2\pi \mathrm{i})-(\alpha_k+1)z)\theta(-z)}{\theta((f_k-\beta_k t)/(2\pi \mathrm{i})-z)\theta(-(\alpha_k+1)z)}\Big)\Big].\end{aligned} \tag{3.3}$$

Here we denote $c(TX) = \prod_l (1+x_l)$ and use $c(TX) = i^*c(TY)/i^*c(M)$. To show that (3.3) is zero, observe that the function whose coefficient at t^{-1} is measured is elliptic in t. Really, $t \to t + 2\pi \mathrm{i}$ obviously keeps it unchanged, and $t \to t + 2\pi \mathrm{i}\tau$ does not change it because $\alpha_{\tilde X} = \sum_k \alpha_k \beta_k + r - 1$. Here we have used the fact that none of the α's is equal to (-1), which follows from the condition that Z is log-terminal (see, e.g., [15]). It remains to show that $t = 0$ is the only pole of this function up to the lattice $2\pi \mathrm{i}(\mathbf{Z}+\mathbf{Z}\tau)$, such that the residue is zero. To do so, observe that the normal crossing condition implies $\beta_k \in \{0, 1\}$, and, moreover, whenever $\beta_k = 1$, the corresponding factor $\theta((f_k - t)/(2\pi \mathrm{i}) - z)$ in the denominator of the last product is offset by a factor $\theta((n_l - t)/(2\pi \mathrm{i}) - z)$ in the numerator of the second product.

We now get rid of assumption (3.1). Indeed, it is easy to see that the difference between $\widehat{\mathrm{Ell}}_Y(Z; y, q)$ and $\widehat{\mathrm{Ell}}_{\tilde Y}(Z; y, q)$ can be written as a degree of an element of $A(\tilde X)$ which is preserved when one deforms $i : X \to Y$ to the embedding of X into the normal cone for which assumption (3.1) is satisfied. □

We have not significantly used the log-terminality condition, except for the fact that we did not have to divide by $\theta(0 \cdot z)$. We now extend our definition of the singular elliptic genus to the category of pairs that consist of an algebraic variety and a $\mathbf{Q}$-Cartier divisor on it. To avoid (-1) discrepancies, we assume that the pair is Kawamata log-terminal.

Definition 3.3
Let Z be a projective variety, and let D be an arbitrary $\mathbf{Q}$-Weil divisor such that $K_Z + D$ is a $\mathbf{Q}$-Cartier divisor on Z. Let $\pi : Y \to Z$ be a desingularization of Z. We denote

by $E = \sum_k E_k$ the exceptional divisor of π plus the sum of the proper preimages of the components of D, and we assume that it has simple normal crossings. The discrepancies α_k of the components E_k are determined by the formula

$$K_Y = \pi^*(K_Z + D) + \sum_k \alpha_k E_k$$

and the requirement that the discrepancy of the proper transform of a component of D be the opposite of the coefficient of D at that component. In what follows, we assume that (Z, D) is a Kawamata log-terminal pair, which means that all discrepancies are greater than (-1).

We introduce Chern roots y_l of Y by $c(TY) = \prod_l (1 + y_l)$ and define

$$\widehat{\mathrm{Ell}}_Y(Z, D; y, q) := \int_Y \Big(\prod_l \frac{(y_l/(2\pi \mathrm{i}))\theta(y_l/(2\pi \mathrm{i}) - z)\theta'(0)}{\theta(-z)\theta(y_l/(2\pi \mathrm{i}))}\Big) \\ \times \Big(\prod_k \frac{\theta(e_k/(2\pi \mathrm{i}) - (\alpha_k + 1)z)\theta(-z)}{\theta(e_k/(2\pi \mathrm{i}) - z)\theta(-(\alpha_k + 1)z)}\Big),$$

where, as usual, $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, the τ-dependence is suppressed, and $e_k = c_1(\mathscr{O}(E_k))$.

THEOREM 3.4
The above-defined elliptic genus does not depend on the choice of the desingularization $\pi : Y \to Z$. We therefore denote it simply by $\widehat{\mathrm{Ell}}(Z, D; y, q)$.

Proof
Any two resolutions of singularities of Z can be connected by a sequence of blow-ups and blow-downs, and the argument of Theorem 3.2 works. Kawamata log-terminality implies that all discrepancies on all intermediate varieties are different from (-1). □

PROPOSITION 3.5
The elliptic genera of two different crepant resolutions of a Gorenstein projective variety coincide.

Proof
We show that the elliptic genus of a crepant resolution Y of a variety X equals the singular elliptic genus of X. If the exceptional set of the morphism $\pi : Y \to X$ is a divisor with simple normal crossings, then it is enough to observe that in Definition 3.1 the second product is trivial. In general, we can further blow up Y to get $\mu : Z \to Y$ so that the exceptional sets of μ and $\pi \circ \mu : Z \to X$ are divisors with simple normal crossings. Then the singular elliptic genera of Y and X calculated via Z are given by the same formula because the discrepancies coincide. □

Remark 3.6
In particular, the above proposition shows that the statement of [35, Theorem 8.1] can be extended to the full elliptic genus.

The following proposition shows that when $q \to 0$, we recover a formula for χ_y genus of (Z, D) which follows from [5].

PROPOSITION 3.7
Let (Z, D) be a Kawamata log-terminal pair. Then

$$\widehat{\mathrm{Ell}}(Z, D; u, q = 0) = (u^{-1/2} - u^{1/2})^{\dim Z} E_{\mathrm{st}}(Z, D; u, 1),$$

where E_{st} is defined as in [5].

Proof
To avoid confusion, we immediately remark that the second arguments in the singular elliptic genus and in Batyrev's E-function have drastically different meanings. The definition of $E_{\mathrm{st}}(Z, D)$ in [5] could be stated as

$$E_{\mathrm{st}}(Z, D; u, v) := \sum_{J \subset I} E(E_J; u, v) \prod_{j \in J} \Big(\frac{uv - 1}{uv^{\alpha_j + 1} - 1} - 1 \Big),$$

where $\sum_{i \in I} \alpha_i E_i$ is the exceptional divisor of a resolution $Y \to Z$ together with proper preimages of the components of D and is assumed to have normal crossings. Polynomials $E(E_J; u, v)$ are defined in terms of mixed Hodge structure on the cohomology of E_J (see [5]). Subvariety E_J is $\bigcap_{j \in J} E_j$, and the sum includes the empty subset J.

