Isolated non-normal crossings

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Abstract

We describe a multivariable polynomial invariant for certain class of non isolated hypersurface singularities generalizing the characteristic polynomial on monodromy. The starting point is an extension of a theorem due to Lê and K.Saito on commutativity of the local fundamental groups of certain hypersurfaces. The description of multivariable polynomial invariants is given in terms of the ideals and polytopes of quasiadjunction generalizing corresponding data used in the study of the homotopy groups of the complements to projective hypersurfaces and Alexander invariants of plane reducible curves.

1 Introduction

One of the central results in the study of the topology of algebraic hypersurfaces near a singular point is the assertion that the complement to the link of a singularity is fibered over the circle with the connectivity of the fiber depending on the dimension of the singular locus of the hypersurface (cf. [20]). More precisely, if $X \subset \mathbf{C}^{n+1}$ is a germ of such a hypersurface, Sing$X$ is the singular locus of X which we shall assume contains the origin 0, $k = \dim X$ and B_ϵ is ball in $\mathbf{C}^{n+1}$ of a sufficiently small radius ϵ centered at 0, then there is a locally trivial fibration $\partial B_\epsilon - \partial B_\epsilon \cap X \to S^1$. Moreover, the fiber F_X of this fibration is $n-k-1$-connected (i.e $\pi_i(F_X) = 0$ for $1 \le i < n-k$). Existence of a locally trivial fibration allows to restate this Milnor's result as follows: the cyclic cover of $\partial B_\epsilon - \partial B_\epsilon \cap X$ corresponding to the kernel of the map $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X) \to \mathbf{Z}$ sending a loop to its linking number with $\partial B_\epsilon \cap X$ is $n-k-1$-connected. In fact this cyclic cover is homotopy equivalent to F_X.

In present papers we consider the *abelian* rather than cyclic covers of the complement $\partial B_\epsilon - \partial B_\epsilon \cap X$ and study their connectivity. It is easy to see (cf. lemma 2.1) that $H_1(\partial B_\epsilon - \partial B_\epsilon \cap X) = \mathbf{Z}^r$ where r is the number of irreducible components of X i.e. a hypersurface must be reducible in order that its complement will admit abelian but non cyclic covers. This forces singularities already in codimension 1. We shall assume that these singularities, inherently present in the cases when non cyclic covers do exist, are normal crossings except perhaps for the origin. In this case the Milnor fiber has the fundamental group isomorphic to $\mathbf{Z}^{r-1}$ (cf. [10] and 4.6) but the universal abelian cover is $(n-1)$-connected i.e. making this situation analogous to the case of isolated singularities in cyclic case. This result is proven in section 3. Standard arguments with hyperplane sections show that if a germ of a singularity has non-normal crossings only in dimension k then $\pi_i(\partial B_\epsilon - \partial B_\epsilon \cap X) = 0$ for $2 \leq i \leq n-k-1$. The fundamental group $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)$ is abelian as was shown by Lê Dung Trang and K.Saito (cf. [10]) and hence is isomorphic to $\mathbf{Z}^r$. Therefore section 3 can be viewed as a generalization of the result of Lê-Saito which takes into account dimension of the locus of non normal crossings.

In the case $n = 1$ the topology of abelian covers of the complements to the union of irreducible germs was studied in [17]. This case is similar to the one considered in this paper but the role of the homotopy group is played by the homology of the universal abelian cover. Indeed in the case of isolated non normal crossings of a union of germs of r hypersurfaces (INNC) the homology and the homotopy of the universal abelian covers are isomorphic (cf. section 2.3). In [17] we described an algebro-geometric way to calculate the support of the homology of universal abelian cover endowed with the structure of $\mathbf{Z}^r$-module. In this paper we generalize this method to the case of isolated non normal crossings. More precisely we consider the $\mathbf{C}[\mathbf{Z}^r]$-module $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes_{\mathbf{Z}} \mathbf{C}$. Its support, can be defined as the set of points $\chi \in \mathrm{Spec}\mathbf{C}[\mathbf{Z}^r] = \mathbf{C}^{*r}$ such that:

$$(\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes_{\mathbf{Z}} \mathbf{C}) \otimes_{\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)} \mathbf{C}_\chi \neq 0$$

(we identify $\mathrm{Spec}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$ and $\mathrm{Char}(\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)$ using isomorphism of their rings of regular function obtained by viewing elements of $\mathbf{Z}^r$ as functions on $\mathrm{Char}(\mathbf{Z}^r)$; here $\mathbf{C}_\chi$ is $\mathbf{C}$ with the action of $\gamma \in \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)$ given by multiplication by $\chi(\gamma)$). Notice that the study of the homotopy groups of the complements with module structure when the fundamental group is $\mathbf{Z}$ started in [13], [14], [1], [2] and in the context of non isolated singularities in [18].

Support loci for homology groups of the universal abelian cover of a CW complex may have arbitrary codimension in $\mathrm{Spec}\mathbf{C}[\mathbf{Z}^r]$ (cf. e.g. calculations in [16]). We show, however, that in the case of isolated non normal crossings $codim \mathrm{Supp} \pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes_{\mathbf{Z}} \mathbf{C}$ is equal to one. In fact we calculate several components of the support of the homotopy group in terms of a resolution of the non normal crossing singularity. These components correspond to faces of polytopes constructed here from resolutions and are called *the principal components*. This calculation is given in terms of certain ideals associated with the singularity generalizing the ideals of quasiadjunction studies in [17] in multivariable version and in [11], [19] in the case of one variable polynomial invariants.

This suggests a polynomial invariant $P(t_1, t_1^{-1}, ..., t_r, t_r^{-1}, X)$ of INNC's defined as a generator of the divisorial hull of the first Fitting ideal of $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes_{\mathbf{Z}} \mathbf{C}$ (it is well defined up to a unit of the ring of Laurent polynomial). In cyclic case this polynomial is the characteristic polynomial of the monodromy. An interesting problem is to isolate the cases then the support of $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X)$ consist entirely of the principal components described in this paper.

The content of the paper is the following. In the next section we discuss technical results used in the paper: (Γ, n)-complexes, Leray-Mayer-Vietoris sequence and other preliminary results. In section 3 we prove mentioned above extension of the theorem of Lê-Saito. Next we show that the zero sets of Fitting ideals of homotopy modules, which we call the characteristic varieties of the homotopy groups (cf. [16]), can be used to calculate the homology of abelian covers of ∂B_ϵ branched over the intersection of the latter with a germ of INNC. Characteristic varieties also determine the homology of unbranched covers of the complement to a germ of INNC in ∂B_ϵ as well as homology of the Milnor fiber. These results generalize corresponding results in dimension $n = 1$ which appear in [12] in unbranched and in [21] in unbranched case. In section 5 we calculate what we call "the principal component" of the support of the characteristic variety in question in terms of ideals and polytopes of quasiadjunciton. In the last section we provide examples and several concluding remarks relating introduced here ideals and polytopes of quasiadjunction to multiplier ideals and log-canonical thresholds. Complete calculation of characteristic varieties remains an interesting problem.

