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GENERALIZATIONS OF THE ODD DEGREE THEOREM AND APPLICATIONS

BY

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ABSTRACT

Let $V \subset \mathbb{PR}^n$ be an algebraic variety, such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is irreducible of codimension $m \geq 1$. We use a sufficient condition on a linear space $L \subset \mathbb{PR}^n$ of dimension $m + 2r$ to have a nonempty intersection with V, to show that any six dimensional subspace of 5×5 real symmetric matrices contains a nonzero matrix of rank at most 3.

1. Introduction

Let $p(x) = x^k + a_1x^{k-1} + \cdots + a_k \in \mathbb{R}[x]$. Then the odd degree theorem states that $p(x)$ has a real root if k is odd. Let $\mathbb{PR}^n$ and $\mathbb{P}^n: = \mathbb{PC}^n$ be the real and the complex projective space of dimension n, respectively. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we view a linear space $L \subset \mathbb{PF}^n$ of dimension m as an element of the Grassmannian manifold $\mathrm{Gr}(m+1, n+1, \mathbb{F})$. Let $V \subset \mathbb{PR}^n$ be an algebraic variety, such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is irreducible and has codimension $m \geq 1$. If $d = \deg V_{\mathbb{C}}$ is odd then for any linear space $L \subset \mathbb{PR}^n$ of dimension m the intersection $V \cap L \neq \emptyset$. Indeed, we have $B(V) = V$, where $B: \mathbb{P}^n \to \mathbb{P}^n$ is the involution $z \mapsto \bar{z}$. For generic L, the set $V_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of exactly d points. As this set is invariant under the involution B, we deduce that there exists $z \in V_{\mathbb{C}} \cap L_{\mathbb{C}}$ such that $B(z) = z \Rightarrow z \in \mathbb{PR}^n$. The continuity argument yields that $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}(m+1, n+1, \mathbb{R})$.

Consider now the case when d is even. Then it is not difficult to find nontrivial examples where $V \cap L' = \emptyset$ for some $L' \in \mathrm{Gr}(m+1, n+1, \mathbb{R})$. We are interested

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in this paper in cases when V is a determinantal variety, i.e., finding nonzero real matrices of rank at most k in linear families. The examples such that for any integer $k \in [0,p)$ there exists $L' \in \mathrm{Gr}(m+k+1, n+1, \mathbb{R})$ satisfying $V \cap L' = \emptyset$, while $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}(m+p+1, n+1, \mathbb{R})$, can be found among determinantal varieties (see §2).

Let $S_n(\mathbb{F})$ be the space of $n \times n$ symmetric matrices with entries in $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Let $V_{k,n}(\mathbb{F})$ be the variety of all matrices in $S_n(\mathbb{F})$ of rank k or less. Then the projectivization $\mathbb{P}V_{k,n}(\mathbb{F})$ is an irreducible variety of codimension $\binom{n-k+1}{2}$ in the projective space $\mathbb{P}S_n(\mathbb{F})$. Note that $V_{k-1,n}(\mathbb{F})$ is the variety of the singular points of $V_{k,n}(\mathbb{F})$ (e.g., [3, II]). Let $d(n,k,\mathbb{F})$ be the smallest integer ℓ such that every ℓ dimensional subspace of $S_n(\mathbb{F})$ contains a nonzero matrix whose rank is at most k. Then

$$d(n,k,\mathbb{C}) = \binom{n-k+1}{2} + 1, \tag{1.1}$$

and the problem is to determine $d(n,k,\mathbb{R})$. The degree of $\mathbb{P}V_{k,n}(\mathbb{C})$ was computed by Harris and Tu in [9],

$$\delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{n+j}{n-k-j}}{\binom{2j+1}{j}}. \tag{1.2}$$

It was shown in [5] that $\delta_{n-q,n}$ is odd if

$$n \equiv \pm q (\operatorname{mod} 2^{\lceil \log_2 2q \rceil}). \tag{1.3}$$

Then $d(n, n-q, \mathbb{R}) = d(n, n-q, \mathbb{C})$ for these values of n and q. It is conjectured in [5] that if $\delta_{n-q,n}$ is odd then (1.3) holds.

In this paper we show that not only the degree of complexification but also the Euler characteristic of the intersection of $\mathbb{P}V_{k,n}(\mathbb{C})$ with a generic linear space of dimension $\binom{n-k+1}{2}+2r$ can be used to get additional information about $d(n,k,\mathbb{R})$. Our estimate of $d(n,k,\mathbb{R})$ from above uses the following result proved in §2.

Corollary 1.1: *Let $V \subset \mathbb{PR}^n$ be an algebraic variety such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is an irreducible variety of codimension m. Assume that $\deg V_{\mathbb{C}}$ is even and let r be a positive integer. Suppose that the codimension of the variety of the singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is at least $2r+1$. Suppose furthermore that for a generic $L \in \mathrm{Gr}(m+2r+1, n+1, \mathbb{C})$ the Euler characteristic of $V_{\mathbb{C}} \cap L$ is odd. Then $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}(m+2r+1, n+1, \mathbb{R})$.*

This corollary applies whenever one has an answer to the following problem:

PROBLEM 1.1: *Assume that $\delta_{k,n}$ is even. Find an integer $r \geq 1$, preferably the smallest possible, such that*

$$2r < \binom{n-k+2}{2} - \binom{n-k+1}{2}, \tag{1.4}$$

and the Euler characteristic of $\mathbb{P}V_{k,n}(\mathbb{C}) \cap L$ is odd for a generic $L \in \mathrm{Gr}(\binom{n-k+1}{2} + 2r + 1, \binom{n+1}{2}, \mathbb{C})$.

For $k = n-1$ there is no r which satisfies the conditions of Problem 1.1, hence Corollary1.1 is not applicable. This follows from the result that the Euler characteristic of a smooth hypersurface of an even degree is even. Let $k = n-2$. The smallest n of interest is $n = 5$ [5]. In §6 we show that the minimal solution to Problem 1.1 is $r = 1$. Hence $d(5, 3, \mathbb{R}) \leq 6$. Numerical evidence supports the conjecture that $d(5, 3, \mathbb{R}) = 6$ [5].

The contents of the paper are as follows. In §2 we give a generalization of the odd degree theorem. It is a straightforward consequence of the Lefschetz fixed point theorem, the Hodge decomposition and the Poincaré duality. We also recall the exact value of the gap $d(n, n-1, \mathbb{R}) - d(n, n-1, \mathbb{C})$. In §3, we recall some known results about the projectivized complex bundles and the corresponding Chern classes of their tangent bundles. Next, we discuss a resolution of the singularities of $V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{k,n}(\mathbb{C})$. Let $\tau, \kappa \to \mathrm{Gr}(k, n, \mathbb{C})$ be the tautological k-bundle and its quotient bundle respectively. Then $\mathrm{Sym}^2 \tau, \mathrm{Sym}^2 \kappa$ are resolutions of $V_{k,n}(\mathbb{C}), V_{n-k,n}(\mathbb{C})$ respectively. The projectivized bundle $\mathbb{P}(\mathrm{Sym}^2 \tau), \mathbb{P}(\mathrm{Sym}^2 \kappa)$ are resolutions of $\mathbb{P}V_{k,n}(\mathbb{C}), \mathbb{P}V_{n-k,n}(\mathbb{C})$ respectively. In §4 we discuss $\mathbb{P}(\mathrm{Sym}^2 \tau)$ for $k = 1$. In §5 we discuss $\mathbb{P}(\mathrm{Sym}^2 \tau)$ for $k = 2$ and mostly for $n = 4$. In §6 we discuss $\mathbb{P}(\mathrm{Sym}^2 \kappa)$ for $k = 2$, $n = 5$ modulo 2.