For each J,

$$E_{\mathrm{st}}(E_J; u, 1) = \int_{E_j} \prod_{i=1}^{\dim E_J} \frac{(1 - u\,\mathrm{e}^{-x_{i,J}}) x_{i,J}}{1 - \mathrm{e}^{-x_{i,J}}},$$

where $c(TE_J) = \prod_i (1 + x_i, J)$. The adjunction formula for complete intersections yields

$$c(TE_J) = i_J^*\big(c(TY)\big) / \prod_{j \in J} \big(1 + i_J^* c_1(E_j)\big),$$

where $i_J : E_J \to Y$ is the closed embedding. We then obtain

$$\begin{aligned} E(E_J; u, 1) &= \int_{E_j} \prod_{i=1}^{\dim Y} \frac{(1 - u\,\mathrm{e}^{-i_J^* x_i}) i_J^* x_i}{1 - \mathrm{e}^{-i_J^* x_i}} \prod_{j \in J} \frac{1 - \mathrm{e}^{-i_J^* e_j}}{(1 - u\,\mathrm{e}^{-i_J^* e_j}) i_J^* e_j} \\ &= \int_Y \prod_{i=1}^{\dim Y} \frac{(1 - u\,\mathrm{e}^{-x_i}) x_i}{1 - \mathrm{e}^{-x_i}} \prod_{j \in J} \frac{1 - \mathrm{e}^{-e_j}}{1 - u\,\mathrm{e}^{-e_j}}, \end{aligned}$$

where $c(TY) = \prod_i (1 + x_i)$. When we plug this result into Batyrev's formula, we get

$$\begin{aligned} E_{st}(Z, D; u, 1) &= \int_Y \prod_{i=1}^{\dim Y} \frac{(1 - u\,\mathrm{e}^{-x_i})x_i}{1 - \mathrm{e}^{-x_i}} \prod_{j \in I} \Big(1 + \frac{(u - u^{\alpha_j + 1})(1 - \mathrm{e}^{-e_j})}{(u^{\alpha_j + 1} - 1)(1 - u\,\mathrm{e}^{-e_j})} \Big) \\ &= \int_Y \prod_{i=1}^{\dim Y} \frac{(1 - u\,\mathrm{e}^{-x_i})x_i}{1 - \mathrm{e}^{-x_i}} \prod_{j \in I} \frac{(u - 1)(1 - u^{\alpha_j + 1}\,\mathrm{e}^{-e_j})}{(u^{\alpha_j + 1} - 1)(1 - u\,\mathrm{e}^{-e_j})} \\ &= (u^{-1/2} - u^{1/2})^{-\dim Z} \lim_{q \to 0} \widehat{\mathrm{Ell}}(Z, D; u, q). \end{aligned}$$

□

The following simple proposition establishes the modular properties of the singular elliptic genus in the Calabi-Yau case.

PROPOSITION 3.8

Let (Z, D) be a Kawamata log-terminal pair which is also a Calabi-Yau pair in the sense that $K_Z + D$ is zero as a $\mathbf{Q}$-Cartier divisor. Then the singular elliptic genus $\widehat{\mathrm{Ell}}(Z, D; y, q)$ has the transformation properties of the Jacobi form of weight $\dim Z$ *and index zero for the subgroup of the full Jacobi group generated by*

$$\begin{aligned} &(z, \tau) \to (z + n, \tau), \qquad (z, \tau) \to (z + n\tau, \tau), \\ &(z, \tau) \to (z, \tau + 1), \qquad (z, \tau) \to \Big(\frac{z}{\tau}, \frac{-1}{\tau} \Big), \end{aligned}$$

where n is the least common denominator of the discrepancies.

Proof

The transformation properties of $\theta(z, \tau)$ under $(z, \tau) \to (z + 1, \tau)$ and $(z, \tau) \to (z + \tau, \tau)$ together with Calabi-Yau condition

$$K_Y = \sum_k \alpha_k E_k$$

assure that

$$\widehat{\mathrm{Ell}}(Z, D; z + n, \tau) = \widehat{\mathrm{Ell}}(Z, D; z + n\tau, \tau) = \widehat{\mathrm{Ell}}(Z, D; z, \tau).$$

We need here the fact that $n\alpha_k \in \mathbf{Z}$. Similarly, the transformation properties of θ under $(z, \tau) \to (z, \tau + 1)$ show that

$$\widehat{\mathrm{Ell}}(Z, D; z, \tau + 1) = \widehat{\mathrm{Ell}}(Z, D; z, \tau).$$

It remains to investigate what happens under $(z, \tau) \to (z/\tau, -1/\tau)$. For this, one considers the change $(e_k, y_l) \to (e_k/\tau, y_l/\tau)$ in the formula of Definition 3.3. A rather lengthy but straightforward calculation, similar to that of [12, Theorem 2.2], shows that

$$\widehat{\mathrm{Ell}}\Big(Z, D; \frac{z}{\tau}, -\frac{1}{\tau}\Big) = \tau^{\dim Z}\widehat{\mathrm{Ell}}(Z, D; z, \tau). \qquad \square$$

Another application of our techniques is the following theorem, which complements similar results for Hodge numbers of Calabi-Yau manifolds (see, e.g., [4] and [17]).

THEOREM 3.9
The elliptic genera of two birationally equivalent Calabi-Yau manifolds coincide. Moreover, the statement is true for smooth projective algebraic manifolds X with $nK_X \sim 0$ for some n.

Proof
Let Z_1 and Z_2 be two birationally equivalent Calabi-Yau manifolds or their generalizations above. Let Y be a desingularization of the closure of the graph of the birational equivalence such that $\pi_{1,2} : Y \to Z_{1,2}$ are regular birational morphisms. Let n be the smallest integer such that $nK_{Z_{1,2}}$ is rationally equivalent to zero and therefore has a global section. Global sections of the pluricanonical bundle are birational invariants, so one can consider the divisor $\sum_k a_k E_k$ of this section on Y. It is easy to see that for both morphisms π_1 and π_2 the exceptional divisor is $\sum_k (a_k/n)E_k$, which we can then assume to have simple normal crossings (perhaps by passing to a new desingularization). Therefore, the elliptic genera of $Z_{1,2}$ are calculated on Y using the same discrepancies. $\square$

Remark 3.10
It is interesting to compare the results of this section with the work of Totaro in [35], where he tried to see which Chern numbers can be meaningfully defined for singular varieties. For varieties that admit IH-small resolutions, the singular elliptic genus provides the maximum possible collection of such numbers. Totaro has shown that every flop-invariant Chern number comes from the elliptic genus, and he obtained partial results in the opposite direction by means of intersection cohomology.

In general, coefficients of the singular elliptic genus of Z at $y^k q^l$ provide analogs of Chern numbers of singular varieties in the following sense.

(1) They are the invariants of the isomorphism class of singular spaces.

(2) For manifolds, these invariants are the usual Chern numbers (i.e., linear combinations of $c_{i_1}(X)\cdots c_{i_N}(X)[X]$, where $\sum_{k=1}^{k=N} i_k = \dim X$ and $[X]$ is the fundamental class of a manifold X).

(3) These invariants are unchanged under small resolutions.

In fact, for singular varieties, elliptic genera may contain more information than in the nonsingular case. For varieties with non-Gorenstein singularities, the singular elliptic genus may depend on rational powers of y. Moreover, there exist examples of Gorenstein varieties whose elliptic genera do not lie in the span of elliptic genera of nonsingular varieties. This can be seen already at the level of the χ_y genus (see [3] for an example of a variety with Gorenstein canonical singularities whose E-function is not a polynomial).

We hope that elliptic genera of singular varieties can be interpreted as nontrivial invariants of a not-yet-defined cobordism theory of singular spaces. Transformations leaving the singular elliptic genus invariant in such a theory for smooth manifolds should include the usual cobordisms as well as flops. It would be interesting to compare our results with the invariants of Witt spaces studied by P. Siegel; the latter, however, were defined in SO rather than in the complex category (see [9], [34]).

Remark 3.11

It is an open question whether the notion of singular elliptic genus can be extended beyond Kawamata log-terminal pairs, and some partial resuts in this direction can be obtained as follows. Assume that some of the discrepancies α_k in Definition 3.3 equal (-1). One can try to define singular elliptic genus by continuity. Namely, for any effective Cartier divisor H on Z that contains all singular points of Z, and whose preimage on Y has simple normal crossings with the exceptional divisor and preimage of of D, we calculate

$$\lim_{n\to\infty} \widehat{\mathrm{Ell}}_Y(Z, D + H/n; z, \tau)$$

for each (z, τ). If such a limit exists and is independent of H, then we call it $\widehat{\mathrm{Ell}}_Y(Z, D; z, \tau)$. Notice that if n is sufficiently big, then the discrepancies of all divisors E_k calculated for the pair $(Z, D + H/n)$ are not equal to (-1). However, we do not know of any necessary or sufficient conditions for the limit to exist. We also cannot prove in general that this limit is independent of Y.