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2 Preliminaries.

2.1 Complements to reducible germs

We continue to use the notions from the Introduction i.e. $X = \bigcup_{i=1}^{i=r} D_i \subset \mathbf{C}^{n+1}$ is a union of germs of singularities and B_ϵ is a small ball about the origin. We shall start with the following:

Lemma 2.1 $H_1(\partial B_\epsilon - \partial B_\epsilon \cap X) = \mathbf{Z}^r$

This is immediate consequence of the Alexander duality and Mayer Vietoris exact sequence since $\partial B_\epsilon \cap X$ is a union of r $(2n-1)$-dimensional manifolds intersecting *pairwise* along $(2n-3)$-submanifolds.

Notice that the identification with $\mathbf{Z}^r = \oplus_{i=1}^r \mathbf{Z} e_i$ depends on the ordering of irreducible components of X. If such selection is made, the isomorphism in 2.1 sends e_i to a loop having the linking number with i-th component equal to 1 and zero linking number with the remaining ones.

Our main result on the homotopy groups $\pi_i(\partial B_\epsilon - \partial B_\epsilon \cap \bigcup D_i)$, which we prove in the next section, is the following:

Theorem 2.2 *Let $X = \bigcup_{i=1}^{i=r} D_i \subset \mathbf{C}^{n+1}$ be is union of r irreducible germs with normal crossings outside of the origin. If $n \geq 2$, then*

$$\pi_1((\partial B_\epsilon - \partial B_\epsilon \cap X) = \mathbf{Z}^r \quad \pi_k((\partial B_\epsilon - \partial B_\epsilon \cap X) = 0 \quad \text{for} \quad 2 \leq k < n$$

The term "normal crossings outside of the origin" is used in the sense that each components is non singular outside of the origin and intersection of components at each point outside of the origin is the normal crossing. The claim about the fundamental group is immediate consequence of the result of Lê and K.Saito (cf. [10]) coupled with lemma 2.1.

Here we just point out that theorem 2.2 yields the following:

Corollary 2.3 *Let $NN(X)$ be the non-normal locus of $X = \bigcup_{i=1}^{i=r} D_i$ i.e. subset of X of points at which X fails to be a normal crossing divisor. If codimension of $NN(X)$ in X is greater than 1 then:*

$$\pi_1((\partial B_\epsilon - \partial B_\epsilon \cap X) = \mathbf{Z}^r \text{ and } \pi_k(\partial B_\epsilon - \partial B_\epsilon \cap X) = 0 \text{ for } 2 \leq k < n - \mathrm{dim} NN(X)$$

Let us consider generic linear subspace L in $\mathbf{C}^{n+1}$ having codimension equal to $\mathrm{dim}NN(X)$ and passing through the origin. Then by a theorem of Lê and Hamm (cf. [9]) $\pi_k(L \cap \partial B_\epsilon - \partial B_\epsilon \cap X) = \pi_k(\partial B_\epsilon - \partial B_\epsilon \cap X)$ for $k < \mathrm{dim}L - 1$. Hence the corollary follows from theorem 2.2.

2.2 Action of $\pi_1(X)$ on higher homotopy groups

Fundamental group of a CW-complex acts on the homotopy groups $\pi_i(X)$ via Whitehead product (cf. [25]). If $n \geq 2$ and one has vanishing: $\pi_i(X) = 0$ for $2 \leq i < n$ then such action on $\pi_n(X)$ coincides with the action of the group of covering transformation on $H_n(\tilde{X}) = \pi_n(\tilde{X}) = \pi_n(X)$ where $\tilde{X}$ is the universal cover of X (the first equality is the Hurewicz isomorphism).

We shall use the functoriality of this action. In particular if $p : Y \to X$ is a circle fibration, $\gamma \in \pi_1(Y)$ is an element corresponding to the fiber and $x \in \pi_j(Y)$ we have: $p_*(\gamma \cdot x) = p_*(\gamma)\dot{p}_*(x) = p_*(x)$. Since $p_* : \pi_j(Y) \to \pi_j(X)$ is injective for $j \geq 2$ the action of γ is trivial.

2.3 Complexes with vanishing low dimensional homotopy

Definition 2.4 *([3]) A complex K is called a (Γ, k) complex if:*

1.$dimK = k$

2. $\pi_1(K) = \Gamma$

3. $\pi_i(K) = 0$ for $2 \leq i \leq k-1$.

For certain Γ the homotopy type of a (Γ, k) complex is determined by the Euler characteristic. Class of such groups includes $\mathbf{Z}^r$ (cf. [3], [24]). In particular a $(\mathbf{Z}^r, n)$ complex is homotopy equivalent to a wedge of the n-skeleton of a n-torus $(S^1)^r$ wedged with several copies of S^n.

2.4 Leray-Mayer-Vietoris spectral sequence

Let X be a paracompact topological space and let $\mathcal{M} = \{M_i | i \in \mathcal{S}\}$ be a locally finite cover of X by closed subsets. For a sheaf $\mathcal{A}$ on X one has the Leray spectral sequence (cf. [4] Th. 5.2.4) which is one of the two spectral sequence corresponding to the double complex: $C^{*,*}(X, \mathcal{M}, \mathcal{A}) = \oplus_{\mathrm{Card}S=p+1} C^q(\cap_{i \in S} M_i, \mathcal{A})$ with Cech and usual cochains differentials. It has as abutment $H^{p+q}(X, \mathcal{A})$ and $E_2^{p,q} = H^p(\mathcal{M}, \mathcal{H}^q(\mathcal{A}))$. Here $\mathcal{H}^q(\mathcal{A})$ is the system of coefficients on the nerve of the cover $\mathcal{M}$ given by $S \to H^q(M_S, \mathcal{A})$ where S is a subset of the set $\mathcal{S}$. Moreover $E_1^{p,q} = C^p(\mathcal{M}, \mathcal{H}^q) = \oplus_{\mathrm{Card}S=p+1} H^q(M_S, \mathcal{A})$. Applying this to the case of finite union of closed subsets $X = \bigcup_{i \in \mathcal{S}} X_i$, a constant sheaf $\mathcal{A}$ and denoting $X^{[p]} = \coprod X_{i_0} \cap .... \cap X_{i_p}$ we obtain the spectral sequence:

$$E_1^{p,q} = H^q(X^{[p]}) \Rightarrow H^{p+q}(X) \tag{1}$$

Example 2.5 *In the case* $\mathrm{Card}\mathcal{S} = 2$ *this spectral sequence is equivalent to the Mayer-Vietoris exact sequence. Indeed, only terms* $E_1^{0,q} = H^q(X_1) \oplus H^q(X_2)$ *and* $E_1^{1,q} = H^q(X_1 \cap X_2)$ *in* E_1*-term of the spectral sequence (1) are non trivial. Moreover* $E_2^{0,q} = \mathrm{Ker} H^q(X_1) \oplus H^q(X_2) \to H^q(X_1 \cap X_2)$ *and* $E_2^{1,q} = \mathrm{Coker} H^q(X_1) \oplus H^q(X_2) \to H^q(X_1 \cap X_2)$. *Since the other differentials must be trivial one has:* $0 \to E_2^{0,n} \to H^n(X_1 \cap X_2) \to E_2^{1,n-1} \to 0$. *The exactness of this sequence is equivalent to the exactness of the Mayer-Vietories.*

3 Homotopy vanishing.

In this section we shall prove the theorem 2.2.

