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Non vanishing loci of Hodge numbers of local systems

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Abstract. We show that closures of families of unitary local systems on quasiprojective varieties for which the dimension of a graded component of Hodge filtration has a constant value can be identified with a finite union of polytopes. We also present a local version of this theorem. This yields the "Hodge decomposition" of the set of unitary local systems with a non-vanishing cohomology extending Hodge decomposition of characteristic varieties of links of plane curves studied by the author earlier. We consider a twisted version of the characteristic varieties generalizing the twisted Alexander polynomials. Several explicit calculations for complements to arrangements are made.

1. Introduction

Let $\rho : \pi_1(X) \to U_N$ be a unitary N-dimensional representation of the fundamental group of a non-singular quasiprojective variety X or, in other words, a unitary rank N local system. It is well known (cf. [1,31]) that with this data one can associate the cohomology groups $H^i(X, \rho)$ and that the cohomology of a unitary local system V_ρ supports canonical mixed Hodge structure, part of which is the Hodge filtration $\ldots F^{p+1}H^n(V_\rho) \subseteq F^p H^n(V_\rho) \subseteq \cdots$.

One of the purposes of this paper is to study the structure of the family $S^{n,p}_{\rho,l}$ of local systems $V_\rho \otimes L_\chi$ with a fixed ρ corresponding to unitary characters $\chi : \pi_1(X) \to \mathbb{C}^*$ for which the dimension of graded components $Gr^p_F = F^p/F^{p+1}$ of the graded space associated with the Hodge filtration on cohomology satisfies $\dim Gr^p_F H^n(X, V_{\rho\otimes\chi}) \geq l$. This structure, it turns out, is rather different in projective and quasiprojective cases and the difference will be explained shortly in this introduction.

Secondly, we derive a *local* version of results on the structure of the sets of local systems with fixed dimension of cohomology and also the sets of local systems with fixed dimension of associated graded for Hodge filtration groups (a construction of a mixed Hodge structure in this case, apparently absent in the literature and therefore we give one in Sect. 3). The local and global situations are closely related and in previous paper [24] we showed how such type of local data can be used to

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derive information about sets of local systems with fixed dimension of cohomology in some quasiprojective cases. For simplicity, we assume in this paper that there are no non-trivial rank one local systems on a non-singular compactification $\bar{X}$ of X i.e. $H^1(\bar{X}, \mathbb{C}^*) = 0$.

Let us first describe our local results (which global counterparts are in [1,9,30]) and consider a pair $(\mathcal{X}, \mathcal{D})$ consisting of a germ of a complex space $\mathcal{X}$ having as its link a simply connected manifold and a divisor $\mathcal{D} = \bigcup D_i$ on $\mathcal{X}$ which is a union of r irreducible components. In this case the family of the rank one local systems on $\mathcal{X} - \mathcal{D}$ is parameterized by a r-dimensional affine torus: $H^1(\mathcal{X} - \mathcal{D}, \mathbb{C}^*) = \mathbb{C}^{*r}$ (cf. [11]). In Sect. 3 we show the following:

Theorem 1.1. *The collection of local systems*

$$S_l^n = \{\chi \in H^1(\mathcal{X} - \mathcal{D}, \mathbb{C}^*) | \dim H^n(\mathcal{X} - \mathcal{D}, L_\chi) \geq l\}$$

is a finite union of translated by points of finite order sub-tori of $H^1(\mathcal{X} - \mathcal{D}, \mathbb{C}^*)$.[1]

This is a local counterpart of the translated subgroup property of the families of rank one local systems with jumping cohomology on quasiprojective manifolds (cf. [1]). Note that $\mathcal{X} - \mathcal{D}$ is Stein but not quasiprojective and so, except for occasional cases when it is homotopy equivalent to a quasiprojective manifold, this does not follow immediately from [1]. Rather, we use a Mayer-Vietoris spectral sequence for the union of tori bundles on quasiprojective manifolds, which, as we show, degenerates in term E_2 and this together with [1] yields the claim. As in [23] (cf. also [12]) we call S_l^n the characteristic variety of $(\mathcal{X}, \mathcal{D})$.

