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HODGE GENERA AND CHARACTERISTIC CLASSES OF COMPLEX ALGEBRAIC VARIETIES

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Abstract. We announce Hodge theoretic formulae of Atiyah-Meyer type for genera and characteristic classes of complex algebraic varieties. Our results are formulated in terms of the generalized (motivic) Hirzebruch characteristic classes, and the arguments used in the proofs rely in an essential way on Saito's theory of algebraic mixed Hodge modules.

1. Introduction

In the mid 1950's, Chern, Hirzebruch and Serre [5] showed that if $F \hookrightarrow E \to B$ is a fiber bundle of closed, coherently oriented, topological manifolds such that $\pi_1(B)$ acts trivially on the cohomology of F, then the signatures of the spaces involved satisfy a simple multiplicative relation:

$$\sigma(E) = \sigma(F) \cdot \sigma(B). \tag{1}$$

A decade later, Kodaira, Atiyah, and respectively Hirzebruch observed that without the assumption on the (monodromy) action of $\pi_1(B)$, the multiplicativity relation fails. In the case when E is the total space of a differentiable fiber bundle of compact oriented manifolds, so that both B and F are even-dimensional, Atiyah [1] obtained a formula for $\sigma(E)$ involving a contribution from the $\pi_1(B)$-action of $H^*(F)$. Let $k = \dim F/2$. Then the flat bundle $\mathcal{V}$ over B with fibers $H^k(F_x; \mathbb{R})$ $(x \in B)$ has a K-theory signature, $[\mathcal{V}]_K \in KO(B)$ for k even (resp. $KU(B)$ for k odd), and the *Atiyah signature theorem* asserts that

$$\sigma(E) = \langle ch^*_{(2)}([\mathcal{V}]_K) \cup L^*(B), [B] \rangle, \tag{2}$$

where $ch^*_{(2)}$ is a twisted Chern character obtained by composing with the second Adams operation and $L^*(B)$ is the total Hirzebruch L-polynomial of B. Meyer [12] extended Atiyah's formula to the case of twisted signatures of closed manifolds endowed with Poincaré local systems (i.e., local systems with duality), not

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necessarily arising from a fibre bundle projection. More precisely, if B is a closed, oriented, smooth manifold of even dimension and $\mathcal{L}$ is a local system equipped with a nondegenerate (anti-)symmetric bilinear pairing $\mathcal{L} \otimes \mathcal{L} \to \mathbb{R}_B$, then the *twisted signature* $\sigma(B;\mathcal{L})$, that is, the signature of the nondegenerate form on the sheaf cohomology group $H^{dim(B)/2}(B;\mathcal{L})$, can be computed by the formula:

$$\sigma(B;\mathcal{L}) = \langle ch^*_{(2)}([\mathcal{L}]_K) \cup L^*(B), [B] \rangle, \tag{3}$$

where $[\mathcal{L}]_K$ is the K-theory signature of $\mathcal{L}$. Extensions of Meyer's formula to the singular setting were studied in [2] by means of intersection homology theory.

In complex algebraic geometry, (twisted) signatures are just special cases of more general Hodge-type invariants. The purpose of this note is to announce Hodge theoretic Atiyah-Meyer type formulae for genera and characteristic classes of complex algebraic varieties. The main results are formulated in terms of the generalized Hirzebruch characteristic classes ([4]) and the arguments used in the proofs rely in an essential way on Saito's deep theory of algebraic mixed Hodge modules ([13]). Complete details including proofs of these results can be found in [8], but see also the survey [11].

2. Hirzebruch characteristic classes

For the convenience of the reader, we will briefly recall the definition of the Hirzebruch characteristic classes in the singular setting (cf. [4]).

Let Z be a complex algebraic variety, and $MHM(Z)$ the abelian category of Saito's algebraic mixed Hodge modules on Z (cf. [13]). Recall that if Z is a point, then $MHM(pt)$ coincides with the category of graded polarizable rational mixed Hodge structures. If Z is smooth, an admissible variation of mixed Hodge structures (e.g., a geometric variation or a pure polarizable variation) defined on all of Z, with underlying local system $\mathcal{L}$, yields an element $\mathcal{L}^H[\mathrm{dim} Z] \in MHM(Z)$ with underlying perverse sheaf $\mathcal{L}[\mathrm{dim} Z]$.