Step 1. The action of $\pi_1(\partial B_\epsilon - \cup_i D_i)$ *on* $\pi_j(\partial B_\epsilon - \cup_i D_i)$ *is trivial if* $j \leq n-1$.

Let $T_{\delta,l}$ be a sufficiently small tubular of one of the hypersurfaces D_l in B_ϵ. Let L be a generic hyperplane passing through the origin which belongs to $T_{\delta,l}$. By a Zariski-Lefschetz type theorem (cf. [9]) we have surjection for $j \leq n-1$ and an isomorphism for $j < n-1$ (the first and last equalities due to the conical structure of singularities and are valid for all i):

$$\pi_j(\partial B_\epsilon - \cup_i D_i) \cap L) = \pi_j((B_\epsilon - \cup_i D_i) \cap L) \to \pi_j((B_\epsilon - \cup_i D_i)) = \pi_j(\partial B_\epsilon - \cup_i D_i) \tag{2}$$

This implies that the second map in the composition:

$$\pi_j(\partial(B_\epsilon - \cup_i D_i) \cap L) \to \pi_j(T_{\delta,l} \cap \partial B_\epsilon - \cup_i D_i) \to \pi_j(\partial B_\epsilon - \cup_i D_i) \tag{3}$$

is surjective for $j \leq n-1$

On the other hand we have a locally trivial fibration:

$$T_{\delta,l} \cap (\partial B_\epsilon - \cup_i D_i) \to \partial B_\epsilon \cap (D_l - D_l \cap_{i \neq l} D_i) \tag{4}$$

which yields that the action on $\pi_j(T_{\delta,l} \cap (\partial B_\epsilon - \cup_i D_i))$ of the element of $\pi_1(T_{\delta,l} \cap (\partial B_\epsilon - \cup_i D_i))$ corresponding to the loop which the boundary of the transversal to D_l disk in ∂B_ϵ is trivial (cf. sect. 2.2) Therefore the action of the element of $\pi_1(\partial B_\epsilon - \cup D_i)$ corresponding to the boundary of a small 2-disk normal to $D_l \cap \partial B_\epsilon$

in ∂B_ϵ on $\pi_i(\partial B_\epsilon - \cup D_i)$ is trivial for $j \le n-1$. Since this is the case for all $l, (l = 1, ..., r)$ and $\pi_1(\partial B_\epsilon - \cup D_i)$ is generated by these loops the claim follows.

Step 2. For $j \le n-1$ one has the isomorphism:

$$H_j(\partial B_\epsilon - \cup_i D_i) = \Lambda^j(\mathbf{Z}^r) \tag{5}$$

and for $j = n$ the surjection:

$$H_n(\partial B_\epsilon - \cup_i D_i) \to \Lambda^n(\mathbf{Z}^r) \tag{6}$$

First notice that exact sequence of the pair $(\partial B_\epsilon, \partial B_\epsilon - \cup_i D_i)$ and the duality yields:

$$H_j(\partial B_\epsilon - \cup_i D_i) = H^{2n-j}(\partial B_\epsilon \cap \cup_i D_i) \tag{7}$$

The spectral sequence

$$E_1^{p,q} = H^q(D^{[p]}) \Rightarrow H^{p+q}(\cup_i \partial B_\epsilon \cap D_i) \tag{8}$$

discussed in 2.4 in the case $X_i = D_i \cap \partial B_\epsilon$ has non zeros in the term E_1 only for $q = 0, n-1-p, n-p, 2n-1-2p$. Moreover $E_1^{p,2n-1-p} = \oplus_{i_0 < ... < i_p} H^{2n-1-p}(D_{i_0} \cap .... \cap D_{i_p}) = \Lambda^{p+1}(\mathbf{Z}^r)$ as can be seen by induction from the Mayer-Vietoris sequence. For $p + q \ge n+1$ the terms in $E_1^{p,q}$ are unaffected by subsequent differentials and hence:

$$\Lambda^p(\mathbf{Z}^r) = E_1^{p-1,2n+1-2p} = E_\infty^{p-1,2n+1-2p} = H^{2n-p}(\cup_i D_i) = H_p(\partial B_\epsilon - \cup_i D_i) \ (p \le n-1) \tag{9}$$

If $r < n$ then $E_1^{p,q} = 0$ for $p \ge r$ and for $p = n$ it follows that $E_1^{n-1,1} = \Lambda^n(\mathbf{Z}^r) = 0$ which proves the claim if the number of components of the divisor is less than the dimension of the ambient space. On the other hand the case of arbitrary r can be reduced to this one as follows. Notice that adding sufficiently high powers of new variables to equations defining D_i's yields the system of r equations defining INNC $\cup \tilde{D}_i$ in $\mathbf{C}^N$ such that all sections of $\cup \tilde{D}_i$ by linear subspace through the origin are diffeomorphic (since by Tougeron's theorem adding terms of sufficiently high degree does not change topological type). By Lefschetz theorem (cf. [9]) one has surjection $\pi_n(\partial B_\epsilon - \cup_i D_i) \to \pi_n(\partial B_\epsilon - \cup_i \tilde{D}_i) = \Lambda^n \mathbf{Z}^r$. Taking $N > r$ we obtain the claim.

Step 3. End of the proof.