2. Generalizations of the odd degree theorem

LEMMA 2.1: *Let $W \subset \mathbb{PR}^n$ be an algebraic variety such that its complexification $W_{\mathbb{C}} \subset \mathbb{P}^n$ is a smooth irreducible variety of (complex) dimension $m \geq 1$. Then for any nonnegative integer r*

$$\begin{aligned} \mathrm{trace}(B^*|H^{2r+1}(W_{\mathbb{C}}, \mathbb{R})) &= 0, \\ \mathrm{trace}(B^*|H^{2r}(W_{\mathbb{C}}, \mathbb{R})) &= \mathrm{trace}(B^*|H^{r,r}(W_{\mathbb{C}})) \\ &= (-1)^m \,\mathrm{trace}(B^*|H^{m-r,m-r}(W_{\mathbb{C}})), \end{aligned} \tag{2.1}$$

where B is conjugation in $\mathbb{P}^n$.

Proof: Since $B^*(H^{p,q}(W_{\mathbb{C}})) = H^{q,p}(W_{\mathbb{C}})$ we have, for $p \neq q$,

$$\mathrm{trace}(B^*|H^{p,q}(W_{\mathbb{C}}) \oplus H^{q,p}(W_{\mathbb{C}})) = 0.$$

The Hodge decomposition of $H^k(W_{\mathbb{C}}, \mathbb{R})$ yields the claim, since B^* reverses the orientation of $W_{\mathbb{C}}$ if m is odd and preserves the orientation of $W_{\mathbb{C}}$ if m is even. ∎

COROLLARY 2.1: *Let the assumptions of Lemma 2.1 hold. Then the Lefschetz number $\lambda(W_{\mathbb{C}})$ of $B\colon W_{\mathbb{C}} \to W_{\mathbb{C}}$ is given by*

$$\lambda(W_{\mathbb{C}}) = 0, \quad \text{if } m \text{ is odd},$$

$$\lambda(W_{\mathbb{C}}) = \mathrm{trace}(B^*|H^m(W_{\mathbb{C}})) + 2\sum_{r=0}^{(m-2)/2} \mathrm{trace}(B^*|H^{2r}(W_{\mathbb{C}})) \in \mathbb{Z}, \tag{2.2}$$

$$\text{if } m \text{ is even.}$$

If $\lambda(W_{\mathbb{C}}) \neq 0$ then $W \cap \mathbb{PR}^n \neq \emptyset$.

Proof: This is a consequence of the last lemma and the Lefschetz fixed point theorem. ∎

COROLLARY 2.2: *Let W be as in Lemma 2.1. Suppose that m is even and $b_m(W_{\mathbb{C}})$ (equivalently the Euler characteristic $\chi(W_{\mathbb{C}})$) is odd. Then $W \cap \mathbb{PR}^n \neq \emptyset$.*

Proof: Since the eigenvalues of $B^*|H^m(W_{\mathbb{C}})$ are ± 1 we have that $b_m(W_{\mathbb{C}}) = \lambda(W_{\mathbb{C}}) \bmod 2$. ∎

THEOREM 2.1: *Let $V \subset \mathbb{PR}^n$ be an algebraic variety such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is an irreducible variety of codimension m. Suppose that the codimension of the variety of the singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is at least k. Then for a generic $L \in \mathrm{Gr}(m+k, n+1, \mathbb{R})$ $\lambda(V_{\mathbb{C}} \cap L_{\mathbb{C}})$ is equal to zero if k is even and is equal to $b_{k-1}(V_{\mathbb{C}} \cap L_{\mathbb{C}}) \bmod 2$ if k is odd. In particular, if $k = 2r+1$ and $b_{2r}(V_{\mathbb{C}} \cap L_{\mathbb{C}})$ is odd, or more generally $\lambda(V_{\mathbb{C}} \cap L_{\mathbb{C}}) \neq 0$, then $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}(m+2r+1, n+1, \mathbb{R})$.*

Proof: For $k = 1$, $V_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of $\deg V_{\mathbb{C}}$ distinct points for a generic L and the theorem follows. Assume that $k > 1$. Let $W = V \cap L$, $W_{\mathbb{C}} = V_{\mathbb{C}} \cap L_{\mathbb{C}}$. The assumptions of the theorem yield that for a generic L, $W_{\mathbb{C}}$ is a smooth irreducible variety. Hence $\lambda(B|W_{\mathbb{C}})$ is given by Corollary 2.1. Other claims of the theorem follow from Corollaries 2.1 and 2.2. ∎

Clearly, Corollary 1.1 follows from Theorem 2.1. The values of $d(n, n-1, \mathbb{R})$ were computed by Adams, Lax and Phillips in [2] using the work of Adams [1] on the maximal number of linearly independent vector fields on the $n-1$ dimensional sphere S^{n-1}. Write $n = (2a+1)2^{c+4d}$, where a and d are nonnegative integers, and $c \in \{0, 1, 2, 3\}$. Then $\rho(n) = 2^c + 8d$ is the Radon–Hurwitz number. Let $\rho(x) = 0$ if x is not a positive integer.

Then

$$d(n, n-1, \mathbb{R}) = \rho(n/2) + 2.$$

Let

$$p: = d(n, n-1, \mathbb{R}) - d(n, n-1, \mathbb{C}) = \rho(n/2). \tag{2.3}$$

Note that either p is even or $p = 1$. Assume that n is even. Let $V = \mathbb{P}V_{n-1,n}(\mathbb{R})$. Then $V_{\mathbb{C}} = \mathbb{P}V_{n-1,n}(\mathbb{C})$. The codimension of the variety of singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is 2. Then for any $k < p$ there exists a linear space $L' \in \mathrm{Gr}(2+k, \binom{n+1}{2}, \mathbb{R})$ such that $V \cap L' = \emptyset$. It is shown in [2] that $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}(2+p, \binom{n+1}{2}, \mathbb{R})$.