In particular, we cannot show in general that two resolutions Y_1 and Y_2 with no (-1) discrepancies give the same singular elliptic genus because the sequence of blow-ups and blow-downs which connects Y_1 to Y_2 may have intermediate varieties with (-1) discrepancies. Provided a single divisor H can be chosen to satisfy the normal crossing condition for both resolutions, one gets

$$\begin{aligned}\widehat{\mathrm{Ell}}_{Y_1}(Z, D; z, \tau) &= \lim_{n\to\infty} \widehat{\mathrm{Ell}}_{Y_1}(Z, D + H/n; z, \tau)\\ &= \lim_{n\to\infty} \widehat{\mathrm{Ell}}_{Y_2}(Z, D + H/n; z, \tau) = \widehat{\mathrm{Ell}}_{Y_2}(Z, D; z, \tau),\end{aligned}$$

where the middle identity follows from the argument of Theorem 3.2 since for large n all intermediate discrepancies will be different from (-1). A similar approach allows one to extend the definition of elliptic genus to arbitrary Q-divisors D on a log-terminal variety Z by looking at $\lim_{n\to\infty} \widehat{\mathrm{Ell}}(Z, (n+1)D/n)$.

4. Orbifold elliptic genus and DMVV formula

In this section we define *orbifold elliptic genus*, which we conjecture to equal the singular elliptic genus of Section 3. We delay the comparison of these two genera until Section 5. Instead, the goal of this section is to show how this definition of orbifold elliptic genus allows one to recover the formula of [19] whose derivation was based partly on heuristic string-theoretic arguments. Our definition of elliptic genus is inspired by the calculations of [10].

Definition 4.1

Let X be a *smooth* algebraic variety acted upon by a finite group G. We assume that the subgroup of elements of G acting trivially on X contains only the identity. We define the following function of two variables, which we call the *orbifold elliptic genus* of X/G:

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) := y^{-\dim X/2} \sum_{\{h\}, X^h} y^{F(h, X^h \subseteq X)} \frac{1}{|C(h)|} \sum_{g \in C(h)} L(g, V_{h, X^h \subseteq X}),$$

where $F(h, X^h \subseteq X)$ is the fermionic shift (see [38], [7]) and $V_{h,X^h \subseteq X}$ is a vector bundle over X^h defined as follows. Let $TX|_{X^h}$ decompose into eigensheaves for h as

$$V_0 \oplus \Big(\bigoplus_{\lambda : \langle h \rangle \to \mathbf{Q}/\mathbf{Z}} V_\lambda \Big). \tag{4.1}$$

We lift $\lambda(h)$ to a rational number in $[0, 1)$. Then $V_{h,X^h \subseteq X}$ is defined as

$$\begin{aligned} V_{h,X^h \subseteq X} := \bigotimes_{k \geq 1} \Big[&(\Lambda^\bullet V_0^* y q^{k-1}) \otimes (\Lambda^\bullet V_0 y^{-1} q^k) \otimes (\mathrm{Sym}^\bullet V_0^* q^k) \otimes (\mathrm{Sym}^\bullet V_0 q^k) \\ &\otimes \Big[\bigotimes_{\lambda \neq 0} (\Lambda^\bullet V_\lambda^* y q^{k-1+\lambda(h)}) \otimes (\Lambda^\bullet V_\lambda y^{-1} q^{k-\lambda(h)}) \\ &\otimes (\mathrm{Sym}^\bullet V_\lambda^* q^{k-1+\lambda(h)}) \otimes (\mathrm{Sym}^\bullet V_\lambda q^{k-\lambda(h)}) \Big] \Big]. \end{aligned}$$

Remark 4.2

Another way to state this definition is

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) := y^{-\dim X/2} \sum_{\{h\}, X^h} y^{F(h, X^h \subseteq X)} \chi\big(H^\bullet(V_{h, X^h \subseteq X}^{C(h)})\big).$$

THEOREM 4.3

Let X and G be as above, and let $X^{g,h}$ be the set of fixed points of a pair of commuting elements $g, h \in G$. Let $TX|_{X^{g,h}} = \bigoplus W_\lambda$ be the decomposition (refinement of (4.1)) *of the restriction on $X^{g,h}$ of the tangent bundle into the direct sum of line bundles on which g (resp., h) acts as multiplication by* $\mathrm{e}^{2\pi\mathrm{i}\lambda(g)}$ *(resp.,* $\mathrm{e}^{2\pi\mathrm{i}\lambda(h)}$*). Denote by x_λ the Chern roots of the bundle W_λ.*

(1) *We have*

$$\mathrm{Ell}_{\mathrm{orb}}(X,G)=\frac{1}{|G|}\sum_{g,h,gh=hg}\Big(\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\Big)\times\prod_{\lambda}\frac{\theta(\tau,x_\lambda/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h)-z)}{\theta(\tau,x_\lambda/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h))}e^{2\pi\mathrm{i}\lambda(h)z}[X^{g,h}].$$

(2) *Let X be a Calabi-Yau of dimension d, such that $H^0(X,K_X)=\mathbf{C}$. Denote by n the order of G in* $\mathrm{Aut}H^0(X,K_X)$*. Then* $\mathrm{Ell}_{\mathrm{orb}}(X,G)$ *is a weak Jacobi form of weight zero and index $d/2$ with respect to the subgroup of the Jacobi group Γ^J generated by transformations*

$$(z,\tau)\to(z+n,\tau),\qquad (z,\tau)\to(z+n\tau,\tau),$$
$$(z,\tau)\to(z,\tau+1),\qquad (z,\tau)\to\Big(\frac{z}{\tau},-\frac{1}{\tau}\Big).$$

In particular, if the action preserves holomorphic volume, then $\mathrm{Ell}_{\mathrm{orb}}(X,G)$ *is a weak Jacobi form of weight zero and index $d/2$ for the full Jacobi group.*

Proof

We replace the contribution of each conjugacy class by an average contribution of its elements to obtain

$$\mathrm{Ell}_{\mathrm{orb}}(X,G)=\frac{1}{|G|}y^{-\dim X/2}\sum_{gh=hg}y^{F(h,X^h\subset X)}L(g,V_{h,X^h\subset X}).$$

Using holomorphic Lefschetz theorem, we obtain

$$\mathrm{Ell}_{\mathrm{orb}}(X,G)=\frac{1}{|G|}y^{-\dim X/2}\sum_{gh=hg}y^{F(h,X^h\subset X)}\frac{\mathrm{ch}(V_{h,X^h\subset X}|_{X^{g,h}})(g)\,\mathrm{td}(T_{X^{g,h}})[X^{g,h}]}{\mathrm{ch}\Lambda_{-1}(N^g_{X^h})^*(g)},$$

where $N^g_{X^h}$ is the normal bundle to $X^{g,h}$ in X^h. An explicit calculation of the Chern and Todd classes then yields

$$\begin{aligned}
\mathrm{Ell}_{\mathrm{orb}}(X,G) &= \frac{1}{|G|}\sum_{gh=hg} y^{F(h,X^h\subset X)-\dim X/2}\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\\
&\quad\times \prod_{k\geq 1,\lambda}\frac{(1-yq^{k-1+\lambda(h)}e^{-x_\lambda-2\pi\mathrm{i}\lambda(g)})(1-y^{-1}q^{k-\lambda(h)}e^{x_\lambda+2\pi\mathrm{i}\lambda(g)})}{(1-q^{k-1+\lambda(h)}e^{-x_\lambda-2\pi\mathrm{i}\lambda(g)})(1-q^{k-\lambda(h)}e^{x_\lambda+2\pi\mathrm{i}\lambda(g)})}\\
&= \frac{1}{|G|}\sum_{gh=hg}\Big(\prod_{\lambda(h)=\lambda(g)=0} x_\lambda\Big)\\
&\quad\times\prod_{\lambda}\frac{\theta(x_\lambda/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h)-z)}{\theta(x_\lambda/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h))}e^{2\pi\mathrm{i}z\lambda(h)}[X^{g,h}],
\end{aligned}$$

which proves the first part of the theorem.