We shall finish the proof of the theorem 2.2 by induction over j in $\pi_j(\partial B_\epsilon - \cup D_i)$. For $j = 1$ the claim is proven in [10]. Next assuming vanishing of the homotopy up to dimension $j-1$ let us consider the five terms exact sequence for a space $\mathcal{X}$ on which a group π acts freely ($(\pi_j)_\pi$ is the quotient of covariants):

$$H_{j+1}(\mathcal{X}) \to H_{j+1}(\pi) \to (\pi_j)_\pi \to H_j(\mathcal{X}) \to H_j(\pi) \to 0 \tag{10}$$

In the case when $\mathcal{X}$ is the universal abelian cover of $\partial B_\epsilon - \partial B_\epsilon \cap \bigcup_i D_i$ and $\pi = \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i))$ it follows from the step 2 and the isomorphism $\pi = \mathbf{Z}^r$ that the right homomorphism is isomorphism for $j \le n-1$, while the left is the isomorphism for $j \le n-2$ and surjective for $j = n-1$. On the other hand the step 1 yields that $(\pi_j(X))_{\pi_1} = \pi_j(X)$. Hence $\pi_j(X) = 0$ as long as $j \le n-1$ which yields the claim of theorem 2.2.

4 Homotopy module and the homology of abelian covers.

We shall describe now the characteristic varieties corresponding to INNC, which are the invariants of the first non-vanishing higher homotopy group of the complement to the link. Then we shall use them to calculate the homology of the covers associated with INNC and the homology of Milnor fibers of INNC's.

Definition 4.1 *Homotopy module of an INNC is* $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ *considered as the module over* $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$.

Let $\mathrm{Char}(\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))) = \mathrm{Hom}(\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)), \mathbf{C}^*)$ be the group of characters of the fundamental group. This group can be identified with $\mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))]$ since any $\gamma \in \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))$ defines the function $\chi \to \chi(\gamma)$ on the group of characters and so does any linear combination of the elements of the group algebra.

Let $R = \mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))]$. Using the identification of $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)) = H_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))$ with $\mathbf{Z}^r$ from 2.1 we obtain identification of R with the ring of Laurent polynomials $\mathbf{C}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$. The k-th Fitting ideal of the homotopy module 4.1 is the ideal generated by the minors of order $(n-k+1) \times (n-k+1)$ of the map $\Phi : R^m \to R^n$ such that $\mathrm{Coker}\Phi = \pi_n(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i) \otimes_{\mathbf{Z}} \mathbf{C}$ (cf. [16]).

Definition 4.2 *k-th characteristic variety $V_k(X)$ of an isolated non-normal crossing $X = \cup_{i=1}^{i=r} D_i$ is the subset of zeros of the k-th Fitting ideal in* $\mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))]$. *Alternatively (cf. [16]) this is the set of P such that*

$$\mathrm{rk}_{\mathbf{C}} \pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes_{\mathbf{Z}} \mathbf{C} \otimes_{\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)} \mathbf{C}_P \ge k$$

with the structure of $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)$-module on $\mathbf{C}_P$ defined as the standard structure on a quotient of $\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$-module (cf. [16]).

Notice that if X_i is obtained from X by deleting the component D_i we have surjections $\psi_i : \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X) \to \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X_i)$ and hence embedding: $\psi_i^* : \mathrm{Spec}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X_i)] \to \mathrm{Spec}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$.

Definition 4.3 *The nonessential part of k-th characteristic variety of INNC X is the subset of this characteristic variety which consist of points which belong to $\mathrm{Im}\psi_i^*(V_k(X_i))$. Complement to the non essential part is called the essential part of characteristic variety.*

Now we shall describe the homology of branched and unbranched covers in terms of characteristic varieties. For each $P \in \mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$ let:

$$f(P, X) = \{\max k \mid P \in V_k(X)\} \tag{11}$$

We have the following:

Proposition 4.4 *Let $U_{m_1,...,m_r}$ be unbranched cover of $\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ corresponding to the homomorphism $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)) = \mathbf{Z}^r \to \oplus_{1\le i\le r}\mathbf{Z}/m_i\mathbf{Z}$. Then*

$$\mathrm{rk}H_p(U_{m_1,...,m_r},\mathbf{C}) = \Lambda^p(\mathbf{Z}^r) \; p \le n-1$$

$$\mathrm{rk}H_n(U_{m_1,...,m_r},\mathbf{C}) = \sum_{(...,\omega_j,...),\omega_j^{m_j}=1} f((...,\omega_j,...),\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i))$$

A proof, similar to the one given in [12] in the case of links in S^3, can be obtained by applying the sequence (10) to the case when $\mathcal{X}$ is the universal abelian cover of $\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ and when π is the kernel of the homomorphism $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)) = \mathbf{Z}^r \to \oplus_{1\le i\le r}\mathbf{Z}/m_i\mathbf{Z}$.

In branched case we have:

Proposition 4.5 *Let $V_{m_1,...,m_r}$ be branched cover of ∂B_ϵ branched over $\partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ with the Galois group $G = \oplus_{1\le i\le r}\mathbf{Z}/m_i\mathbf{Z}$. For each $\chi \in \mathrm{Char}G$ let $I_\chi = \{i | 1 \le i \le r, \chi(\mathbf{Z}_{m_i}) \ne 1\}$ where $\mathbf{Z}_{m_i}$ is the i-th summand of G. Such χ can also be considered as a character of $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{i\in I_\chi} D_i))$ in which case it will be called reduced and denoted χ_{red}. Let V_χ be the branched cover of ∂B_ϵ branched over $\partial B_\epsilon \cap (\bigcup_{i\in I_\chi} D_i)$ and having $\mathrm{Im}\chi = G/\mathrm{Ker}\chi$ as its Galois group. Then*

$$\pi_p(V_{m_1,...,m_r}) = 0 \;\; 1 \le p \le n-1$$

$$\mathrm{rk}H_n(V_{m_1,...,m_r},\mathbf{C}) = \sum_{\chi\in\mathrm{Char}} f(\chi_{red}, \bigcup_{i\in I_\chi} D_i)$$

Proof. The abelian cover $V_{m_1,...,m_r}$ is the link of an isolated complete intersection singularity:

$$z_1^{m_1} = f_1(x_1,...,x_n), ..., z_r^{m_r} = f_r(x_1,...,x_n) \tag{12}$$

where $f_i(x_1,...,x_n) = 0, 1 \le i \le r$ is an equation of the divisor D_i. Hence we obtain the first equality in the proposition (cf. [6]).