Let us consider $d(n, k, \mathbb{R})$ for $k = 1$. We have $\mathbb{P}V_{1,n}(\mathbb{C}) \subset \mathbb{P}S_n(\mathbb{C}) \sim \mathbb{P}^{\binom{n+1}{2}-1}$. The variety $\mathbb{P}V_{1,n}(\mathbb{C})$ is biholomorphic to $\mathbb{P}^{n-1}$. Indeed, identify $\mathbb{P}^{n-1}$ with the lines in $\mathbb{C}^n$ spanned by the nonzero column vectors $x \in \mathbb{C}^n$. Then

$$q: \mathbb{P}^{n-1} \to \mathbb{P}V_{1,n}(\mathbb{C}), \quad q(x) = xx^T \tag{2.4}$$

is a biholomorphism.

In [6] the linear subspace $L_0 \subset \mathbb{P}S_n(\mathbb{C})$ (of codimension 1) of matrices of trace 0 was considered. Clearly $\mathbb{P}V_{1,n}(\mathbb{R}) \cap L_0 = \emptyset$. Hence [6]

$$d(n, 1, \mathbb{R}) = \binom{n+1}{2}.$$

Corollary 1.1 yields that for any generic complex linear subspace $L \subset \mathbb{P}S_n(\mathbb{C})$ of codimension m, $1 \leq m \leq n-1$ the middle Betti number of $L \cap \mathbb{P}V_{1,n}(\mathbb{C})$ is even. (Since $L \cap \mathbb{P}V_{1,n}(\mathbb{C})$ is biholomorphic to a nonsingular quadric this Betti number is either 0 or 2 depending on parity of n.) Similarly for $n > 1$, $\mathbb{P}V_{1,n}(\mathbb{R}) \cap L_0 = \emptyset$ yields that $\deg \mathbb{P}V_{1,n}(\mathbb{C})$ is even. (This fact follows also from the formula (1.2).)

Since for an odd n the middle Betti number of $\mathbb{P}V_{1,n}(\mathbb{C})$ is 1, we see that the parity of the Euler characteristic of smooth variety in $\mathbb{P}^n$ is independent of the parity of its degree, though a complete intersection of even degree has an even Euler characteristic.

3. Chern classes for desingularizations of determinantal varieties

In this section we shall collect the formulas for the Chern classes of projectivizations of certain bundles. The main reference is [7]. We also specify how such projectivizations come up as desingularizations of determinantal varieties.

Let E be an ℓ-bundle over smooth complex manifold M with the Chern classes $c_1(E),\ldots,c_\ell(E)$. Let u_i, $i=1,\ldots,\ell$ be the roots of the Chern polynomial

$$c(E,t)=\sum_{j=0}^{\ell} c_j(E)t^j$$

of E, i.e.,

$$c(E,t)=\prod_{i=1}^{\ell}(1+u_it).$$

We have (cf. [4, §4.20])

$$c(\operatorname{Sym}^2 E,t)=\prod_{1\le i\le j\le \ell}(1+(u_i+u_j)t). \tag{3.1}$$

Let $\mathbb{P}(E)$ be the projectivization of E. (As a set it consists of the pairs $(x,[v])$, where $x\in M$ and $[v]$ is a line in E over x spanned by a nonzero point $v\in E$ over x.) Let $\tilde{E}$ be the tautological line bundle over $\mathbb{P}(E)$ (given by the line $[v]$ over the point $(x,[v])$). Let E^* be the pull back of E to $\mathbb{P}(E)$ induced by the projection $\pi_1\colon \mathbb{P}(E)\to M$. $\tilde{E}$ is a subbundle of E^* (cf. [7, B.5.5]).

LEMMA 3.1: *Let M be a complex manifold of dimension n. Let $E\to M$ be a complex vector bundle vector of rank $\ell\ge 1$ and $\pi\colon \mathbb{P}(E)\to M$ be its projectivization. Let $\tilde{E}$ be the tautological line bundle over $\mathbb{P}(E)$, and $q=c_1(\tilde{E})$ be its first Chern class (resp., $h=-q$ is the first Chern class of $\tilde{E}'$, which is the dual to $\tilde{E}$). Then the cohomology ring $\mathrm{H}^*(\mathbb{P}(E),\mathbb{C})$ is $\mathrm{H}^*(M,\mathbb{C})[q]$ together with the relation*

$$q^\ell+\sum_{i=1}^{\ell}(-1)^i c_i(E)q^{\ell-i}=0. \tag{3.2}$$

Let

$$c(T_M,t)=\sum_{i=0}^{n} c_i(T_M)t^i,\quad c_0(T_M)=1$$

be the Chern polynomial of the tangent bundle of M. Then the Chern polynomial of the tangent bundle of $\mathbb{P}(E)$ is given by

$$c(T_{\mathbb{P}(E)},t)=c(T_M,t)\Big(\sum_{j=0}^{\ell} c_j(E)t^j(1-qt)^{\ell-j}\Big). \tag{3.3}$$

Proof: For the proof of (3.2) see [10], [8, §4.6, pp. 606] or [4, §4.20]. On the other hand, for the relative tangent bundle $T_{\mathbb{P}(E)/M}$, which fits into exact sequence

$$0 \to T_{\mathbb{P}(E)/M} \to T_{\mathbb{P}(E)} \to \pi^*(T_M) \to 0,$$

we have

$$T_{\mathbb{P}(E)/M} = \tilde{E} \odot Q, \tag{3.4}$$

where Q is the universal quotient bundle: $E^*/\tilde{E}$ (cf. [7, B.5.8]). This yields (3.3). ∎

For example, if E is trivial and has rank m then $\mathbb{P}(E) = M \times \mathbb{P}^{m-1}$ and (3.3) becomes

$$c(T_{\mathbb{P}(E)}) = c(T_M)(1-qt)^m, \quad q^m = 0. \tag{3.5}$$

In the next sections the following situation will arise:

LEMMA 3.2: *Let M be a complex manifold of dimension n and $E \to M$ be a trivial complex vector bundle vector of rank $m \geq 2$. Denote by $\tilde{E}'$ the dual to the tautological bundle $\tilde{E}$. Let $U \subset \mathbb{P}(E)$ be a connected complex submanifold of dimension d. Consider hypersurfaces $\tilde{H}_i$ $i = 1, \dots, k$ in $\mathbb{P}(E)$ each being the zero set of a generic section of $\tilde{E}'$. Let $W = U \cap \bigcap_{i=1}^{i=k} H_i$ and ι be the embedding W in U. Then*

$$c(T_W, t) = \iota^* c(T_U|_W, t)(1-tq)^{-k}, \tag{3.6}$$

and

$$\chi(W) = h^d c(T_U)(1-tq)^{-k}[U], \tag{3.7}$$

where $[U]$ is the fundamental class of U and h is the restriction on U of the first Chern class $c_1(\tilde{E}')$.