The Theorem 1.1 was conjectured in [23] where we also conjectured a procedure for calculation of the translated sub-tori from the Theorem 1.1 in the case when $\mathcal{X}$ is non-singular and $\mathcal{D}$ is an isolated non-normal crossing divisor (see Sect. 4 for discussion of this case). The idea in [23] was to calculate the dimensions $h_\chi^{p,q,n}$ of the vector spaces defined in terms of the mixed Hodge structure on the cohomology of a local system L_χ, using their relation to the homology of abelian covers and then to apply the results to obtain the translated sub-tori in the Theorem 1.1 as the Zariski closure in $H^1(\mathcal{X} - \mathcal{D}, \mathbb{C}^*)$ of the image of the jumping loci of the Hodge group under the exponential map. A special case of this approach is the algebro-geometric calculation of the zeros of multivariable Alexander polynomials carried out in [22]. In Sect. 3 we address the issue of the existence of a Mixed Hodge structure on the cohomology of local systems on $\mathcal{X} - \mathcal{D}$, which admit logarithmic extension with eigenvalues belonging to $[0, 1)$. This is a local version of the result of Timmescheidt [31] (also [1]) and in the case of trivial local systems reduces to construction in [13]. The Hodge numbers of local systems corresponding to χ's having finite order in $H^1(\mathcal{X} - \mathcal{D}, \mathbb{C}^*)$ obtained from the construction in Sect. 3 coincide with the Hodge numbers $h_\chi^{p,q,n}$ which are the dimensions of χ-eigenspaces of the covering group acting on the Deligne's Hodge groups of finite abelian covers of $\mathcal{X} - \mathcal{D}$. This allows us in Sect. 4 to use the information obtained in Sect. 3 to study the Hodge decomposition of abelian covers corresponding to isolated non normal crossings.

[1] The assumption of simply connectedness of the link of $\mathcal{X}$ is used only to simplify the exposition and can be dropped after some modification in the statement. Cf. work [3] for such modification in related context.

Now let us explain the difference between the loci with fixed dimension of $Gr_F^p H^n$ in projective and quasiprojective cases. On compact Kahler manifolds, the families of local systems can be identified with the families of topologically trivial[2] holomorphic line bundles (cf. [30] or [3]) and those with jumping Hodge numbers are the unions of translated abelian subvarieties of the Picard groups (cf. [9]). As an example in quasiprojective case, let us look at the cohomology of local systems on $\mathbb{P}^1 - (0, 1, \infty)$. The unitary local systems are parameterized by the maximal compact subgroup in $\mathbb{C}^* \times \mathbb{C}^*$. One can show that the cohomology $H^1(\mathbb{P}^1 - (0, 1, \infty), L_\chi)$ of a local system corresponding to $\chi : \pi_1(\mathbb{P}^1 - (0, 1, \infty)) \to \mathbb{C}^*$ and having a finite order n is isomorphic to the χ-eigenspace $H^1(X_{n,n}, \mathbb{C})_\chi \subset H^1(X_{n,n}, \mathbb{C})$ where $X_{n,n}$ is the abelian cover of $\mathbb{P}^1 - (0, 1, \infty)$ corresponding to the reduction modulo n:

$$H_1(\mathbb{P}^1 - (0, 1, \infty), \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$$