To verify the modular property, we denote

$$\Phi(g,h,\lambda,z,\tau,x) := \frac{\theta(x/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h)-z)}{\theta(x/(2\pi\mathrm{i})+\lambda(g)-\tau\lambda(h))}e^{2\pi\mathrm{i}z\lambda(h)},$$

where λ is a character of the subgroup of G generated by g and h. Then

$$E_{\mathrm{orb}}(z,\tau) = \frac{1}{|G|}\sum_{gh=hg}\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\prod_\lambda \Phi(g,h,z,\tau,x_\lambda)[X^{g,h}], \tag{4.2}$$

where we suppress (X,G) from the notation for the sake of brevity. We have

$$\Phi(g,h,\lambda,z+1,\tau,x) = -\,\mathrm{e}^{2\pi\mathrm{i}\lambda(h)}\cdot\Phi(g,h,\lambda,z,\tau,x),$$

and hence $\mathrm{Ell}_{\mathrm{orb}}(z+n,\tau) = (-1)^{dn}\,\mathrm{Ell}_{\mathrm{orb}}(z,\tau)$ since by assumption $n\cdot\sum\lambda(h)\in\mathbf{Z}$. It is clear that

$$\Phi(g,h,\lambda,z,\tau+1,x) = \Phi(gh^{-1},h,\lambda,z,\tau,x),$$

and hence $\mathrm{Ell}_{\mathrm{orb}}(z,\tau+1) = \mathrm{Ell}_{\mathrm{orb}}(z,\tau)$. We have

$$\Phi(g,h,\lambda,z+n\tau,\tau,x) = (-1)^n\,\mathrm{e}^{-2\pi\mathrm{i}nz-\pi\mathrm{i}n^2\tau}\,\mathrm{e}^{nx+2\pi\mathrm{i}n\lambda(g)}\cdot\Phi(g,h,\lambda,z,\tau,x),$$

and hence

$$\mathrm{Ell}_{\mathrm{orb}}(z+n\tau,\tau) = (-1)^{dn}e^{-2\pi\mathrm{i}dnz-\pi\mathrm{i}dn^2\tau}\,\mathrm{Ell}_{\mathrm{orb}}(z,\tau)$$

since X is a Calabi-Yau and $n\lambda(g) \in \mathbf{Z}$. Finally,

$$\begin{aligned}
&\Phi\Big(g, h, \lambda, \frac{z}{\tau}, -\frac{1}{\tau}, \frac{x}{\tau}\Big) \\
&\qquad = \frac{\theta(-z/\tau + x/(2\pi i\tau) + \lambda(g) + \lambda(h)/\tau, -1/\tau)}{\theta(x_\lambda/(2\pi \mathrm{i}) + \lambda(g) + \lambda(h)/\tau, -1/\tau)} e^{2\pi \mathrm{i} z\lambda(h)/\tau} \\
&\qquad = e^{\pi \mathrm{i} z^2/\tau - 2\pi \mathrm{i} z/\tau (x/(2\pi \mathrm{i}) + \lambda(g)\tau + \lambda(h))} \\
&\qquad\quad \times \frac{\theta(-z + x/(2\pi \mathrm{i}) + \lambda(g)\tau + \lambda(h), \tau)}{\theta(x/(2\pi \mathrm{i}) + \lambda(g)\tau + \lambda(h), \tau)} e^{2\pi \mathrm{i} z\lambda(h)/\tau} \\
&\qquad = e^{\pi \mathrm{i} z^2/\tau - zx/\tau} \cdot \frac{\theta(-z + x/(2\pi \mathrm{i}) + \lambda(g)\tau + \lambda(h), \tau)}{\theta(x/(2\pi \mathrm{i}) + \lambda(g)\tau + \lambda(h), \tau)} e^{2\pi \mathrm{i} z(-\lambda(g))} \\
&\qquad = e^{\pi \mathrm{i} z^2/\tau - zx/\tau} \cdot \Phi(h, g^{-1}, \lambda, z, \tau, x).
\end{aligned}$$

Then the Jacobi transformation properties follow easily from (4.2), similarly to [12, proof of Theorem 2.2].

It is straightforward to see from (4.2) that the orbifold elliptic genus is holomorphic and has the Fourier expansion with nonnegative powers of q. □

We apply our definition of the orbifold elliptic genus to symmetric products of a smooth variety. This gives a mathematical justification of the physical calculation performed in [19]. More precisely, we calculate the generating function for the orbifold elliptic genera introduced above for the action of the symmetric groups. To a certain extent, our calculation follows [19], but we now have precise mathematical definitions.

THEOREM 4.4
Let X be a smooth variety with elliptic genus $\sum_{m,l} c(m,l) y^l q^m$, where the elliptic genus is normalized as in [19] and [12]. Then

$$\sum_{n\geq 0} p^n \operatorname{Ell}_{\mathrm{orb}}(X^n, \Sigma_n; y, q) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}}.$$

We start with the following lemma, essentially contained in [19, Section 2.2], which we include only for completeness.

LEMMA 4.5
Let $V = V_{\mathrm{even}} \oplus V_{\mathrm{odd}}$ be a supersymmetric space, and let A and B be two commuting operators preserving parity decomposition of V, such that B has only nonnegative integer eigenvalues. We assume that V splits into a direct sum of eigenspaces V_m of

the operator B and that each V_m is finite-dimensional. Define

$$\begin{aligned}\chi(V)(y,q) &= \mathrm{Supertrace}_V\, y^A q^B := \mathrm{tr}_{V_{\mathrm{even}}}(y^A q^B) - \mathrm{tr}_{V_{\mathrm{odd}}}(y^A q^B)\\ &= \sum_{m,l} d(m,l) q^m y^l,\end{aligned}$$

where $d(m,l)$ is the superdimension of the space $V_{m,l} = \{v \in V | Av = lv, Bv = mv\}$. The operators A and B act on the space of invariants of the symmetric group acting on $V^{\otimes^N}$ and

$$\sum_N p^N \mathrm{Supertrace}_{\mathrm{Sym}^N(V)}\, y^A q^B = \prod_{m,l} \frac{1}{(1-pq^m y^l)^{d(m,l)}},$$

where the right-hand side is expanded as a power series in q and p.