Proof: (3.6) is a consequence of the exact sequence

$$0 \to T_W \to T_U|_W \to \bigoplus_{i=1}^{k} N_{\tilde{H}_i}|_W \to 0.$$

(3.7) is similar to [10, 9.3]. ∎

Let $E \to M$ a trivial m-bundle, and $F \to M$ is an ℓ-subbundle of E. As above q_E (resp., q_F) is the first Chern class of the tautological bundle $\tilde{E}$ (resp., $\tilde{F}$) on $\mathbb{P}(E)$ (resp., $\mathbb{P}(F)$). Then $\mathbb{P}(F) \subset \mathbb{P}(E)$, and if ι is the embedding then

$$q_F = \iota^* q_E. \tag{3.8}$$

We describe now a smooth resolutions of $V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{k,n}(\mathbb{C})$ for $1 \leq k \leq n-1$. This construction is similar to the one described in [3, II]. We have the following exact sequence of three bundles over $\mathrm{Gr}(k,n,\mathbb{C})$:

$$0 \to \tau \to \mathbb{C}^n \to \kappa \to 0. \tag{3.9}$$

Here τ is the tautological k-bundle, $\mathbb{C}^n$ is the n-trivial bundle and $\kappa: = \mathbb{C}^n/\tau$ the $n-k$ quotient bundle.

LEMMA 3.3: *Let $1 \leq k < n$. Then the bundles $\mathrm{Sym}^2\,\tau$ and $\mathrm{Sym}^2\,\kappa$ are smooth resolutions of $V_{k,n}(\mathbb{C})$ and $V_{n-k,n}(\mathbb{C})$, respectively. Furthermore, the projectivized bundles $\mathbb{P}(\mathrm{Sym}^2\,\tau)$ and $\mathbb{P}(\mathrm{Sym}^2\,\kappa)$ are smooth resolutions of $\mathbb{P}V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{n-k,n}(\mathbb{C})$, respectively.*

Proof: Viewing A as a linear operator $A: \mathbb{C}^n \to \mathbb{C}^n$ yields the two linear subspaces: Range A and Ker A of $\mathbb{C}^n$, which are the range and kernel of the operator A, respectively. Note that if $a \in \mathbb{C}^*$ then $\mathrm{Range}\,A = \mathrm{Range}\,aA$ and $\mathrm{Ker}\,A = \mathrm{Ker}\,aA$. Let

$$\begin{aligned}
&X: = S_n(\mathbb{C}) \times \mathrm{Gr}(k,n,\mathbb{C}), \quad \tilde{X}: = \mathbb{P}S_n(\mathbb{C}) \times \mathrm{Gr}(k,n,\mathbb{C}),\\
&Y: = \{(A,V) \in X: \quad \mathrm{Range}\,A \subset V\},\\
&\tilde{Y}: = \{(A,V) \in \tilde{X}: \quad \mathrm{Range}\,A \subset V\},\\
&Z: = \{(B,V) \in X: \quad \mathrm{Kernel}\,B \supset V\},\\
&\tilde{Z}: = \{(B,V) \in \tilde{X}: \quad \mathrm{Kernel}\,B \supset V\}.
\end{aligned} \tag{3.10}$$

Let $\pi_1: X \to S_n(\mathbb{C})$, $\pi_2: X \to \mathrm{Gr}(k,n,\mathbb{C})$ be the projections on the first and second coordinates, respectively. Clearly

$$\begin{aligned}
&\pi_1(Y) = V_{k,n}(\mathbb{C}), \quad \pi_2(Y) = \mathrm{Gr}(k,n,\mathbb{C}),\\
&\pi_1(Z) = V_{n-k,n}(\mathbb{C}), \quad \pi_2(Z) = \mathrm{Gr}(k,n,\mathbb{C}).
\end{aligned}$$

The map π_1 is a resolution. Indeed, it is birational of degree one since it is $1-1$ on

$$\pi_1^{-1}(V_{k,n}(\mathbb{C})\backslash V_{k-1,n}(\mathbb{C})) \subset Y \quad \text{and} \quad \pi_1^{-1}(V_{n-k,n}(\mathbb{C})\backslash V_{n-k-1,n}(\mathbb{C})) \subset Z.$$

A similar situation takes place for $\tilde{\pi}_1\colon \tilde{X} \to \mathbb{P}S_n(\mathbb{C})$.

Finally, the fiber of the projection of Y on $\mathrm{Gr}(k,n,\mathbb{C})$ over V can be identified with the space of symmetric transformations of V which yields the identification of Y with $\mathrm{Sym}^2\tau$. Similarly, Z can be identified with $\mathrm{Sym}^2\kappa$. Hence $\mathbb{P}(\mathrm{Sym}^2\tau)$ and $\mathbb{P}(\mathrm{Sym}^2\kappa)$ are smooth resolutions of $\mathbb{P}V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{n-k,n}(\mathbb{C})$, respectively. ∎

We review now some known facts about the cohomology of Grassmannians used in the rest of the paper. Let $c_1,\dots,c_k$ and $s_1,\dots,s_{n-k}$ be the Chern classes of τ and κ, respectively. Denote by $c(\tau,t)$, $c(\kappa,t)$ the Chern polynomials

$$c(\tau,t)=1+\sum_{i=1}^{\infty}c_it^i,\quad c(\kappa,t)=1+\sum_{j=1}^{\infty}s_jt^j,$$

where $c_i=s_j=0$ for $i>k$, $j>n-k$. Recall that

$$c(\tau,t)c(\kappa,t)=1. \tag{3.11}$$

Then the cohomology ring of $\mathrm{Gr}(k,n,\mathbb{C})$ has the following representation, [7, Ex. 14.6.6] or [4, §4.23],

$$\mathrm{H}^*(\mathrm{Gr}(k,n,\mathbb{C}),\mathbb{C})=\mathbb{C}[c_1,\dots,c_k]/(s_{n-k+1},\dots,s_n). \tag{3.12}$$

Here we use the formula

$$c(\kappa,t)=\frac{1}{1+c_1t+\cdots+c_kt^k}. \tag{3.13}$$

With the help of these formulas we can compute the Chern classes of

$$\mathrm{Sym}^2\tau,\mathrm{Sym}^2\kappa\subset E$$

as polynomials in $c_1,\dots,c_k$ and $s_1,\dots,s_{n-k}$, respectively. Here

(3.14) $E\to\mathrm{Gr}(k,n,\mathbb{C})$ is a trivial bundle with the fiber $S_n(\mathbb{C})=\mathrm{Sym}^2\mathbb{C}^n$.