(in other words: $H^1(X_{n,n}, \mathbb{C})_\chi = \{a \in H^1(X_{n,n}, \mathbb{C}) | g \cdot a = \chi(g)a \text{ where } g \in \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}\}$). Such an abelian cover has as a model the complement to the fixed points of non identity elements in the group $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ acting on a Fermat curve:

$$F_n : \ x^n + y^n = z^n, \quad (x, y, z) \to (\zeta_n^a x, \zeta_n^b y, z)$$

(ζ_n is a primitive root of degree n, $a, b \in \mathbb{Z}/n\mathbb{Z}$). In particular, each $H^1(\mathbb{P}^1 - (0, 1, \infty), L_\chi)$ acquires the Hodge and weight filtrations induced from $H^1(X_{n,n}, \mathbb{C})_\chi$. We shall focus on the weight 1 component $Gr_1^W H^1(X_{n,n}) = H^1(F_n)$ and will look for those χ for which

$$Gr_F^1 Gr_1^W H^1(\mathbb{P}^1 - (0, 1, \infty), L_\chi) = Gr_F^1 Gr_1^W H^1(F_n, \mathbb{C})_\chi \neq 0$$

Now the cohomology classes $H^{1,0}(F_n, \mathbb{C})$ are represented by the residues of meromorphic 2-forms on $\mathbb{P}^2$ with poles *having order one* along F_n cf. [10]:

$$\frac{P(x, y, z) z dx \wedge dy}{x^n + y^n - z^n}$$

where $P(x, y, z)$ is a homogeneous polynomial of degree $n - 3$. Since the group $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ acts via multiplication of the coordinates (in particular generators act as follows: $g_x : (x, y, z) \to (\zeta_n x, y, z)$, $g_y : (x, y, z) \to (x, \zeta_n y, z)$) each non zero eigenspace of the character $\chi_{a,b}$ such that $\chi_{a,b}(g_a) = e^{\frac{2\pi i(a+1)}{n}}$, $\chi_{a,b}(g_b) = e^{\frac{2\pi i(b+1)}{n}}$ is generated by the monomial forms $\frac{x^a y^b z^{n-3-a-b} z dx \wedge dy}{x^n + y^n - z^n}$ $(0 \leq a + b \leq n - 3)$. On the universal cover of the space of unitary characters, they are represented in terms of $\bar{a} = a + 1, \bar{b} = b + 1$ by the triangle

$$\left\{ \frac{\bar{a}}{n}, \frac{\bar{b}}{n} | \bar{a} > 0 \ \bar{b} > 0, \bar{a} + \bar{b} < n \right\}$$ [3]

[2] One can also work with bundles for which the first Chern class is a torsion.

[3] One can check that weight two part $Gr_F^1 Gr_2^W H^1(X_{n,n})_\chi$ adds characters on the boundary of this triangle.

The situation described in this example is quite general. In Sect. 2.1 we prove the following:

Theorem 1.2. *Let X be a quasiprojective manifold without non-trivial rank one local systems on a non singular compactification, $\rho : \pi_1(X) \to U_N$ be a N-dimensional unitary representation ,* $\mathrm{Char}\pi_1(X)$ *be the torus of characters of the fundamental group and* $\mathrm{Char}^u\pi_1(X)$ *be the subgroup of unitary characters. Let $\mathcal{U}$ be the fundamental domain of $\pi_1(\mathrm{Char}^u\pi_1(X))$ acting on the universal cover* $\widetilde{\mathrm{Char}^u\pi_1(X)}$ *of the torus* $\mathrm{Char}^u\pi_1(X)$ *and $exp : \mathcal{U} \to \mathrm{Char}^u\pi_1(X)$ be the universal covering map.*[4] *Then*

$$\mathcal{S}^{n,p}_{\rho,l} = \{\chi \in \mathrm{Char}^u\pi_1(X) | \dim Gr^p_F H^n(V_\rho \otimes L_\chi) \geq l\} \tag{1}$$

is a finite union of polytopes in $\mathcal{U}$, i.e., the subsets, each of which is the set of solutions for a finite set of inequalities $L \geq 0$ where L is a linear function.

The argument is based on a study of Deligne extensions and yields an independent proof of quasiprojective version of Theorem 1.1 (i.e., the result of [1]).