For any variety Z, and for any $p \in \mathbb{Z}$, Saito [13] constructed a functor of triangulated categories

$$gr_p^F DR : D^b MHM(Z) \to D^b_{coh}(Z) \tag{4}$$

commuting with proper push-down, where $D^b_{coh}(Z)$ is the bounded derived category of sheaves of $\mathcal{O}_Z$-modules with coherent cohomology sheaves. If $\mathbb{Q}^H_Z \in D^b MHM(Z)$ denotes the constant Hodge module on Z and if Z is smooth and pure dimensional, then $gr^F_{-p} DR(\mathbb{Q}^H_Z) \simeq \Omega^p_Z[-p]$. The transformations $gr^F_p DR(M)$ induce functors on the level of Grothendieck groups. Therefore, if $G_0(Z) \simeq K_0(D^b_{coh}(Z))$ denotes the Grothendieck group of coherent sheaves on Z, we get a group homomorphism

$$\begin{aligned} gr^F_{-*} DR &: K_0(MHM(Z)) \to G_0(Z) \otimes \mathbb{Z}[y, y^{-1}], \\ [M] &\mapsto \sum_{i,p} (-1)^i [\mathcal{H}^i(gr^F_{-p} DR(M))] \cdot (-y)^p. \end{aligned} \tag{5}$$

We let $td_{(1+y)}$ be the natural transformation

$$\begin{aligned} td_{(1+y)} &: G_0(Z) \otimes \mathbb{Z}[y, y^{-1}] \to H^{BM}_{2*}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \\ [\mathcal{F}] &\mapsto \sum_{k \geq 0} td_k([\mathcal{F}]) \cdot (1+y)^{-k}, \end{aligned} \tag{6}$$

where H_*^{BM} stands for Borel-Moore homology, and td_k is the degree k component of the Todd class transformation td_* of Baum-Fulton-MacPherson [3], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$.

Definition 2.1. The motivic Hirzebruch class transformation MHT_y is defined by the composition (cf. [4]):

(7)
$$MHT_y := td_{(1+y)} \circ gr^F_{-*}DR : K_0(MHM(Z)) \to H_{2*}^{BM}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}].$$

The motivic Hirzebruch class $T_{y_*}(Z)$ of a complex algebraic variety Z is defined by

$$T_{y_*}(Z) := MHT_y([\mathbb{Q}_Z^H]). \tag{8}$$

Similarly, if Z is a n-dimensional complex algebraic manifold and $\mathcal{L}$ is a local coefficient system on all of Z underlying an admissible variation of mixed Hodge structures, we define the twisted Hirzebruch characteristic class by

$$T_{y_*}(Z; \mathcal{L}) := MHT_y([\mathcal{L}^H]), \tag{9}$$

where $\mathcal{L}^H[n]$ is the (smooth) mixed Hodge module on Z whose underlying perverse sheaf is $\mathcal{L}[n]$.

By definition, the transformation MHT_y commutes with proper push-forwards. Moreover, the following normalization property holds (cf. [4]): if Z is smooth and pure dimensional, then

$$T_{y_*}(Z) = T_y^*(T_Z) \cap [Z],$$

where $T_y^*(T_Z)$ is the Hirzebruch (or generalized Todd) cohomology class of Z appearing in the generalized Hirzebruch-Riemann-Roch theorem (cf. [9]).

For a complete (possibly singular) variety Z with $k : Z \to pt$ the constant map to a point, $k_* T_{y_*}(Z)$ is the *Hodge genus*

$$\chi_y(Z) := \sum_{i,p} (-1)^i dim_{\mathbb{C}}(gr_F^p H^i(Z; \mathbb{C})) \cdot (-y)^p.$$

For Z smooth, $k_* T_{y_*}(Z; \mathcal{L})$ is the *twisted* χ_y*-genus* $\chi_y(Z; \mathcal{L})$ defined in a similar manner (cf. [8]).