For $\chi \in \mathrm{Char}G$ and a G-vector space W let $W^\chi = \{w \in W | gw = \chi(g)w\}$. We also will denote by U_χ the unbranched cover of $\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ corresponding to the subgroup $\mathrm{Im}\chi \subset G$. We have $U_\chi = U_{m_1,...,m_r}/\mathrm{Ker}\chi$ and $V_\chi = V_{m_1,...,m_r}/\mathrm{Ker}\chi$. Hence

$$H_p(V_\chi,\mathbf{C}) = H_p(V_{m_1,...,m_r},\mathbf{C})_{\mathrm{Ker}\chi} \tag{13}$$

(on the right is the quotient group of covariants:

$$H_p(V_{m_1,...,m_r},\mathbf{C})/(1-g)H_p(V_{m_1,...,m_r},\mathbf{C})$$

with $g \in \mathrm{Ker}\chi$). The latter implies that

$$H_p(U_{m_1,...,m_r})^\chi = H_p(U_\chi)^\chi \quad H_p(V_{m_1,...,m_r})^\chi = H_p(V_\chi)^\chi \tag{14}$$

If a character χ is such that $\chi(t_i) \ne 1, q \le i \le r$ where t_i's are the standard generators of $H_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ (cf. section 2.1) then

$$H_n(U_{m_1,...,m_r})^\chi = H_n(V_{m_1,...,m_r})^\chi \tag{15}$$

Indeed in the exact sequence of the pair:

$$H_{n+1}(V_{m_1,...,m_r}, U_{m_1,...,m_r}) \to H_n(U_{m_1,...,m_r}) \to H_n(V_{m_1,...,m_r}) \to H_n(V_{m_1,...,m_r}, U_{m_1,...,m_r})$$

the terms on the ends are isomorphic to the cohomology of preimage of branching locus of the cover. All eigenspaces there have characters trivial on one of t_i which yields (15). Finally for a character χ such that $\chi(t_{i_1}) = ... = \chi(t_{i_k}) = 1$ we have:

$$H_n(V_{m_1,...,m_r})^{\chi} = H_n(V_{\chi})^{\chi} = H_n(V_{m_1,...,\hat{m}_i,..m_r})^{\chi} = H_n(U_{m_1,...\hat{m}_i,..,m_r})^{\chi}) \ (i = i_1, ..., i_k) \tag{16}$$

(the second equality takes place since for selected χ the space V_{χ} constructed for $V_{m_1,...,m_r}$ and $V_{m_1,...,\hat{m}_i,..m_r}$ are the same. Now the conclusion follows form the proposition 4.4

Similar to the above approach can be used to describe the homology of the Milnor fiber of the singularity of X in terms of the homotopy module.

Proposition 4.6 *The homology of the Milnor fiber F_X of the INNC singularity X is given by:*

$$H_p(F_X, \mathbf{Z}) = \Lambda^p(\mathbf{Z}^{r-1}) \quad \text{for } 1 \le p < n$$

The action of the monodromy of this homology is trivial. The multiplicity of $\omega \neq 1$ as a root of the characteristic polynomial $\Delta_n(X, t)$ of the Milnor's monodromy on $H_n(F_X, \mathbf{C})$ is equal to:

$$m_{\omega} = f((..., \omega, ...), X) = \max\{i | (\omega, ..., \omega) \in V_i(X) \subset \mathrm{Spec}\mathbf{C}[\pi_1(\partial B_{\epsilon} - \partial B_{\epsilon} \cap X)]\}$$

Proof. As follows the from Milnor's fibration theorem (cf. [20]), the Milnor fiber is homotopy equivalent to the cyclic cover of $\partial B_{\epsilon} - \partial B_{\epsilon} \cap X$ corresponding to the subgroup $\mathbf{Z}^{r-1} = \mathrm{Ker}\pi \subset \pi_1(\partial B_{\epsilon} - \partial B_{\epsilon} \cap X)$ where the homomorphism $\pi : \pi_1(\partial B_{\epsilon} - \partial B_{\epsilon} \cap X) \to \mathbf{Z}$ is given by $e_i \to 1$ with generators e_i described in 2.1. The fibration of the classifying spaces corresponding to the homomorphism π: $S^{1^r} \to S^1$ is trivial and hence the action of the monodromy yielded by S^1 on the cohomology of the classifying space of $\mathrm{Ker}\pi$ is trivial as well. We have $\pi_1(F_X) = \mathrm{Ker}\pi = \mathbf{Z}^{r-1}$. We can calculate the homology of the Milnor fiber using the induction starting with $j = 2$ and the exact sequence (10) corresponding to the group acting freely on the universal abelian cover with the quotient F_X. In this case it looks as follows:

$$H_{j+1}(F_X) \to H_{j+1}(\mathbf{Z}^{r-1}) \to (\pi_j(\partial B_{\epsilon} - \partial B_{\epsilon} \cap X)_{\mathbf{Z}^{r-1}}) \to H_j(F_X) \to H_j(\mathbf{Z}^{r-1}) \to 0 \tag{17}$$

As long as $j < n$ the middle term is trivial by vanishing theorem of section 3 and we obtains the first claim of the proposition. For $j = n$ the second from the left term may affect only the root $t = 1$ of $\Delta_n(X, t)$. Other roots of $\Delta(X, t)$ are the elements of characteristic variety of $\mathbf{C}[t, t^{-1}]$-module $H_n(F_X, \mathbf{C})$ and the second claim follows since the map $\mathrm{Spec}\mathbf{C}[t, t^{-1}] \to \mathrm{Spec}\mathbf{C}[\pi_1(\partial B_{\epsilon} - \partial B_{\epsilon} \cap X)]$ corresponding to homomorphism π is given by $\omega \to (\omega, ..., \omega)$ (cf. [16]).

Proposition 4.7 *Let X be a germ of INNC having $r-1$ irreducible components and let D be a hypersurface with isolated singularity at the origin. Let i_D be the surjection: $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (X \cup D)) \to \pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)$ inducing the map of the groups of characters:*

$$i_D^* : \mathrm{Char}\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X) \to \mathrm{Char}\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (X \cup D))$$

Then

$$i_D^*(V_i(X) \cap \mathrm{Tor}\ \mathrm{Char}\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X) \subset V_i(X \cup D)$$

Proof. First notice that if Y is a link of INCC and h is a diffeomorphism of Y leaving irreducible components and their orientations invariant then

$$h_*|_{H_i(Y,\mathbf{Z})} = id \ \text{ for } [\frac{\mathrm{dim}Y}{2}] + 1 < i \leq \mathrm{dim}Y \tag{18}$$

(here [...] denotes the integer part). This follows from the Mayer-Vietoris spectral sequence considered in section 3. Indeed, the action of h_* is trivial on the terms $E_1^{p,q}$ with $p+q > [\frac{\mathrm{dim}Y}{2}]+1$ since each such non zero $E_1^{p,q}$ is generated by the fundamental class of a link of isolated hypersurface singularity.