Then $\mathbb{P}(E)$ is identified with $\mathbb{P}S_n(\mathbb{C})\times\mathrm{Gr}(k,n,\mathbb{C})$. Furthermore, $q=-h$ is the first Chern class of the tautological line bundle over $\mathbb{P}(E)$. Thus

$$\mathrm{H}^*(\mathbb{P}S_n(\mathbb{C})\times\mathrm{Gr}(k,n,\mathbb{C}),\mathbb{C})=\mathrm{H}^*(\mathrm{Gr}(k,n,\mathbb{C}),\mathbb{C})[q],\quad q^{\binom{n+1}{2}}=0. \tag{3.15}$$

From the proof of Lemma 3.3 it follows that $\mathbb{P}(\mathrm{Sym}^2\tau)$, $\mathbb{P}(\mathrm{Sym}^2\kappa)$ are subvarieties of $\mathbb{P}(E)$, which can be identified with the smooth subvarieties $\tilde{Y},\tilde{Z}\subset$

$\mathbb{P}S_n(\mathbb{C}) \times \mathrm{Gr}(k,n)$. Then on $\tilde{Y}$, $\tilde{Z}$ the generator q satisfies the corresponding relation

$$q^{\binom{k+1}{2}} + \sum_{i=1}^{\binom{k+1}{2}} (-1)^i c_i(\mathrm{Sym}^2 \tau) q^{\binom{k+1}{2}-i} = 0, \tag{3.16}$$
$$q^{\binom{n-k+1}{2}} + \sum_{j=1}^{\binom{n-k+1}{2}} (-1)^j c_j(\mathrm{Sym}^2 \kappa) q^{\binom{n-k+1}{2}-j} = 0.$$

To find the Chern classes of the tangent bundles of $T_{\tilde{Y}}$, $T_{\tilde{Z}}$ we use Lemma 3.1. To find the Chern class of the tangent bundle of $\mathrm{Gr}(k,n,\mathbb{C})$ recall the following identity (cf. [7, §B.6]):

$$T_{Gr(k,n,\mathbb{C})} \sim \kappa \otimes \tau'. \tag{3.17}$$

Then

$$c(\tau',t) = 1 + \sum_{i=1}^{k}(-1)^i c_i(\tau) t^i = \prod_{i=1}^{k}(1+\alpha_i t),$$
$$c(\kappa,t) = 1 + \sum_{j=1}^{n-k} s_j t^j = \prod_{j=1}^{n-k}(1+\beta_j t), \tag{3.18}$$
$$c(\kappa\otimes\tau',t) = \prod_{i,j=1}^{k,n-k}(1+(\alpha_i+\beta_j)t) = 1 + \sum_{\ell=1}^{k(n-k)} v_\ell t^\ell.$$

4. $\mathrm{Gr}(1,n,\mathbb{C})$

As an illustration of the above formulas, in particular (3.3), let us consider the case $\mathrm{Gr}(1,n,\mathbb{C}) = \mathbb{P}^{n-1}$. The Chern class of the tautological line bundle τ of $\mathrm{Gr}(1,n,\mathbb{C})$ is c_1. The basic relation is $c_1^n = 0$. Note that $-c_1$ is the dual class of the hyperplane section. So $c(\tau,t) = 1 + c_1 t$. The Chern polynomial of $T_{\mathbb{P}^{n-1}}$ is $(1-c_1t)^n$, e.g., [8, §3.3]. Let $E = \mathrm{Sym}^2 \tau$. Then

$$c(\mathrm{Sym}^2 \tau, t) = 1 + w_1 t, \quad w_1 = 2c_1.$$

Let $q = -h$ be the first Chern class of the tautological line bundle of $\mathbb{P}(E)$ (cf. Lemma 3.1). Then $-h = q = w_1 = 2c_1$. The equality (3.3) yields the obvious equality

$$c(T_{\mathbb{P}V_{1,n}(\mathbb{C})}) = (1-c_1t)^n((1-tq)+w_1t) = (1-c_1t)^n,$$

as $\mathbb{P}V_{1,n}(\mathbb{C}) \sim \mathbb{P}^{n-1}$. We now compute the degree of $\mathbb{P}V_{1,n}(\mathbb{C})$. It is equal to the self intersection index of the hyperplane section

$$h^{n-1} = (-q)^{n-1} = (-2c_1)^{n-1} = 2^{n-1}(-c_1)^{n-1}.$$

Since $-c_1$ is the class of the hyperplane section in $\mathbb{P}^{n-1}$ it follows that deg $\mathbb{P}V_{1,n}(\mathbb{C}) = 2^{n-1}$, which agrees with the formula (1.2). We now compute the Euler characteristic of the intersection of $\mathbb{P}V_{1,n}(\mathbb{C})$ with a generic linear subspace of codimension $k \geq 1$. Let $U = \mathbb{P}(\mathrm{Sym}^2 \tau)$. Then by (3.6)

$$c(T_W, t) = (1 - c_1 t)^n (1 - 2c_1 t)^{-k}.$$

Hence

$$c_{n-1-k}(T_W) = (-c_1)^{n-k-1} \sum_{j=0}^{n-1-k} \binom{n}{j} \binom{-k}{n-1-k-j} 2^{n-1-k-j}.$$

(3.6) yields

$$\chi(W) = 2^k \sum_{j=0}^{n-1-k} \binom{n}{j} \binom{-k}{n-1-k-j} 2^{n-1-k-j}.$$

For $k = n-2$, W is a smooth curve with the Euler characteristic

$$\chi(W) = 2^{n-2}(4-n).$$

5. Gr(2, 4, $\mathbb{C}$)

We now consider $\mathrm{Gr}(2, n, \mathbb{C})$ for $n \geq 3$. Then

$$\begin{aligned}
&c(\tau, t) = 1 + c_1 t + c_2 t^2, \\
&c(\tau', t) = 1 - c_1 t + c_2 t^2 = (1 + \alpha_1 t)(1 + \alpha_2 t), \\
&\alpha_1 + \alpha_2 = -c_1, \quad \alpha_1 \alpha_2 = c_2, \\
(5.1) \qquad &c(\kappa, t) = 1 + \sum_{j=1}^{\infty} s_j t^j = \prod_{j=1}^{n-2} (1 + \beta_j t) \\
&\qquad\qquad = \frac{1}{1 + c_1 t + c_2 t^2} = \frac{1}{(1 - \alpha_1 t)(1 - \alpha_2 t)}, \\
&s_p = \sum_{i=0}^{p} \alpha_1^i \alpha_2^{p-i}, \quad p = 1, \ldots .
\end{aligned}$$

A straightforward calculation shows (cf. [10])

$$s_1 = -c_1,\ s_2 = c_1^2 - c_2,\ s_3 = -c_1^3 + 2c_1c_2,$$
$$s_4 = c_1^4 - 3c_1^2c_2 + c_2^2,\ s_5 = -c_1^5 + 4c_1^3c_2 - 3c_1c_2^2. \tag{5.2}$$

Thus

$$H^*(\mathrm{Gr}(2,4,\mathbb{C}),\mathbb{C}) = \mathbb{C}[c_1,c_2]/(-c_1^3 + 2c_1c_2, c_1^4 - 3c_1^2c_2 + c_2^2),$$
$$H^*(\mathrm{Gr}(2,5,\mathbb{C}),\mathbb{C}) = \mathbb{C}[c_1,c_2]/(c_1^4 - 3c_1^2c_2 + c_2^2, -c_1^5 + 4c_1^3c_2 - 3c_1c_2^2). \tag{5.3}$$