It was shown in [4] that for any variety Z,

$$T_{-1_*}(Z) = c_*(Z) \otimes \mathbb{Q}$$

is the total (rational) Chern class of MacPherson [10]. Moreover, for a variety Z with at worst Du Bois singularities (e.g., toric varieties), we have that

$$T_{0*}(Z) = td_*(Z) := td_*([\mathcal{O}_Z]) ,$$

for td_* the Baum-Fulton-MacPherson transformation [3]. If Z is smooth and projective, then $T_1^*(T_Z)$ is the total Hirzebruch L-polynomial of Z, so in this case $\chi_1(Z) = \sigma(Z)$.

3. Statement of results and applications

The central result of this note is the following:

Theorem 3.1. *([8]) Let Z be a complex algebraic manifold of pure dimension n, and $\mathcal{L}$ an admissible variation of mixed Hodge structures on Z with associated flat bundle with Hodge filtration $(\mathcal{V}, \mathcal{F}^\bullet)$. Then*

$$T_{y_*}(Z; \mathcal{L}) = \left(ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cup T_y^*(T_Z) \right) \cap [Z] = ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cap T_{y_*}(Z), \tag{10}$$

where

$$\chi_y(\mathcal{V}) := \sum_p [Gr^p_{\mathcal{F}}\mathcal{V}] \cdot (-y)^p \in K^0(Z)[y, y^{-1}]$$

is the K-theory χ_y-characteristic of $\mathcal{V}$ (with $K^0(Z)$ the Grothendieck group of algebraic vector bundles on Z), and $ch^*_{(1+y)}$ is a twisted Chern character whose value on a complex vector bundle ξ is

$$ch^*_{(1+y)}(\xi) = \sum_{j=1}^{rk\xi} e^{\beta_j(1+y)},$$

for $\{\beta_j\}_j$ the Chern roots of ξ.

The proof uses Saito's description of the smooth mixed Hodge module $\mathcal{L}^H[n]$ associated to $\mathcal{L}$, together with the classical Grothendieck-Riemann-Roch theorem for the Todd class transformation. For complete details, see [8, 11].

Corollary 3.2. *If the variety Z in Theorem 3.1 is also complete, then by pushing down to a point, we obtain a Hodge theoretic Meyer-type formula for the twisted χ_y-genus:*

$$\chi_y(Z; \mathcal{L}) = \langle ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cup T^*_y(T_Z), [Z] \rangle. \tag{11}$$

Remark 3.3. If Z is smooth and projective and the local system $\mathcal{L}$ underlies a polarizable variation of pure Hodge structures on Z, then the choice of such a polarization defines a duality structure on $\mathcal{L}$, i.e. it makes it into a Poincaré local system, and it is easy to see that in this case our formula (11) specializes for $y = 1$ to Meyer's signature formula (3).

Without the compactness assumption on Z, we can obtain directly a formula for $\chi_y(Z; \mathcal{L})$ by noting that the twisted logarithmic de Rham complex $\Omega^\bullet_X(\log D) \otimes \bar{\mathcal{V}}$ associated to the Deligne extension of $\mathcal{L}$ on a good compactification (X, D) of Z (with X smooth and compact, and D a simple normal crossing divisor), with its Hodge filtration induced by Griffiths' transversality, is part of a cohomological mixed Hodge complex that calculates $H^*(Z; \mathcal{L} \otimes \mathbb{C})$. In the above notation, we obtain (cf. [8]):

$$\chi_y(Z; \mathcal{L}) = \langle ch^*(\chi_y(\bar{\mathcal{V}})) \cup ch^*\left(\lambda_y(\Omega^1_X(\log D))\right) \cup td^*(X), [X] \rangle, \tag{12}$$

where $\langle , \rangle$ denotes the Kronecker pairing on X, $td^*(X)$ is the total Todd class of X (in cohomology),

$$\lambda_y\left(\Omega^1_X(\log D)\right) := \sum_i \Omega^i_X(\log D) \cdot y^i, \quad \text{and} \quad \chi_y(\bar{\mathcal{V}}) = \sum_p \left[Gr^p_{\bar{\mathcal{F}}}\bar{\mathcal{V}}\right] \cdot (-y)^p,$$

with $(\bar{\mathcal{V}}, \bar{\mathcal{F}}^\bullet)$ the unique extension of $(\mathcal{V}, \mathcal{F}^\bullet)$ to X corresponding to the Deligne extension of $\mathcal{L}$.