The map i_D^*, in coordinates dual to coordinates from the identification described in 2.1: $H_1(\partial B_\epsilon - \partial B_\epsilon \cap X, \mathbf{Z}) = \mathbf{Z}^{r-1}$, takes $(\omega_1, ..., \omega_{r-1}) \to (\omega_1, ..., \omega_{r-1}, 1)$ (where ω_i is a root unity of degree m_i). Let $\pi_{m_1,...,m_{r-1}} : U_{m_1,...,m_{r-1}} \to \partial B_\epsilon - \partial B_\epsilon \cap X$ be the covering map. We have the exact sequence:

$$H_n(U_{m_1,...,m_{r-1}} - \pi^{-1}_{m_1,...,m_{r-1}}(D \cap (\partial B_\epsilon - \partial B_\epsilon \cap X))) \to H_n(U_{m_1,...,m_{r-1}}) \to$$

$$\to H_n(U_{m_1,...,m_{r-1}} - \pi^{-1}_{m_1,...,m_{r-1}}(D \cap (\partial B_\epsilon - \partial B_\epsilon \cap X)), U_{m_1,...,m_{r-1}}) \to \tag{19}$$

Using Lefschetz duality one obtains that the above relative homology group is isomorphic to

$$H^{n+1}(\bar{\pi}^{-1}_{m_1,...,m_{r-1}}(D \cap B_\epsilon), \bar{\pi}^{-1}_{m_1,...,m_{r-1}}(D \cap X \cap B_\epsilon)) \tag{20}$$

$\pi^{-1}_{m_1,...,m_{r-1}}(D \cap \partial B_\epsilon)$, which is the abelian cover of $D \cap \partial B_\epsilon$ branched over $D \cap X \cap \partial B_\epsilon$ can be identified with the link of singularity

$$u_1^{m_1} = g_1(x_1, ..., x_{n+1}), ..., u_r^{m_r} = g_r(x_1, ..., x_{n+1}), f(x_1, ..., x_{n+1}) = 0$$

where g_i are the equations of the components of X and f is the equation of D. Similarly the preimage of the ramification locus $\bar{\pi}_{m_1,..,m_{r-1}}$ can be identified with the link of INNC. Hence the exact sequence of pair and (18) yield that the action of the covering transformation on the last term in (19) is trivial. Therefore the multiplicity of the character corresponding to $(\omega_1, ..., \omega_{r-1})$ in $H_n(U_{m_1,..,m_{r-1}})$ does not exceed the multiplicity of $(\omega_1, .., \omega_{r-1}, 1)$ in $H_n(U_{m_1,..,m_{r-1}} - \pi^{-1}_{m_1,...,m_{r-1}}(D))$. Now the result follows from Prop. 4.4.

Remark 4.8 *The proposition 4.7 shows that non empty components of an INNC X containing the torsion points (in particular principal components described in the next section) always define components of characteristic varieties of homotopy modules of X' obtained from X by adding extra-components.*

5 Polynomial invariants of INNC.

In this section we show how one can calculate the components of characteristic varieties using resolution of singularities.

Recall first the results on the mixed Hodge structure on the cohomology of links of isolated singularities ([22]). Let V be a germ of a complex m-dimensional space having isolated singularity at $p \in V$ and L be the link of this singularity. Let $\pi : \tilde{V} \to V$ be a resolution such that the exceptional locus E is a divisor with normal crossings. The cohomology groups $H^*(L)$, which can also be viewed as the cohomology with support $H^{*+1}_{\{p\}}(V)$, carry the canonical mixed Hodge structure (cf. [22] (1.7)). Its Hodge numbers we denote

$$h^{ipq}(L) = \dim Gr^q_F Gr^p_{\bar{F}} Gr^{p+q}_W H^i(L) = \dim Gr^q_F Gr^p_{\bar{F}} Gr^{p+q}_W H^{i+1}_{\{p\}}(V)$$

We will need the expression of h^{ipq} in terms of differentials on the smooth part of V. We have:

$$h^{(m-1)pq}(L) = h^{(m-1)pq}(E) \ for \ \ p+q < m-1$$

$$h^{(m-1)p(m-1-p)}(L) = h^{(m-1)p(m-1-p)}(E) - h^{(m+1)(m-p)(p+1)}(E) \tag{21}$$

(cf. [22] Cor.(1.12), the first exact sequence and $Gr^W_{p+q} H^i_E(\tilde{V}) = 0$ for $p+q \neq i, i \leq m$). Notice that Mayer Vietors exact sequence and induction over the number components of E yield that $h^{m+1,i,j}(E) = 0$ if $i > m-1$ or $j > m-1$ since each irreducible component of E has dimension equal to $m-1$. Therefore:

$$h^{(m-1)k0}(L) = h^{(m-1)k0}(E) \qquad 0 \leq k \leq m-1 \tag{22}$$

Moreover, the Fujiki duality $\mathrm{Hom}(H^i(E), \mathbf{Q}(-m)) = H^{2m-i}_E(\tilde{V})$ (cf. [22] (1.6)) allows to interpret the cohomology of E in terms of cohomology of with support on E:

$$h^{(m-1)pq}(E) = h^{(m+1)(m-p)(m-q)}_E(\tilde{V})$$

where

$$h^{(m+1)ab}_E = \dim Gr^a_F Gr^b_{\bar{F}} Gr^{a+b}_W H^{m+1}_E(\tilde{V})$$

Filtrations on H^{m+1}_E have the form:

$$H^{m+1}_E = F^0 \supseteq F^1 \supseteq ... \supseteq F^{m+1} \supseteq 0$$

$$0 = W_m \subseteq W_{m+1} ... \subseteq W_{2m} = H^{m+1}_E$$

Therefore, since $\sum_{p \geq m+2, a+b=w} h^{(m+1)ab} = \mathrm{rk} F^{m+2}(W_w(H^{m+1}_E)/W_{w-1}(H^{m+1}_E)) = 0$, we have:

$$F^{m+1}(W_w(H^{m+1}_E)/W_{w-1}(H^{m+1}_E)) = h^{(m+1)(m+1)(w-m-1)}_E \quad (m+1 \leq w \leq 2m) \tag{23}$$

Hence:

$$h^{(m+1)(m+1)0}_E + ... + h^{(m+1)(m+1)(l-m-1)}_E = \dim W_{m+l} F^{m+1} H^{m+1}_E = \dim H^0(W_l \Omega^m_{\tilde{V}}(\log E)/\Omega^m_{\tilde{V}})$$

The last equality is a consequence of the bifiltered isomorphism $H^*_E(\tilde V) = \mathbf{H}^*(\Omega^{*-1}_{\tilde V}(\log E)/\Omega^{*-1}_{\tilde V})$ (i.e. respecting weight and Hodge filtrations) with the Hodge filtration on the latter obtained from the spectral sequence starting with $H^q(W_l\Omega^p_{\tilde V}(\log E)/\Omega^m_{\tilde V})$ and abuting to $\mathbf{H}^*(W_l\Omega^{*-1}_{\tilde V}(\log E)/\Omega^{*-1}_{\tilde V})$ and degeneration of this spectral sequence (cf. [22],(1.6)). We have $H^1(\Omega^m_{\tilde V}) = 0$ by a theorem of Grauert and Riemenschneider ([5]). Hence we obtain:

$$h_E^{(m+1)(m+1)l} = \text{dimCoker}(H^0(\Omega^m_{\tilde V}) \to H^0(W_l\Omega^m(\log E))) - \text{dimCoker}(H^0(\Omega^m_{\tilde V}) \to \\ \to H^0(W_{l-1}\Omega^m(\log E))) \quad (24)$$

We shall apply this identity to calculation of the Hodge numbers of branched covers $V_{m_1,...,m_r}$ of ∂B_ϵ (cf. Prop. 4.5).