We also prove that in the local situation of Theorem 1.1 the structure of the families local system with fixed Hodge numbers is similar to the one in quasiprojective case as described in Theorem 1.2. More precisely, we show the following:

Theorem 1.3. *Let $(\mathcal{X}, \mathcal{D})$ be a germ of a divisor with r irreducible components on a germ of a complex space as in Theorem 1.1. Let $X = \mathcal{X} - \mathcal{D}$ and $Char X = Char\pi_1(X)$ be the space of characters of the fundamental group of X.*

Let $Char^u\pi_1(X) \subset Char\pi_1(X)$ be the maximal torus of unitary characters and let $\mathcal{U} \subset \widetilde{\mathrm{Char}^u\pi_1(X)}$ be a fundamental domain of the covering group of the universal cover $\widetilde{\mathrm{Char}^u\pi_1(X)}$ of $Char^u\pi_1(X)$. For each pair (p, n) and $l \in \mathbb{N}$ there exists a collection of polytopes $\mathcal{S}^{p,n}_l \subset \mathcal{U}$ such that the image of $\mathcal{S}^{p,n}_l$ under the covering map $\mathcal{U} \to Char^u\pi_1(X)$ consists of the local systems L satisfying $rk F^p/F^{p+1}H^n(L) = l$. The image of the union $\cup_{p,(\ldots,l_i,\ldots),\sum l_p = l}\mathcal{S}^{n,p}_{l_i}$ under the covering map $\widetilde{\mathrm{Char}^u\pi_1(X)} \to Char^u X$ is the unitary part of characteristic variety: $S^n_l \cap \mathrm{Char}^u\pi_1(X)$.

One of the tools used in the Proof of theorem 1.3 is the Mayer Vietoris spectral sequence for the cohomology of rank one local systems on the union of quasiprojective manifolds with normal crossings which is discussed in Lemma 3.5.

In the case when $(\mathcal{X}, \mathcal{D})$ is an isolated non-normal crossing (which is studied in [11,23]) this has as a consequence the fact that the support of the homotopy groups of the complements has canonical decomposition into a union of polytopes. These polytopes can be related to the distribution of the Hodge numbers of abelian covers and will be discussed in Sect. 4.

[4] We identify the universal cover with the tangent space to $\mathrm{Char}^u\pi_1(X)$, the universal covering map with the exponential map and the fundamental domain $\mathcal{U}$ with the unit cube in the tangent space.

The next section studies an abelian generalization of the twisted Alexander polynomials studied in [4]. We also prove a generalisation of the cyclotomic property of the roots of Alexander polynomials (cf. Theorem 5.4) which lead to a restrictions on class of groups which are isomorphic to the fundamental groups of the complements to plane algebraic curves (cf. Theorem 5.7). Finally, in the last section, we make explicit calculations of polytopes $\mathcal{S}_l^{n,p}$ in several examples.

The author wants to thank N.Budur and J.I.Cogolludo for their comments on a draft of this paper. Note that the polytopes[5] in this circle of questions were introduced in [21] (cf. also [22]). Polytopes in the quasiprojective case were studied recently by Nero Budur in [3] using Mochizuki's work.

2. Hodge numbers of local systems on quasiprojective varieties

Let X be a quasiprojective manifold and let $\bar{X}$ be a compact projective manifold such that $H^1(\bar{X}, \mathbb{C}^*) = 0$, $\bar{X} - D = X$ where D is a divisor with normal crossings. Denote by $\Omega^1_{\bar{X}}(\log D)$ the sheaf of logarithmic 1-forms. We shall fix a N-dimensional unitary representation $\rho : \pi_1(X) \to U_N$ such that there is a locally trivial bundle $\mathcal{V}$ on $\bar{X}$, a meromorphic connection $\nabla : \mathcal{V} \to \Omega^1_{\bar{X}}(\log D) \otimes \mathcal{V}$ for which the restriction on X is flat and the corresponding holonomy representation is ρ (cf. [6]). Let $\chi : \pi_1(X) \to \mathbb{C}^*$ be a unitary character of the fundamental group. Denote by $V_{\rho\otimes\chi}$ the local system corresponding to representation $\rho \otimes \chi$. It follows from [1,31] that the cohomology groups $H^i(X, V_{\rho\otimes\chi})$ support three filtrations $F, \hat{F}, W$ ($F, \hat{F}$ being decreasing and W increasing) such that