In the relative setting, as an application of Theorem 3.1, we obtain the following Atiyah-type result:

Theorem 3.4. *([8]) Let $f : E \to B$ be a projective morphism of complex algebraic varieties, with B smooth and connected, such that the sheaves $R^s f_* \mathbb{Q}_E$, $s \in \mathbb{Z}$, are locally constant on B, e.g., f is a locally trivial topological fibration. Then*

$$f_* T_{y_*}(E) = ch^*_{(1+y)}(\chi_y(f)) \cap T_{y_*}(B), \tag{13}$$

where

$$\chi_y(f) := \sum_{i,p} (-1)^i [Gr^p_{\mathcal{F}} \mathcal{H}_i] \cdot (-y)^p \in K^0(B)[y]$$

is the K*-theory* χ_y*-characteristic of* f*, for* $\mathcal{H}_i$ *the flat bundle with connection* $\bigtriangledown_i : \mathcal{H}_i \to \mathcal{H}_i \otimes_{\mathcal{O}_B} \Omega^1_B$*, whose sheaf of horizontal sections is* $R^i f_* \mathbb{C}_E$*.*

If, moreover, B *is complete, then by pushing down to a point, we obtain:*

$$\chi_y(E) = \langle ch^*_{(1+y)}(\chi_y(f)) \cup T^*_y(T_B), [B] \rangle. \tag{14}$$

Remark 3.5. If the action of $\pi_1(B)$ on $H^*(F)$ is *trivial* (e.g., $\pi_1(B) = 0$), then

$$ch^*_{(1+y)}(\chi_y(f)) = \chi_y(F) \in H^0(B; \mathbb{Q}),$$

and formula (14) yields in this case the following Hodge-theoretic analogue of the Chern-Hirzebruch-Serre result (1):

$$\chi_y(E) = \chi_y(F) \cdot \chi_y(B).$$

If f is a smooth proper map of smooth projective varieties, formula (14) allows us to express $\chi_y(E)$ in terms of *higher* χ_y*-genera* of Novikov type, so that if the action of $\pi_1(B)$ on $H^*(F)$ is trivial or, more generally, if it preserves the Hodge filtration, this expression reduces to ordinary multiplicativity. Let $D_{\pm,\eta}$ be the Griffiths classifying space of pure Hodge structures on a vector space V, polarized via a $\pm$-symmetric bilinear form, and corresponding to a partition $\eta : \dim V = \sum h^{p,q}$. A subgroup Γ of finite index in the total monodromy group, i.e., in the image of the map $\pi_1(B) \to \prod_k \mathrm{Aut} H^k(F)$, acts freely on $\prod_k D_{(-1)^k, \eta_k(F)}$, for the partition $\eta_k(F)$ of $\mathrm{rank} H^k(F)$ given by the Hodge decomposition. Let

$$\pi : B \to (\prod_k D_{(-1)^k, \eta_k})/\Gamma$$

be the corresponding period map.

Definition 3.6. To each $\alpha \in H^*((\prod_k D_{(-1)^k, \eta_k})/\Gamma)$ we associate a higher χ_y-genus by

$$\chi_y^{[\alpha]}(B, \pi) := \langle \pi^*(\alpha) \cup T^*_y(T_B), [B] \rangle. \tag{15}$$

For some η the period domain $\prod_k D_{(-1)^k, \eta_k}$ is contractible, and if this is the case then, for $y = 1$ one obtains the Novikov higher signature. As a consequence of (14) we obtain:

Theorem 3.7. *([8]) There exist universal classes* $[\alpha^{p,q}] \in H^*((\prod_k D_{(-1)^k, \eta_k})/\Gamma)$ *such that*

$$\chi_y(E) = \sum_{p,q} (-1)^q \chi_y^{[\alpha^{p,q}]}(B, \pi) \cdot y^p. \tag{16}$$

Remark 3.8. If the monodromy action is *trivial*, then

$$\chi_y^{[\alpha^{p,q}]}(B, \pi) = h^{p,q}(F) \cdot \chi_y(B),$$

and (16) again yields the multiplicativity relation $\chi_y(E) = \chi_y(F) \cdot \chi_y(B)$. In the special case $y = 1$, formula (16) generalizes Atiyah's formula (2) for the signature of fiber bundles.