Definition 5.1 *Let $f_i = 0$ be the equation of divisor D_i and let $\pi : \tilde{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ be a resolution of the singularities of $\bigcup D_i$ (i.e. the proper preimage of the latter is a normal crossings divisor). Let $\mathcal{V}_{m_1,...,m_r}$ be the singularity (12) having $V_{m_1,...,m_r}$ as its link. Let $\tilde V$ be a normalization of $\tilde{\mathbf{C}}^{n+1} \times_{\mathbf{C}^{n+1}} \mathcal{V}_{m_1,..,m_r}$ (cf. (12)) The ideal of quasiadjunction of type $(j_1, ..., j_r|m_1, ..., m_r)$ is the ideal $\mathcal{A}(j_1, ..., j_r|m_1, ..., m_r)$ of germs $\phi \in \mathcal{O}_{0,\mathbf{C}^{n+1}}$ such that $(n+1)$-form:*

$$\omega_\phi = \frac{\phi z_1^{j_1} \cdot ... \cdot z_r^{j_r} dx_1 \wedge ... \wedge dx_{n+1}}{z_1^{m_1-1} \cdot ... \cdot z_r^{m_r-1}} \quad (25)$$

on the non singular locus of $V_{m_1,...,m_r}$ after the pull back on $\tilde V$ extends over the exceptional set.

The l-th ideal of log-quasiadjunction $\mathcal{A}_l(\log E)(j_1, ..., j_r|m_1, ..., m_r)$ is the ideal of $\phi \in \mathcal{O}_{0,\mathbf{C}^{n+1}}$ such that the the pull back of the corresponding form ω_ϕ on $\tilde V$ is log-*form on $(\tilde V, E)$ having weight at most l.*

Collection of ideals of log-quasiadjunction (which is finite, cf. [16]) allows to define the following collection of subsets in the unit cube.

Proposition 5.2 *There exist a collection of subsets $\mathcal{P}_\kappa, (\kappa \in \mathcal{K})$ in the unit cube*

$$\mathcal{U} = \{(x_1, ..., x_r)|0 \le x_i \le 1\}$$

in $\mathbf{R}^r$ and a collection of affine hyperplanes $l_i(x_1, ..., x_r) = \alpha_i$ such that each $\mathcal{P}_\kappa$ is the boundary of the polytope consisting of solutions to the system of inequalities:

$$l_i(x_1, ..., x_r) \ge \alpha_i$$

and such that

$$(\frac{j_1+1}{m_1}, ..., \frac{j_r+1}{m_r}) \in \mathcal{U} \quad (26)$$

belongs to $\mathcal{P}_\kappa$ if and only if

$$\dim\mathcal{A}(\log E)(j_1,...,j_r|m_1,....,m_r)/\mathcal{A}(j_1,...,j_r|m_1,....,m_r) \geq 1 \quad (27)$$

Moreover

$$\dim\mathcal{A}_l(\log E)(j_1,...,j_r|m_1,....,m_r)/\mathcal{A}_{l-1}(j_1,...,j_r|m_1,....,m_r) \geq k \quad (28)$$

if only if (26) belongs to a collection of certain faces $\mathcal{P}^{k,l}_{\kappa,\iota}(\iota \in \mathcal{I}^{k,l})$ of polytopes $\mathcal{P}_\kappa$.

The proof is completely analogous to the proof of the proposition 2.2 in [17] or proposition 2.3.1 in [16] and details will be omitted. We shall record, however, the equations for the affine hyperplanes mentioned in proposition 5.2. Let $\pi : \tilde{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ be a resolution of singularities of $\bigcup D_i$ i.e. the irreducible components E_j of the exceptional set of π and the proper π-transforms of D_i's form a normal crossing divisor. Let, as in (12), f_i be the defining equations of D_i. Let $\mathrm{ord}_{E_j}\pi^*(f_i) = a_{i,j}$ and $\mathrm{ord}_{E_j}\pi^*(dx_1 \wedge ... \wedge dx_{n+1}) = c_j$. Finally for

$$\phi \in \mathcal{A}(\log E)(j_1,...,j_r|m_1,....,m_r)/\mathcal{A}(j_1,...,j_r|m_1,....,m_r)$$

for some array $(j_1,...,j_r|m_1,...,m_r)$ let $e_j(\phi) = \mathrm{ord}_{E_j}\pi^*(\phi)$. Then (cf. [17]):

$$a_{1,j}(1-x_1) + ...a_{r,j}(1-x_r) = e_j(\phi) + c_j + 1 \quad (29)$$

is equation of the hyperplane containing a face of a polytope of quasiadjunction. Moreover, $\phi \in \mathcal{A}(\log E)(j_1,...,j_r|m_1,....,m_r)$ if and only if:

$$a_{1,j}(1-\frac{j_1}{m_1}) + ... + a_{r,j}(1-\frac{j_r}{m_r}) \leq e_j(\phi) + c_j + 1 \quad (30)$$

In the case of ideal of quasiadjunction we have: $\phi \in \mathcal{A}(j_1,...,j_r|m_1,....,m_r)$ if and only if:

$$a_{1,j}(1-\frac{j_1}{m_1}) + ... + a_{r,j}(1-\frac{j_r}{m_r}) < e_j(\phi) + c_j + 1 \quad (31)$$

Hence the existence of ϕ satisfying (27) is equivalent to $(\frac{j_1+1}{m_1},...,\frac{j_r+1}{m_r})$ being solution to all inequalities (30) with at least one of them being solution to equality (29). Moreover, for $\phi \in \mathcal{A}(\log E)(j_1,...,j_r|m_1,....,m_r)$ the corresponding form $\omega_\phi \in \Omega_{\tilde{V}}(logE)$ has weight at most l if and only if $(\frac{j_1+1}{m_1},...,\frac{j_r+1}{m_r})$ satisfies at most l equalities corresponding to components of exceptional divisor having l-fold intersections. This yields the second claim of the propositon.

Definition 5.3 *The polytopes $\mathcal{P}_\kappa$ existence of which is asserted in the Proposition 5.2 are called the polytopes of quasiadjunction.*

One has the following description of the smallest element in the Hodge filtration of the cohomology of a link of singularity (12) in terms of polytopes of quasiadjunciton and their faces.