Proof

It is easy to see that it is enough to check the lemma for a one-dimensional space $V = V_{m,l}$. If V is even, then

$$\sum_N p^N \mathrm{Supertrace}_{\mathrm{Sym}^N(V)}\, y^A q^B = \sum_{N \geq 0} p^N y^{Nl} q^{Nm} = (1-pq^m y^l)^{-\mathrm{superdim} V}.$$

If V is odd, then

$$\sum_N p^N \mathrm{Supertrace}_{\mathrm{Sym}^N(V)}\, y^A q^B = 1 - py^l q^m = (1-pq^m y^l)^{-\mathrm{superdim} V}. \qquad \square$$

Proof of Theorem 4.4

We observe that for a fixed k the conjugacy classes of Σ_k are indexed by the numbers a_i of cycles of length i in the permutation. For each $h \in \Sigma_k$, the fixed-point set $(X^k)^h$ consists of the Cartesian products of several copies of X, one for each cycle. For a cycle of length i, the corresponding X is embedded into X^i. The centralizer group is a semidirect product of its normal subgroup $\prod_i (\mathbf{Z}/i\mathbf{Z})^{a_i}$, which acts by cyclic permutations inside cycles of h, and the product of symmetric groups $\prod_i \Sigma_{a_i}$, which act by permuting cycles of the same length.

Our definition of the elliptic genus then gives

$$
\begin{aligned}
&\sum_{n\geq 0} p^n \operatorname{Ell}_{\mathrm{orb}}(X^n, \Sigma_n; y, q)\\
&\quad= \sum_{a_1, a_2, \ldots, a_n} p^{a_1+2a_2+\cdots+na_n} y^{-(\dim X/2)(a_1+2a_2+\cdots+na_n)}\\
&\qquad\times \prod_{i=1}^{n} y^{a_i F(i-\text{cycle}, X\subseteq X^i)} \chi\big((H^\bullet(V_{i-\text{cycle}, X\subseteq X^i})^{\bigotimes a_i})^{\Sigma_{a_i} \rtimes (\mathbf{Z}/i\mathbf{Z})^{a_i}}\big)\\
&\quad= \prod_{i=1}^{\infty} \chi\big(\operatorname{Sym}^\bullet(p^i y^{-i\dim X/2+F(i-\text{cycle}, X\subseteq X^i)} H^\bullet(V_{i-\text{cycle}, X\subseteq X^i})^{\mathbf{Z}/i\mathbf{Z}})\big). \qquad (4.3)
\end{aligned}
$$

The symbol Sym should be interpreted here as the supersymmetric product where the cohomology of $V_{h, X^h\subseteq X}$ is given parity by the sum of the cohomology number and the parity of the exterior algebras.

We now calculate

$$
\chi_i(y, q) = \chi\big(p^i y^{-i\dim X/2+F(i-\text{cycle}, X\subseteq X^i)} H^\bullet(V_{i-\text{cycle}, X\subseteq X^i})^{\mathbf{Z}/i\mathbf{Z}}\big).
$$

We denote the i-cycle by h and observe that

$$
TX^i|_X = \bigoplus_{j=0,\ldots,i-1;\lambda(h)=j/i} TX_j.
$$

This implies $F(h, X\subseteq X^i) = \dim X \sum_{j=0}^{i-1} j/i = \dim X((i-1)/2)$, which allows us to write

$$
\begin{aligned}
&\chi_i(y, q)\\
&\quad= p^i y^{-\dim X/2} \chi\Big[\big[H^\bullet(\bigotimes_{k\geq 1}\big[(\Lambda^\bullet T^* y q^{k-1}) \otimes (\Lambda^\bullet T y^{-1} q^k) \otimes (\operatorname{Sym}^\bullet T^* q^k)\\
&\qquad\qquad \otimes (\operatorname{Sym}^\bullet T q^k) \otimes \big[\bigotimes_{j=1,\ldots,i-1} (\Lambda^\bullet T^* y q^{k-1+j/i})\\
&\qquad\qquad \otimes (\Lambda^\bullet T y^{-1} q^{k-j/i}) \otimes (\operatorname{Sym}^\bullet T^* q^{k-1+j/i})\\
&\qquad\qquad \otimes (\operatorname{Sym}^\bullet T q^{k-j/i})\big]\big]\big]^{\mathbf{Z}/i\mathbf{Z}}\Big]\\
&\quad= p^i y^{-\dim X/2} \frac{1}{i} \sum_{r=0}^{i-1} \int_X \prod_{l=1}^{\dim X} x_l\\
&\qquad\times \prod_{k\geq 1} \prod_{m=0}^{i-1} \frac{(1 - y q^{k-1+m/i} \xi^{mr} \mathrm{e}^{-x_l})(1 - y^{-1} q^{k-m/i} \xi^{-mr} \mathrm{e}^{x_l})}{(1 - q^{k-1+m/i} \xi^{mr} \mathrm{e}^{-x_l})(1 - q^{k-m/i} \xi^{-mr} \mathrm{e}^{x_l})}
\end{aligned}
$$

$$= p^i y^{-\dim X/2} \frac{1}{i} \sum_{r=0}^{i-1} \int_X \prod_{l=1}^{\dim X} x_l$$
$$\times \prod_{j\geq 1} \frac{(1 - yq^{(j-1)/i} \xi^{(j-1)r} \mathrm{e}^{-x_l})(1 - y^{-1} q^{j/i} \xi^{jr} \mathrm{e}^{x_l})}{(1 - q^{(j-1)/i} \xi^{(j-1)r} \mathrm{e}^{-x_l})(1 - q^{j/i} \xi^{jr} \mathrm{e}^{x_l})}$$
$$= p^i 1/i \sum_{r=0}^{i-1} \mathrm{Ell}(X; y, q^{1/i} \xi^r) = \sum_{m,l} c(mi, l) y^l q^m.$$

Here we have denoted the primitive ith root of unity by ξ. Now Lemma 4.5 finishes the proof. □

Remark 4.6

In [37] the authors conjectured an equivariant version of Theorem 4.4. Its proof follows using the same arguments as above. More precisely, we have the following. Let X and G be as above, and let $G \wr \Sigma_n$ be the wreath product (consisting of pairs $((g_1, \dots, g_n); \sigma)$, $g_i \in G, \sigma \in \Sigma_n$, with multiplication $((g_1, \dots, g_n); \sigma_1) \cdot ((h_1, \dots, h_n); \sigma_2) = ((g_1 \cdot h_{\sigma_1^{-1}(1)}, \dots, g_n \cdot h_{\sigma_1^{-1}(n)}); \sigma_1\sigma_2)$). $G \wr \Sigma_n$ acts in an obvious way on X^n, and if $\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) = \sum c_G(m, l) y^l q^m$, then

$$\sum_{n\geq 0} p^n \,\mathrm{Ell}_{\mathrm{orb}}(X^n, G \wr \Sigma_n; y, q) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c_G(mi,l)}}. \tag{4.4}$$

To obtain a proof of this formula, one should make the following changes in the above proof of Theorem 4.4. Using the description of the conjugacy classes in wreath products (see, e.g., [29]), $\sum_{n\geq 0} p^n \,\mathrm{Ell}_{\mathrm{orb}}(X^n, G \wr \Sigma_n; y, q)$ can be rewritten as the right-hand side of the first row of (4.3) with summation taken over collections $\{h\}, a_1, \dots, a_n$, where a_i, as earlier, are positive integers and $\{h\}$ runs through all conjugacy classes in G. The same transformation used in (4.3) now yields the product over i and $\{h\}$ of terms in which invariants are taken for the semidirect product of the centralizer of h and $\mathbf{Z}/i\mathbf{Z}$ with the sheaf V constructed for X^h. Finally, each term in this product is the graded dimension of a supersymmetric algebra, which Lemma 4.5 expresses in terms of $\chi_{i,\{h\}}$. A calculation similar to the calculation of χ_i above identifies $\chi_{i,\{h\}}$ with

$$\sum_{m,l} c_{\{h\}}(mi, l) y^l q^m = y^{-\dim X/2 + F(h, X^h \subseteq X)} \frac{1}{|C(h)|} \sum_{g\in C(h)} L(g, V_{h, X^h \subseteq X})$$

(the component of the orbifold elliptic genus corresponding to the conjugacy class $\{h\}$). This yields (4.4).