Theorem 3.1 can also be used for computing invariants arising from intersection homology. Let $IC_Z^H \in MHM(Z)$ be the intersection homology Hodge module on a pure-dimensional variety Z. Define new Hirzebruch-type characteristic classes by

$$IT_{y_*}(Z) := MHT_y(\big[IC_Z^H[-\mathrm{dim}Z]\big]). \tag{17}$$

If Z is complete, then by pushing down to a point we recover $I\chi_y(Z)$, a polynomial in the Hodge numbers of $IH^*(Z;\mathbb{Q})$ defined by analogy with $\chi_y(Z)$. (Note: If Z is projective, then $I\chi_1(Z)$ is the Goresky-MacPherson signature of Z.) Standard calculus in the Grothendieck group $K_0(MHM(B))$ yields the following:

Theorem 3.9. *Let $f : E \to B$ be a proper map, with E pure-dimensional and B smooth and connected so that f is a locally trivial topological fibration with fiber F. Then*

$$f_*IT_{y_*}(E) = \sum_i (-1)^{dimF+i}\; T_{y_*}(B;\mathcal{L}_i) \in K_0(MHM(B)), \tag{18}$$

where $\mathcal{L}_i$ is the admissible variation of mixed Hodge structures on B with stalk $IH^{dimF+i}(F;\mathbb{Q})$, and associated mixed Hodge module $H^i(f_*IC_E^H) \in MHM(B)$.

Each term in the sum of (18) can be further computed by formula (10). In particular, if $\pi_1(B) = 0$, then

$$f_*IT_{y_*}(E) = I\chi_y(F) \cdot T_{y_*}(B).$$

Similar considerations apply to genera. This is a very special case of the *stratified multiplicative property* studied in detail in [6, 7].

Meyer type formulae for Hodge-theoretic invariants defined by means of intersection homology will be studied elsewhere (but see [11] for a preliminary result).

References

[1] M. F. Atiyah, *The signature of fiber bundles*, in "Global Analysis (Papers in Honor of K. Kodaira)," 73–84, Univ. Tokyo Press, Tokyo, 1969. MR 0254864 (40 8071)

[2] M. Banagl, S. E. Cappell and J. L. Shaneson, *Computing twisted signatures and L-classes of stratified spaces*, Math. Ann., **326** (2003), 589-623. MR 1992279 (2004i:32047)

[3] P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch for singular varieties*, Publ. Math. I.H.E.S., Publ. Math. No. 45, (1975), 101-145. MR 0412190 (54 317)

[4] J. P. Brasselet, J. Schürmann and S. Yokura, *Hirzebruch classes and motivic Chern classes of singular spaces*. arXiv:math/0503492

[5] S. S. Chern, F. Hirzebruch and J.-P. Serre, *On the index of a fibered manifold*, Proc. Amer. Math. Soc., **8** (1957), 587–596. MR 0087943 (19,441c)

[6] S. E. Cappell, L. G. Maxim and J. L. Shaneson, *Euler characteristics of algebraic varieties*, Comm. Pure Appl. Math. 61 (2008), no. 3, 409-421. arXiv:math/0606654

[7] S. E. Cappell, L. G. Maxim and J. L. Shaneson, *Hodge genera of algebraic varieties, I*, Comm. Pure Appl. Math. 61 (2008), no. 3, 422-449. arXiv:math/0606655

[8] S. E. Cappell, A. Libgober, L. G. Maxim and J. L. Shaneson, *Hodge genera of algebraic varieties, II*. arXiv:math/0702380

[9] F. Hirzebruch, *Topological methods in algebraic geometry*, Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131, Springer-Verlag New York, Inc., New York, 1966. MR 0202713 (34 2573)

[10] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2), **100** (1974), 423–432. MR 0361141 (50 13587)

[11] L. G. Maxim and J. Schürmann, *Hodge-theoretic Atiyah-Meyer formulae and the stratified multiplicative property*. arXiv:0707.0129

[12] W. Meyer, *Die Signatur von lokalen Koeffizientensystemen und Faserbündeln*, Bonner Mathematische Schriften 53, 1972. MR 0305402 (46 4532)

[13] M. Saito, *Mixed Hodge Modules*, Publ. RIMS, **26** (1990), 221–333. MR 1047415 (91m:14014)

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