Theorem 5.4 *A character of $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1\le i\le r} D_i)$ acting on $W_l(F^n H^n(V_{m_1,....,m_r}))$ via the action of the Galois group has the eigenspace of dimension at least k if and only if it has the form:*

$$(exp 2\pi\sqrt{-1}a_1, ..., exp 2\pi\sqrt{-1}a_r)$$

where $(a_1, ..., a_r)$ belongs to one of the faces $\mathcal{P}^{k,l}_{\kappa,\iota}$ of a polytope $\mathcal{P}_\kappa$ of quasiadjunction of $\bigcup D_i$.

The proof is similar to the proof in the case of reducible curves given in [17] and will be omitted.

Definition 5.5 *Zariski closure of the image of a face of quasiadjunction $\mathcal{P}^{k,l}_{\kappa,\iota}$ will be called a principal component of the characteristic variety V_k. Polynomial invariant of X is a generator of the divisorial hull of the first Fitting ideal of $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes \mathbf{C}$.*

For $k > 1$ the varieties V_k may have codimension greater than one. We don't know if the union of principal components is equal to the characteristic variety V_k of $\pi_n(\partial B_\epsilon - \partial B_\epsilon \cap X) \otimes \mathbf{C}$. However we shall conjecture the following (i.e. that the situation is similar to the one in the case of curves (cf. [17])):

Conjecture 5.6 *Characteristic variety is a union of translated subtori of* $\mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$ *with each translations given by a point of finite order.*

This is the true in irreducible case since the roots of the characteristic polynomial of the monodromy are roots of unity. Conjecture is also true in the case of curves essentially since algebraic links are iterated torus links (cf. [17]).

6 Examples and Remarks

Example 6.1 *Let $L_i(x_0, ..., x_n) = 0 \quad i = 1, ...r$ be generic linear forms. Then the subset $\bigcup D_i$ in $\mathbf{C}^{n+1}$ given by:*

$$L_1 \cdot ... \cdot L_r = 0 \tag{32}$$

is the divisor with normal crossings except for the origin. The complement to divisor (32) is $\mathbf{C}^*$-fibration over the complement in $\mathbf{P}^n$ to the divisor $\bigcup P(D_i)$ given by the same equation (32) in homogeneous coordinates of $\mathbf{P}^n$. Hence

$$\pi_n(\mathbf{C}^{n+1} - \bigcup D_i) = \pi_n(\mathbf{P}^n - \bigcup PD_i) \tag{33}$$

Here the left hand side is considered as the module over $\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$ while the right hand side is the $\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1...t_r - 1)$-module.

The complement $\mathbf{P}^n - \bigcup PD_i$ latter has the homotopy type of the n-skeleton of the torus $(S^1)^{r-1}$ (cf. [7]) and hence is a $(\mathbf{Z}^r, n)$ complex (cf. section 2.3). The *universal abelian cover has cell structure obtained by removing from* $\mathbf{R}^r$ *the* $\mathbf{Z}^r$*-orbits of interiors of faces of dimension greater than* n *in the unit cube. This shows that the chain complex of the universal abelian cover is the truncated Koszul complex corresponding to the system of parameters* $(t_-1, ..., t_r - 1)$ *over the ring of Laurent polynomials* $R = \mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1...t_r - 1)$*:*

$$0 \to \Lambda^n(R^r) \to \Lambda^{n-1}(R^r) \to ....\Lambda^1(R) \to (t_1 - 1, ..., t_r - 1)R \to 0 \qquad (34)$$

This yields that

$$\pi_n(S^{2n+1} - \bigcup D_i) = H_n(S^{2n+1} - \bigcup D_i) = \mathrm{Ker}\Lambda^n(R^r) \to \Lambda^{n-1}(R^r) \qquad (35)$$

Hence we obtain the presentation for the homotopy group:

$$\Lambda^{n+1}(\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1..., t_r - 1)^r) \to \Lambda^n(\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1..., t_r - 1)^r) \to$$
$$\pi_n(\mathbf{C}^{n+1} - \bigcup D_i) \to 0 \qquad (36)$$

In particular the support of $\pi_n(\mathbf{C}^{n+1} - \bigcup D_i)$ *is given by* $t_1....t_r = 1$

Example 6.2 *More generally, consider the cone over union of non singular hypersurfaces* $D_{i_0}, ..., D_{i_r}$ *in* $\mathbf{P}^n$ *having the degrees* $\mathrm{deg} D_{i_k} = d_k$ *and forming there a divisor with normal crossings. Indeed,* $\mathbf{P}^n$ *is* $(\mathbf{Z}^r/(d_1, ..., d_r), n)$ *complex such that passing to finite cover yields the* K*-complex over a finite abelian cover which by [24] is homotopy equivalent to the wedge of spheres* S^n*. Hence the support of the homotopy group is given by*

$$t_1^{d_1} \cdot ... \cdot t_r^{d_r} - 1 = 0 \qquad (37)$$

This result can be obtained using the method of previous section. Indeed, the cone singularity can be resolved by single blow up which yields the polytope of quasi-adjunction with the faces:

$$d_1 x_1 + ... d_r x_r \geq l \qquad (38)$$

Indeed if f_i *are the defining equations of* D_i *and* $\phi \in \mathcal{M}^{r-l-1}$ *we obtain* $\mathrm{mult}_E \pi^*(f_i) = d_i$, $\mathrm{mult}_E \phi = r - l - 1$, $\mathrm{mult} \pi^* dx_1 \wedge ... \wedge dx_{n+1} = n$*. It follows that the ideals of quasiadjunction are the powers of the maximal ideal and hence the claim (38). The exponential map takes the union of the faces of quasiadjunction (38) into the set having as its Zariski closure (37).*

Remark 6.3 Multiplier ideals and log-canonical thresholds in terms of ideals of quasiadjunction. *Recall that for a divisor* D *in* X *one defines the multiplier ideal as* $\mathcal{J}(D) = f_*(\mathcal{O}(K_Y - f^*(K_X) - \lfloor E \rfloor))$ *where* $f : Y \to X$ *is an embedded resolution of pair* (X, D)*. Consider divisor* $D_{\gamma_1,....,\gamma_r} = f_1^{\gamma_1} \cdot ... \cdot f_r^{\gamma_r}$ *on* $\mathbf{C}^n$*. We have* $\mathcal{J}(D_{\gamma_1,...,\gamma_r}) = \mathcal{A}(j_1, ..., j_r | m_1, ..., m_r)$ *where* $\gamma_i = 1 - \frac{j_i+1}{m_i}$

The face of quasiadjunction corresponding to $\phi = 1$ *can be described as the closure of the set of points* $(\gamma_1, ..., \gamma_r)$ *in the unit cube such that the divisor* $\sum_i (1 - \gamma_i) D_i$ *is log canonical i.e. the face is formed by log-canonical thresholds.*

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