5. Comparison of different notions of elliptic genera

It is natural to ask how the orbifold elliptic genus of X/G is related to its singular elliptic genus. To begin, even in the case $|G| = 1$, these two genera differ by a normalization factor. In addition, when $\mu : X \to X/G$ has a ramification divisor $D = \sum_i (\nu_i - 1) D_i$, one has to compare $\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q)$ not to $\widehat{\mathrm{Ell}}(X/G; y, q)$ but rather to $\widehat{\mathrm{Ell}}(X/G, \Delta_{X/G}; y, q)$, where

$$\Delta_{X/G} := \sum_j \Big(\frac{\nu_j - 1}{\nu_j}\Big)\mu(D_j)$$

with the sum taken among representatives D_j of the orbits of the action of G on the components of the ramification divisor.

CONJECTURE 5.1

Let X be a smooth algebraic variety equipped with an effective action of a finite group G. Then

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) = \Big(\frac{2\pi \mathrm{i}\, \theta(-z, \tau)}{\theta'(0, \tau)}\Big)^{\dim X} \widehat{\mathrm{Ell}}(X/G, \Delta_{X/G}; y, q),$$

where $\Delta_{X/G}$ is defined as above.

We now present some evidence to support this conjecture.

PROPOSITION 5.2

Conjecture 5.1 holds in the limit $\tau \to \mathrm{i}\infty$.

Proof

At $q = 0$, the function $\mathrm{Ell}_{\mathrm{orb}}$ specializes to $E_{\mathrm{orb}}(y, 1)$ of [5]. Then the result of [18] allows one to rewrite it in terms of $E_{\mathrm{st}}(y, 1)$, and Proposition 3.7 finishes the proof. □

PROPOSITION 5.3

Conjecture 5.1 holds in the case when X is a smooth toric variety and G is a subgroup of the big torus of X.

Proof

Let Σ be the defining cone of X in the lattice N (see, e.g., [16]). Let n_i be the generators of one-dimensional cones of Σ. The group G can be identified with N'/N, where N' is a sublattice of N of finite coindex. Then the variety X/G is given by the same cone Σ in the new lattice N'. The map $\mu : X \to X/G$ has ramification if and only if for some one-dimensional rays of Σ points n_i are no longer minimal in the new lattice.

Torus-invariant divisors on a toric variety correspond to piecewise linear functions on the fan. It is easy to see that the definition of $\Delta_{X/G}$ assures that the piecewise linear function that takes values (-1) on all n_i gives the divisor $K_{X/G} + \Delta_{X/G}$. We denote this piecewise linear function by deg. One can show that

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; y, q) = \Big(\frac{2\pi \mathrm{i}\theta(-z, \tau)}{\theta'(0, \tau)}\Big)^{\dim X} f_{N', \deg z}(q),$$

where $f_{N', \deg z}(q)$ is the function defined in [11]. More explicitly,

$$f_{N', \deg z}(q) = \sum_{m \in (N')^*} \Big(\sum_{C \in \Sigma} (-1)^{\operatorname{codim} C} \text{a.c.} \sum_{n \in C \cap N'} q^{m \cdot n} \, \mathrm{e}^{2\pi \mathrm{i} z \deg(n)}\Big),$$

where a.c. means analytic continuation. The proof of this fact is based on explicit calculation of the Euler characteristics of the bundles $V_{X^h \subseteq X}$ by means of Čech cohomology. The calculation is very similar to that of [11, Theorem 3.4] and is left to the reader. We remark that the sum over h in Definition 4.1 facilitates the change from N to N', while taking $C(h)$-invariants is responsible for the switch from N^* to its sublattice $(N')^*$.

Now let $Y \to X/G$ be a toric desingularization of X/G given by the subdivision Σ_1 of Σ. We denote the codimension one strata of Y by E_k and the generators of the corresponding one-dimensional cones of Σ_1 by r_k. We also denote the first Chern classes of the corresponding divisors by e_k, and we get

$$\begin{aligned}\widehat{\mathrm{Ell}}(X/G, \Delta_{X/G}; y, q) = \int_Y \Big(\prod_l \frac{(y_l/(2\pi \mathrm{i}))\theta(y_l/(2\pi \mathrm{i}) - z)\theta'(0)}{\theta(-z)\theta(y_l/(2\pi \mathrm{i}))}\Big) \\ \times \Big(\prod_k \frac{\theta(e_k/(2\pi \mathrm{i}) - (\alpha_k + 1)z)\theta(-z)}{\theta(e_k/(2\pi \mathrm{i}) - z)\theta(-(\alpha_k + 1)z)}\Big),\end{aligned}$$

where $c(TY) = \prod_l (1 + y_l)$ and $\alpha_k = \deg(r_k) - 1$. We use $c(TY) = \prod_k (1 + e_k)$ to rewrite $\widehat{\mathrm{Ell}}(X/G, \Delta_{X/G}; y, q)$ as

$$\int_Y \Big(\prod_k \frac{(e_k/(2\pi \mathrm{i}))\theta(e_k/(2\pi \mathrm{i}) - \deg(r_k)z)\theta'(0)}{\theta(-\deg(r_k)z)\theta(e_k/(2\pi \mathrm{i}))}\Big),$$

which equals $f_{N', \deg z}(q)$ by [11, Theorem 3.4]. We have used here the fact that f does not change when the fan is subdivided. □

Remark 5.4

Proposition 5.3 was the main motivation behind our definition of the singular elliptic genus.

PROPOSITION 5.5

Conjecture 5.1 holds for $\dim X = 1$.

Proof
Expanding θ functions as (linear) polynomials in cohomology classes, one obtains that the singular genus is equal to $(2g-2)\theta'(-z)/(2\pi\mathrm{i}\theta(-z))$ plus the sum of contributions of singular points that depend on the ramification numbers only. Here g is the genus of X/G. For the orbifold genus, one needs to notice that the $h = \mathbf{id}$ term gives $(2g-2)\theta'(-z)/(2\pi\mathrm{i}\theta(-z))$ plus contributions of points because it is the Euler characteristic of the bundle on the quotient that equals the usual elliptic genus bundle twisted at the ramification points. Since the equality holds in the toric case of the d-fold covering of $\mathbf{P}^1$ by $\mathbf{P}^1$, which has two points of ramification $(d-1)$, the extra terms for the two genera coincide, which finishes the proof. $\square$

One would also want to compare the singular elliptic genus to the elliptic genus defined for toric varieties and Calabi-Yau hypersurfaces in toric varieties in [12]. It turns out that these definitions agree, up to a normalization. We explain the Calabi-Yau case in more detail and leave the toric case to the reader. We need to recall the combinatorial description of Calabi-Yau hypersurfaces in toric varieties and the previous definition of their elliptic genera.