We now compute the four Chern classes v_1, v_2, v_3, v_4 of the tangent bundle of $\mathrm{Gr}(2,4,\mathbb{C})$ in terms of c_1, c_2 using (3.18). Note that the power series corresponding to terms contributed by only α and β respectively correspond to the polynomials

$$(1 - c_1t + c_2t^2)^2 = 1 - 2c_1t + (c_1^2 + 2c_2)t^2 - 2c_1c_2t^3 + c_2^2t^4,$$
$$(1 + s_1t + s_2t^2)^2 = 1 + 2s_1t + (s_1^2 + 2s_2)t^2 + 2s_1s_2t^3 + s_2^2t^4.$$

Hence

$$\begin{aligned}
v_1 =& 2(-c_1 + s_1) = -4c_1,\\
v_2 =& c_1^2 + 2c_2 + s_1^2 + 2s_2 + 3(\alpha_1+\alpha_2)(\beta_1+\beta_2) = 7c_1^2,\\
v_3 =& -2c_1c_2 + 2s_1s_2\\
&+ (\alpha_1^2 + \alpha_2^2 + 4\alpha_1\alpha_2)(\beta_1+\beta_2) + (\alpha_1+\alpha_2)(\beta_1^2 + \beta_2^2 + 4\beta_1\beta_3)\\
=& -6c_1^3\\
v_4 =& c_2^2 + s_2^2 + \alpha_1\alpha_2(\alpha_1+\alpha_2)(\beta_1+\beta_2) + (\alpha_1+\alpha_2)(\beta_1+\beta_2)\beta_1\beta_2\\
&+ (\alpha_1^2+\alpha_2^2)\beta_1\beta_2 + \alpha_1\alpha_2(\beta_1^2+\beta_2^2) + 2\alpha_1\alpha_2\beta_1\beta_2\\
=& c_1^4 + 4c_2^2 = 3c_1^4.
\end{aligned} \tag{5.4}$$

Here we used the two identities in $H^*(\mathrm{Gr}(2,4,\mathbb{C}),\mathbb{C})$ given in (5.3). This agrees with the following well-known computation of the tangent bundle of $\mathrm{Gr}(2,4,\mathbb{C})$. Recall the classical result that $\mathrm{Gr}(2,4,\mathbb{C})$ imbeds as a smooth quadric in $\mathbb{P}^5$. The tangent bundle of $\mathbb{P}^5$ is $(1+h)^6$, while the normal bundle of the quadric is $(1+2h)$. Hence the tangent bundle of the quadric is given by $(1+h)^6/(1+2h)$. So $c_1 = -h$.

We now consider the 3-bundle $\mathrm{Sym}^2\,\tau$. Let w_1, w_2, w_3 be its Chern classes. Then

$$\begin{aligned}
c(\mathrm{Sym}^2\,\tau, t) =& 1 + \sum_{i=1}^{3} w_it^i = (1 - 2\alpha_1t)(1 - 2\alpha_2t)(1 - (\alpha_1+\alpha_2)t)\\
=& (1 + 2c_1t + 4c_2t^2)(1 + c_1t) = (1 + 3c_1t + (2c_1^2 + 4c_2)t^2 + 4c_1c_2t^3.
\end{aligned} \tag{5.5}$$

Then the cohomology ring of $\mathbb{P}(\mathrm{Sym}^2\tau)$ is $\mathrm{H}^*(\mathrm{Gr}(2,4,\mathbb{C})[h]$ $(h=-q)$ with the relation

$$(5.6)\quad h^3+3c_1h^2+(2c_1^2+4c_2)h+4c_1c_2=h^3+3c_1h^2+(2c_1^2+4c_2)h+2c_1^3=0.$$

Use (3.4) to deduce that

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\tau)/\mathrm{Gr}(2,4,\mathbb{C})},t)=1+(3h+3c_1)t+(3h^2+3c_1h+2c_1^2+4c_2)t^2.$$

Hence

$$(5.7)\quad \begin{aligned} c(T_{\mathbb{P}(\mathrm{Sym}^2\tau)},t)=(1+(3h+3c_1)t+&(3h^2+3c_1h+2c_1^2+4c_2)t^2)\\ &\times(1-4c_1t+7c_1^2t^2-6c_1^3t^3+3c_1^4t^4).\end{aligned}$$

Observe also that any monomial in c_1,c_2 of total degree greater than 4 is zero, since the dimension of $\mathrm{Gr}(2,4,\mathbb{C})$ is 4. Consider the intersection of $\mathbb{P}V_{2,4}(\mathbb{C})$ with a linear subspace of codimension 6. This is equivalent to the class of h^6 in $\mathbb{P}(\mathrm{Sym}^2\tau)$. We want to find out the generator of the top cohomology of $\mathbb{P}(\mathrm{Sym}^2\tau)$ and the class of h^6 in terms of this generator. Using the equation (5.6) we can express h^6 as a quadratic polynomial in q with polynomial coefficients in c_1,c_2:

$$\begin{aligned} h^3&=-3c_1h^2-(2c_1^2+4c_2)h-2c_1^3,\\ h^4&=-3c_1(-3c_1h^2-(2c_1^2+4c_2)h-2c_1^3)-(2c_1^2+4c_2)h^2-2c_1^3h\\ &=(7c_1^2-4c_2)h^2+10c_1^3h+6c_1^4,\\ h^5&=(7c_1^2-4c_2)(-3c_1h^2-(2c_1^2+4c_2)h-2c_1^3)+10c_1^3h^2+6c_1^4h\\ &=-5c_1^3h^2-10c_1^4h-10c_1^5=-5c_1^3h^2-10c_1^4h,\\ h^6&=-5c_1^3(-3c_1h^2-(2c_1^2+4c_2)h-2c_1^3)-10c_1^4h^2\\ &=5c_1^4h^2=10c_1^2c_2h^2=10c_2^2h^2.\end{aligned}$$

Multiply h^5 by c_1, h^4 by c_1^2 and h^3 by c_1^3, respectively, to conclude the following relations:

$$(5.8)\qquad h^6=-c_1h^5=c_1^2h^4=5c_1^4h^2=10c_1^2c_2h^2=10c_2^2h^2,\quad c_1^3h^3=-3c_1^4h^2.$$

Recall the result of Harris and Tu [9] that the degree of $\mathbb{P}V_{2,4}(\mathbb{C})$ is 10. Hence $c_2^2h^2$ is the generator in the top cohomology of $\mathbb{P}(\mathrm{Sym}^2\tau)$. (This can be concluded directly.)

We now compute the Euler characteristic of the smooth curve W, obtained by a generic plane section of codimension 5 with $\mathbb{P}V_{2,4}(\mathbb{C})$. Consider the class of h^5 times the first Chern class a (the coefficient of t) in the product

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\tau)},t)(1+ht)^{-5}.$$

A straightforward calculation shows $a = -2h - c_1$. Hence

$$h^5 a = -2h^6 - h^5 c_1 = -h^6,$$

and $\chi(W) = -10$.

We now compute the Euler characteristic of the smooth surface W obtained by a generic plane section of codimension 4 with $\mathbb{P}V_{2,4}(\mathbb{C})$. Consider the class of h^4 times the second Chern class b (the coefficient of t) in the product

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\tau)}, t)(1 + ht)^{-4}.$$

A straightforward calculation shows $b = h^2 - 5c_1 h - 3c_1^2$. Hence

$$h^4 b = h^6 - 5c_1 h^5 - c_1^2 h^4 = 7h^6,$$

and $\chi(W) = 70$.

COROLLARY 5.1: *A generic linear space of codimension* 5 *in* $\mathbb{P}S_4(\mathbb{C})$ *intersects* $\mathbb{P}V_{2,4}(\mathbb{C})$ *at a smooth curve of degree* 10 *and Euler characteristic* -10. *A generic linear space of codimension* 4 *in* $\mathbb{P}S_4(\mathbb{C})$ *intersects* $\mathbb{P}V_{2,4}(\mathbb{C})$ *at a smooth surface of degree* 10 *and Euler characteristic* 70.

Hence we cannot conclude from these results that any linear subspace $L \subset S_4(\mathbb{R})$ of dimension 6 contains a nonzero matrix of rank 2 at most. In [6] we show (using different topological methods) the sharp result that any linear subspace $L \subset S_4(\mathbb{R})$ of dimension 5 contains a nonzero matrix of rank 2 at most. It is of interest to check if the conjugation map $z \to \bar{z}$, described in the beginning of this paper, for $L \cap \mathbb{P}V_{2,4}(\mathbb{C})$, where $L \subset \mathbb{P}S_4(\mathbb{C})$ is a generic linear space of dimension 5, has a nonzero Lefschetz number.

6. $\mathrm{Gr}(2, 5, \mathbb{C})$ modulo 2

THEOREM 6.1: *Let* $L \subset \mathbb{P}S_5(\mathbb{C})$ *be a generic linear space of dimension* 5. *Then* $L \cap \mathbb{P}V_{3,5}(\mathbb{C})$ *is a smooth surface with an odd Euler characteristic.*

Proof: Let τ, κ be the tautological and the quotient bundles of $\mathrm{Gr}(2, 5, \mathbb{C})$. Then $\mathrm{Sym}^2 \kappa \to \mathrm{Gr}(2, 5, \mathbb{C})$ is the subbundle of the trivial bundle $E \to \mathrm{Gr}(2, 5, \mathbb{C})$ given in (3.14). By Lemma 3.3,

$$\tilde{Z} = \mathbb{P}(\mathrm{Sym}^2 \kappa) \subset \mathbb{P}(E) = \mathbb{P}S_5(\mathbb{C}) \times \mathrm{Gr}(2, 5, \mathbb{C})$$

is a resolution of $\mathbb{P}V_{3,5}(\mathbb{C})$. Then $\mathrm{H}^*(\tilde{Z}, \mathbb{C}) = \mathrm{H}^*(\mathrm{Gr}(2, 5, \mathbb{C}), \mathbb{C})[q])$, where q satisfied the second identity of (3.16). Recall that the tangent bundle of $\mathrm{Gr}(2, 5, \mathbb{C})$

is isomorphic to $\kappa \odot \tau'$. The tangent bundle of $\mathbb{P}(\mathrm{Sym}^2 \kappa)$ is given by the formulas (3.3). As the singular points of $\mathbb{P}V_{3,5}(\mathbb{C})$ comprise the variety $\mathbb{P}V_{2,5}(\mathbb{C})$ of codimension $\binom{4}{2} = 6$, it follows that $L \cap \mathbb{P}V_{2,5}(\mathbb{C}) = \emptyset$. Hence $L \cap \mathbb{P}V_{3,5}(\mathbb{C})$ is a smooth surface. It then follows that

$$L \cap \mathbb{P}V_{3,5}(\mathbb{C}) = \tilde{Z} \cap \bigcap_{k=1}^{9} \tilde{H}_i,$$

where $\tilde{H}_i$, $i = 1, \ldots, 9$ are 9 linearly independent fiber hyperplanes in general position, as in Lemma 3.2.

Let b be the coefficient of t^2 in the product

$$c(\kappa \odot \tau', t)c(T_{\mathbb{P}(\mathrm{Sym}^2 \kappa)/\,\mathrm{Gr}(2,5,\mathbb{C})}, t)(1 + ht)^{-9}. \tag{6.1}$$

Then Lemma 3.2 yields that

$$\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C})) = h^9 b[\tilde{Z}]. \tag{6.2}$$

Since we are interested in the parity of $\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C}))$ we will do all the computations modulo 2. (That is, our computations are in $\mathrm{H}^*(\tilde{Z}, \mathbb{Z}_2)$.) This will simplify our computations significantly.

We first consider $\mathrm{H}^*(\mathrm{Gr}(2, 5, \mathbb{C}), \mathbb{Z}_2)$. It is generated by c_1, c_2 with the two simpler relations induced by the second part of (5.3),

$$c_1^4 + c_1^2 c_2 + c_2^2 = 0, \quad c_1^5 = c_1 c_2^2. \tag{6.3}$$

Multiply the first equality by c_1 and use the second identity to deduce

$$c_1^3 c_2 = 0 \Rightarrow c_1^4 c_2 = 0. \tag{6.4}$$

Multiply the first equality in (6.3) by c_2 and use (6.4). Multiply the second equality of (6.3) by c_1. Then

$$c_1^6 = c_1^2 c_2^2 = c_2^3. \tag{6.5}$$

Hence the generator of the top cohomology in $\mathrm{H}^*(\mathrm{Gr}(2, 5, \mathbb{C}), \mathbb{Z}_2)$ is any class in (6.5).

Recall (5.1) for $n = 5$. The equalities (5.2) modulo 2 yield

$$s_1 = c_1, \quad s_2 = c_1^2 + c_2, \quad s_3 = c_1^3.$$

We now compute the first two Chern classes of $\kappa \odot \tau'$, which gives the first two Chern classes v_1, v_2 of the tangent bundle of $\mathrm{Gr}(2, 5, \mathbb{C})$. Observe that the terms in v_1, v_2, expressed either in terms of α or β, are coming from either $c(\tau', t)^3$ or $c(\kappa, t)^2$:

$$\begin{aligned} c(\tau', t)^3 =& (1 - c_1 t + c_2 t^2)^3 = 1 - 3c_1 t + 3(c_2 + c_1^2)t^2 + \text{ higher order terms}, \\ c(\kappa, t)^2 =& (1 + s_1 t + s_2 t^2 + s_3 t^3)^2 = 1 + 2s_1 t + (2s_2 + s_1^2)t^2 \\ & + \text{ higher order terms}. \end{aligned}$$

Using the equalities in (5.2) we obtain

$$\begin{aligned} v_1 =& -3c_1 + 2s_1 = -5c_1, \\ v_2 =& 3(c_2 + c_1^2) + (2s_2 + s_1^2) + 5(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 + \beta_3) = c_2 + c_1^2. \end{aligned}$$

The coefficient 5 in the product of α's and β's is obtained as follows. Consider the product $\alpha_1\beta_1$. It comes twice from the terms $(\alpha_1 + \beta_1)(\alpha_1 + \beta_i)$, $i = 2, 3$ and three times from the terms $(\alpha_1 + \beta_i)(\alpha_2 + \beta_1)$, $i = 1, 2, 3$.