Let M_1 and N_1 be dual free abelian groups of rank $d+1$. Denote by M and N the dual free abelian groups $M = M_1 \oplus \mathbf{Z}$ and $N = N_1 \oplus \mathbf{Z}$. Element $(0,1) \in M$ is denoted by deg, and element $(0,1) \in N$ is denoted by $\deg^*$. There are dual reflexive polytopes $\Delta \in M_1$ and $\Delta^* \in N_1$ which give rise to dual cones $K \subset M$ and $K^* \subset N$. Namely, K is a cone over $(\Delta, 1)$ with vertex at $(0,0)_M$, and similarly for K^*. There is a complete fan Σ_1 on N_1 whose one-dimensional cones are generated by some lattice points in Δ^* (in particular, by all vertices). This fan induces the decomposition of the cone K^* into subcones, each of which includes $\deg^*$. Let us denote this decomposition by Σ. A generic Calabi-Yau hypersurface X_f of the family given by the above combinatorial data is determined by a choice of coefficients f_m for each $m \in (\Delta, 1)$.

The elliptic genus of X_f was defined in [12] as the graded Euler characteristic of a certain sheaf of vertex algebras on X_f. We do not need to recall the definition of this sheaf in view of the following combinatorial formula for the elliptic genus.

PROPOSITION 5.6
The elliptic genus $\mathrm{Ell}(X_f; y, q)$ *of the Calabi-Yau hypersurface* X_f*, as defined in [12], is given by*

$$\mathrm{Ell}(X_f; y, q) = y^{-d/2} \sum_{m \in M} \text{a.c.}\Big(\sum_{n \in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^{d+2} \Big),$$

where a.c. *stands for analytic continuation and*

$$G(y,q)=\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}.$$

Proof
Combine [12, Proposition 4.2] and [12, Definition 5.1]. □

THEOREM 5.7
The elliptic genus of the Calabi-Yau hypersurface X_f of dimension d defined above and its singular elliptic genus are related by the formula

$$\mathrm{Ell}(X_f;y,q)=\Big(\frac{2\pi\mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)}\Big)^d\widehat{\mathrm{Ell}}(X_f;y,q).$$

Proof
First of all, observe that

$$y^{-1/2}G(y,q)=\frac{2\pi\mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)},$$

due to the product formulas for $\theta(z,\tau)$ and $\theta'(0,\tau)$ (see [14]). Therefore, we only need to show that

$$\widehat{\mathrm{Ell}}(X_f;y,q)=\sum_{m\in M}\mathrm{a.c.}\Big(\sum_{n\in K^*}y^{n\cdot\deg-m\cdot\deg^*}q^{m\cdot n+m\cdot\deg^*}G(y,q)^2\Big).$$

Denote by $\deg_1$ the piecewise linear function on N_1 whose value on the generators of the one-dimensional cones of Σ_1 is 1. Notice that K^* consists of all points $(n_1,l)\in N$ such that $l\geq\deg_1(l)$. In addition, one can replace $\sum_{n\in K}\dots$ by $\sum_{C\in\Sigma}(-1)^{\mathrm{codim}\,\Sigma}\dots$ to get

$$\begin{aligned}
&\sum_{m\in M}\mathrm{a.c.}\Big(\sum_{n\in K^*}y^{n\cdot\deg-m\cdot\deg^*}q^{m\cdot n+m\cdot\deg^*}G(y,q)^2\Big)\\
&\quad=\sum_{k\in\mathbf{Z}}\sum_{m_1\in M}\sum_{C_1\in\Sigma_1}(-1)^{\mathrm{codim}\,C_1}\mathrm{a.c.}\sum_{n_1\in C_1}\sum_{l\geq\deg_1(n_1)}y^{l-k}q^{m_1\cdot n_1+lk+k}G(y,q)^2\\
&\quad=\sum_{k\in\mathbf{Z}}\sum_{m_1\in M}\sum_{C_1\in\Sigma_1}(-1)^{\mathrm{codim}\,C_1}\\
&\qquad\times\mathrm{a.c.}\sum_{n_1\in C_1}\sum_{l\geq\deg_1(n_1)}y^{\deg_1(n_1)-k}q^{m_1\cdot n_1+\deg_1(n_1)k+k}(1-yq^k)^{-1}G(y,q)^2\\
&\quad=G(y,q)^2\sum_{k\in\mathbf{Z}}\frac{y^{-k}q^k}{(1-yq^k)}f_{N_1,\deg_1 z}(yq^k,q).
\end{aligned}$$

Let Σ_1' be a refinement of the fan Σ_1 in N_1 such that the corresponding toric variety $\mathbf{P}_{\Sigma_1'}$ is smooth. Coefficients f_m define a hypersurface X_f' in $\mathbf{P}_{\Sigma_1'}$ which is a resolution of singularities X_f. We denote the codimension one strata of $\mathbf{P}_{\Sigma_1'}$ by D_j, their first Chern classes by d_j, and the corresponding generators of one-dimensional cones of Σ_1' by n_j. By [11, Theorem 3.4], we get

$$\begin{aligned}
G(y,q)^2 \sum_{k\in\mathbf{Z}} \frac{y^{-k}q^k}{(1-yq^k)} f_{N_1,\deg_1 z}(yq^k,q) &= G(y,q)^2 \sum_{k\in\mathbf{Z}} \frac{y^{-k}q^k}{(1-yq^k)} \\
&\quad \times \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(d_j/(2\pi\mathrm{i}))\theta(d_j/(2\pi\mathrm{i}) - \deg_1(n_j)(z+k\tau))\theta'(0)}{\theta(-\deg_1(n_j)(z+k\tau))\theta(n_j/(2\pi\mathrm{i}))} \\
&= \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(d_j/(2\pi\mathrm{i}))\theta(d_j/(2\pi\mathrm{i}) - \deg_1(n_j)z)\theta'(0)}{\theta(-\deg_1(n_j)z)\theta(n_j/(2\pi\mathrm{i}))} \\
&\quad \times \Big(\sum_{k\in\mathbf{Z}} G(y,q)^2 \frac{y^{-k}q^k}{(1-yq^k)} \mathrm{e}^{k\sum_j d_j \deg_1(n_j)}\Big).
\end{aligned}$$

We denote $D = \sum_j \deg_1(n_1) D_j$ and $d = c_1(D)$. Because of [12, Proposition 3.2], we get

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} G(y,q)^2 \frac{y^{-k}q^k}{(1-yq^k)} \mathrm{e}^{k\sum_j d_j \deg_1(n_j)} &= \frac{G(e^d q,q)G(y,q)}{G(y^{-1}e^d q,q)} \\
&= \frac{2\pi\mathrm{i}\theta(d/(2\pi\mathrm{i}))\theta(-z,\tau)}{\theta(d/(2\pi\mathrm{i}) - z)\theta'(0)},
\end{aligned}$$

which gives

$$\begin{aligned}
\sum_{m\in M} \mathrm{a.c.}\Big(\sum_{n\in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^2\Big) &= \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(d_j/(2\pi\mathrm{i}))\theta(d_j/(2\pi\mathrm{i}) - \deg_1(n_j)z)\theta'(0)}{\theta(-\deg_1(n_j)z)\theta(n_j/(2\pi\mathrm{i}))} \\
&\quad \times \Big(\frac{2\pi\mathrm{i}\theta(d/(2\pi\mathrm{i}))\theta(-z,\tau)}{\theta(d/(2\pi\mathrm{i}) - z)\theta'(0)}\Big).
\end{aligned}$$

Observe now that $D = \pi^*(-K_{\mathbf{P}_{\Sigma_1}})$, where $\pi : \mathbf{P}_{\Sigma_1'} \to \mathbf{P}_{\Sigma_1}$ is the resolution induced by the subdivision of the fan. In addition, X_f is a zero set of a section of D. Hence, the adjunction formula gives

$$c(T_{X_f'}) = i^* c(T\mathbf{P}_{\Sigma_1'})/(1 + i^* d),$$

where $i : X'_f \to \mathbf{P}_{\Sigma'_1}$ is the embedding. The exceptional divisors of $X'_f \to X_f$ are $D_j \cap X'_f$ (unless $\dim \pi(D_j) = 0$), and their discrepancies are equal to $\deg(n_j) - 1$. Then it is easy to see that the above expression is precisely the singular elliptic genus of X_f. □

Remark 5.8
The case of toric varieties is a straightforward application of [11, Theorem 3.4] and is left to the reader.

Remark 5.9
The above calculations indicate that for any smooth variety $\mathbf{P}$ of dimension $d + 1$ one can define a weak Jacobi form of weight d and index zero which coincides with the singular elliptic genus of the Calabi-Yau hypersurface in $\mathbf{P}$ if $\mathbf{P}$ has smooth anticanonical divisors. Otherwise, the formula gives the elliptic genus of the "virtual" Calabi-Yau hypersurface in $\mathbf{P}$. One can also interpret this Jacobi form as an elliptic genus of a $(d + 1, 1)$-dimensional Calabi-Yau supermanifold $\Pi K X$ (a canonical line bundle over X, considered as an odd bundle).

6. Cobordism invariance of orbifold elliptic genus

We view $\mathrm{Ell}_{\mathrm{sing}}(X/G)$ and $\mathrm{Ell}_{\mathrm{orb}}(X, G)$ as invariants of G-action on X and work in the category of stably almost complex manifolds.

LEMMA 6.1
A singular elliptic genus is an invariant of a complex G-cobordism.

Proof
We consider cobordisms of pairs (X, D) (see [36]), where X is a stably almost complex manifold (i.e., a C^∞-manifold such that a direct sum of a trivial bundle ϵ with the differentiable tangent bundle T_X admits a complex structure) and $D = \bigcup D_i$ is a finite union of codimension one stably almost complex submanifolds (i.e., $T_{D_i} \oplus \epsilon$ is a complex subbundle in $\epsilon \oplus T_X$) satisfying the following normal crossing condition: at each point of $D_{i_1} \cap \cdots \cap D_{i_k}$, the union of (stabilized by adding trivial bundles) tangent spaces $T_{D_{i_j}} \oplus \epsilon$ is given in the (stabilized) tangent space to X by $l_1 \cdots l_k = 0$, where l_i are linearly independent complex linear forms. A pair (X, D) is cobordant to zero if there exist a C^∞-manifold Y with a complex structure on the stable tangent bundle and a system of submanifolds $\bigcup E_i$ such that $\partial Y = X$ and $\bigcup \partial E_i = \bigcup D_i$. As usual, the disjoint union and product provide the ring structure on cobordism classes.

Notice that the numbers $c_{i_1} \cup \cdots \cup c_{i_k} \cup [D_{j_1}] \cup \cdots \cup [D_{j_k}]([X])$, where $[D_i]$ are the classes in $H^2(X, \mathbf{Z})$ dual to submanifolds D_i, $[X] \in H_{2\dim_{\mathbf{C}} X}$ is the fundamental class of X, and $\sum_j i_j + k = \dim_{\mathbf{C}} X$, are invariants of cobordism of such pairs. (Indeed, if $X = \partial Y$ and $j : X \to Y$, then this number is $j^*(c_{i_1} \cup \cdots \cup c_{i_k} \cup [E_{j_1}] \cup \cdots \cup [E_{j_k}])([X]) = c_{i_1} \cup \cdots \cup c_{i_k} \cup [D_{j_1}] \cup \cdots \cup [D_{j_k}](j_*[X]) = 0$ since X is homologous to zero in Y.) The lemma therefore follows if we show that for an almost complex null-cobordant G-manifold X the quotient X/G admits a resolution of singularities $(\widetilde{X/G}, D)$, where $D = \bigcup D_i$ is the exceptional locus such that this pair is cobordant to zero.

If $X = \partial Y$, where Y is a G-manifold, we can construct a resolution of Y/G as follows. Let H be a subgroup of G, and let $Y_H = \{y \in Y | \operatorname{Stab} y = H\}$. Then Y_H are smooth submanifolds of Y (possibly with boundary) providing a stratification of Y. Let $C(H)$ be the union of subgroups of G conjugate to H. Then $Y_{C(H)} = \bigcup_{H' \in C(H)} Y_{H'}$ is still a submanifold of Y and the group G acts on $Y_{C(H)}$ so that $Y_{C(H)} \to Y_{C(H)}/G$ is an unramified cover (of degree $[G : H]$). In particular, $Y_{C(H)}/G$ is a smooth manifold and these manifolds for all $H \subset G$ provide a stratification of Y/G such that Y/G is equisingular along each stratum $Y_{C(H)}/G$. A small regular neighborhood of each stratum in Y/G is isomorphic to a bundle ξ_H over $Y_{C(H)}/G$ with the fiber isomorphic to V/H, where V is a fiber of the normal bundle to Y_H in Y over a point of Y_H. (This presentation is independent of a point in Y_H, and representations at points of Y_H and $Y_{H'}$ are isomorphic for conjugate H and H'.)

For each quotient singularity V/H, let us fix the universal desingularization constructed by Bierstone and Milman (see [8, Theorem 13.2]). Its universality assures that it is equivariant with respect to the centralizer of H in $GL(V)$. Hence, one can use the transition functions of ξ_H to construct the fibration $\tilde{\xi}_H$ with the same base as ξ_H and having as its fiber the universal resolution of V/H. Moreover, due to universality of canonical resolution (see [8, Theorem 13.2]), this property assures that $\tilde{\xi}_H$ corresponding to different classes of conjugate subgroups H can be glued together, yielding an almost complex manifold whose boundary is the pair $(\widetilde{X/G}, D)$, where D is the exceptional set of the universal resolution of X/G. This proves the lemma. □

LEMMA 6.2
The orbifold elliptic genus is an invariant of G-cobordism.

Proof
Let X be a null-cobordant G-manifold. Then for each $g \in G$ the pair $X^g, \nu(X^g, X)$, where $\nu(X^g, X)$ is the normal bundle of the fixed-point set X^g in X, is cobordant to zero as well. Since the contribution of the term in $\mathrm{Ell}_{\mathrm{orb}}$ corresponding to a conjugacy class $[g]$ is a combination of the products of Chern classes of X^g and $\nu(X^g, X)$ evaluated on the fundamental class of X^g, this contribution is zero. This yields the lemma. □

Remark 6.3
Unfortunately, the orbifold elliptic genus is not multiplicative. Rather, for G-manifolds X and Y with an action of G as above, one has $\mathrm{Ell}_{\mathrm{orb}}(X \times Y, G \times G) = \mathrm{Ell}_{\mathrm{orb}}(X, G) \cdot \mathrm{Ell}_{\mathrm{orb}}(Y, G)$.

COROLLARY 6.4
Conjecture 5.1 is true for $G = \mathbf{Z}/2\mathbf{Z}$.

Proof
This follows from the result of Kosniowski (see [30]) describing generators of $\mathbf{Z}/p\mathbf{Z}$-cobordisms. If $p = 2$, then additive generators of the cobordism group in any dimension are toric varieties with the group being a subgroup of the big torus. Hence Proposition 5.3 yields the claim. □

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