Modulo 2 we get

$$v_1 = c_1, \quad v_2 = c_2 + c_1^2. \tag{6.6}$$

We next compute the Chern polynomial of $\mathrm{Sym}^2 \kappa$ modulo 2. Then

$$c(\mathrm{Sym}^2 \kappa, t) = \prod_{1 \le i \le j \le 3} (1 + (\beta_i + \beta_j)t) = c(\kappa, 2t) \prod_{1 \le i < j \le 3} (1 + (\beta_i + \beta_j)t).$$

Hence modulo 2

$$c(\mathrm{Sym}^2 \kappa, t) = \prod_{1 \le i < j \le 3} (1 + (\beta_i + \beta_j)t) = 1 + w_1 t + w_2 t^2 + w_3 t^3.$$

Then modulo 2

$$\begin{aligned} w_1 =& 2\sum_{i=1}^{3} \beta_i = 0, \\ w_2 =& (\beta_1 + \beta_2)(\beta_1 + \beta_3 + \beta_2 + \beta_3) + (\beta_1 + \beta_3)(\beta_2 + \beta_3) \\ =& \beta_1^2 + \beta_2^2 + \beta_3^2 + s_2 = s_1^2 - s_2 = c_2, \\ w_3 =& (s_1 - \alpha_1)(s_1 - \alpha_2)(s - \alpha_3) = s_1^3 - s_1 s_1^2 + s_2 s_1 - s_3 = c_2 c_1. \end{aligned}$$

Use (3.4) modulo 2 to get

$$\begin{aligned} c(T_{\mathbb{P}(\mathrm{Sym}^2 \kappa)/\mathrm{Gr}(2,5)}, t) =& (1 + ht)^6 + c_2 t^2 (1 + ht)^4 + c_2 c_1 t^3 (1 + ht)^3 \\ =& 1 + (c_2 + h^2)t^2 + \text{ higher order terms}. \end{aligned}$$

Then the coefficient b of t^2 in (6.1) is equal modulo 2 to the coefficient of t^2 in the product

$$\begin{aligned}&(1+c_1t+(c_2+c_1^2)t^2+\cdots)(1+(c_2+h^2)t^2+\cdots)(1+ht+h^2t^2+\cdots)\\ =&1+(c_1+h)t+(c_1^2+c_1h)t^2+\cdots\end{aligned}$$

Hence modulo 2

$$b=c_1^2+c_1h. \tag{6.7}$$

We now consider the second identity of (3.16) for $q=-h$ modulo 2,

$$h^6=c_2h^4+c_2c_1h^3. \tag{6.8}$$

Multiply by h,h^2,h^3,h^4,h^5 the above equality, use (6.3)–(6.5) and the fact that any form in c_1,c_2 of degree greater than 6 equals 0, to obtain

$$\begin{aligned}h^7=&c_2h^5+c_2c_1h^4,\\ h^8=&c_2h^6+c_2c_1h^5=c_2(c_2h^4+c_2c_1h^3)+c_2c_1h^5\\ =&c_2c_1h^5+c_2^2h^4+c_2^2c_1h^3,\\ h^9=&c_2c_1h^6+c_2^2h^5+c_2^2c_1h^4=c_2c_1(c_2h^4+c_2c_1h^3)+c_2^2h^5+c_2^2c_1h^4\\ =&c_2^2h^5+c_2^2c_1^2h^3,\\ h^{10}=&c_2^2h^6+c_2^2c_1^2h^4=c_2^2(c_2h^4+c_2c_1h^3)+c_2^2c_1^2h^4=0,\\ h^{11}=&0.\end{aligned} \tag{6.9}$$

The equality $h^{11}=0 \pmod 2$ means that $h^{11}[\tilde{Z}]$ is an even number. By Harris–Tu this number, the degree of $\mathbb{P}V_{3,5}(\mathbb{C})$, is equal to 20. We claim that the generator of the top cohomology of $\mathrm{H}^*(\tilde{Z},\mathbb{Z}_2)$ is

$$c_1^6h^5=c_1^2c_2^2h^5=c_2^3h^5. \tag{6.10}$$

First, consider all the monomials in h,c_1,c_2 of degree 6 in h and total degree 11,

$$c_1^5h^6,\ c_1^3c_2h^6=0,\ c_1c_2^2h^6.$$

We used here (6.4). Multiply (6.8) by c_1^5 and $c_1c_2^2$ respectively to deduce

$$c_1^5h^6=c_1^3c_2h^6=c_1c_2^2h^6=0.$$

Second, consider all the monomials in h,c_1,c_2 of degree 7 in h and total degree 11,

$$c_1^4h^7,\ c_1^2c_2h^7,\ c_2^2h^7.$$

Multiply h^7 in (6.9) by an appropriate monomial of c_1, c_2 to get

$$c_1^4 h^7 = 0, \quad c_1^2 c_2 h^7 = c_1^2 c_2^2 h^5, \quad c_2^2 h^7 = c_2^3 h^5.$$

Hence all the nonzero terms are equal to the terms in (6.10). Third, consider all the monomial in h, c_1, c_2 of degree 8 in h and total degree 11,

$$c_1^3 h^8, \; c_1 c_2 h^8.$$

Multiply h^8 in (6.9) by an appropriate monomial of c_1, c_2 to get

$$c_1^3 h^8 = 0, \quad c_1 c_2 h^8 = c_1^2 c_2^2 h^5.$$

Thus the nonzero term is equal to the terms in (6.10). Fourth, consider all the monomials in h, c_1, c_2 of degree 9 in h and total degree 11,

$$c_1^2 h^9, \; c_2 h^9.$$

Multiply h^8 in (6.9) by an appropriate monomial of c_1^2 and c_2 to get

$$c_1^2 h^9 = c_1^2 c_2^2 h^5, \quad c_2 h^9 = c_2^3 h^5.$$

Hence all the terms are equal to the terms in (6.10). As $h^{10} = 0$ we deduce that $c_1^2 h^9$ is the generator of the top cohomology in $\mathrm{H}^*(\tilde{Z}, \mathbb{Z}_2)$. Clearly, mod 2

$$h^9 b = c_1^2 h^9 + c_1 h^{10} = c_1^2 h^9.$$

Hence $\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C}))$ is odd. ∎

COROLLARY 6.1: $d(5,3,\mathbb{R}) \leq 6$. *That is, every six dimensional real subspace $L' \subset S_5(\mathbb{R})$ contains a nonzero matrix of rank 3 or less.*

In [5] the authors give an example of five dimensional subspace $L_1 \subset S_5(\mathbb{R})$, for which numerical evidence suggests that every nonzero matrix is of rank 4 at least. Hence the above Corollary suggests that $d(5,3,\mathbb{R}) = 6$.

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