$$\dim Gr_n^W H^i(X, V_\rho \otimes L_\chi) = \dim F^p Gr_n^W + \dim \hat{F}^q Gr_n^W \quad (p+q=n)$$

Moreover, these filtrations on the cohomology are independent of a particular compactification $\bar{X}$ used in their construction but rather depend only on X and ρ. Note that F-filtration is the one resulting from the degenerating in E_1-term Hodge–deRham spectral sequence (cf. [31]):

$$E_1^{p,q} = H^p(\Omega^q(\log D) \otimes \mathcal{V}_{\rho\otimes\chi}) \Rightarrow H^{p+q}(V_{\rho\otimes\chi}). \tag{2}$$

Here $\mathcal{V}_{\rho\otimes\chi}$ is a vector bundle on $\bar{X}$ with flat logarithmic connection ∇ such that the eigenvalues of the residues matrices $Res\nabla$ along components of D satisfy $0 \le \mathrm{Re} < 1$ and such that on X the holonomy of ∇ is the representation $\rho \otimes \chi$. We call such $\mathcal{V}$ the Deligne's (canonical) extension of a bundle on X supporting the local system.

The set of unitary rank one local systems $\mathrm{Char}^u(\pi_1(X))$ is parameterized by the maximal compact subgroup of the torus $Hom(\pi_1(X), \mathbb{C}^*)$. We assume for simplicity *throughout this paper* that this torus is connected (i.e., $H_1(X, \mathbb{Z})$ is torsion free) leaving to an interested reader the general case. As already was mentioned, we view the universal cover $\widetilde{\mathrm{Char}^u(\pi_1(X))}$ as the tangent space to $\mathrm{Char}^u(\pi_1(X))$ at the identity with the covering map being the exponential map. A choice of generators

[5] Called "polytopes of quasiadjunction".

in $H_1(X,\mathbb{Z}) = \oplus\mathbb{Z}^r$ allows one to identify the universal cover $\widetilde{\mathrm{Char}^u(\pi_1(X))}$ with $\mathbb{R}^n$ so that the generator of the i-th summand acts as $(x_1,\ldots,x_r) \to (x_1,\ldots,x_i+1,\ldots,x_r)$. An exponential map yields the lattice $\widetilde{\mathrm{Char}^u(\pi_1(X))}_{\mathbb{Z}} = \mathrm{Ker}(exp) \subset \widetilde{\mathrm{Char}^u(\pi_1(X))}$. We use the unit cube

$$\mathcal{U}: \quad \{(x_1,\ldots,x_r)|0 \le x_i < 1\} \tag{3}$$

as the fundamental domain for the action of $\widetilde{\mathrm{Char}^u(\pi_1(X))}_{\mathbb{Z}} = H_1(X,\mathbb{Z})$ on the universal cover $\widetilde{\mathrm{Char}^u(\pi_1(X))}$. A subset of $\mathcal{U}$ is called *a polytope* if there exists a collection of linear functions $l_j \in Hom(\widetilde{\mathrm{Char}^u(\pi_1(X))}_{\mathbb{Z}},\mathbb{Q})$ on the universal cover $\widetilde{\mathrm{Char}^u(\pi_1(X))}$ so that this subset is the set of solutions of a finite collection of inequalities $a_j \le l_j(u) < a'_j \ (a_j, a'_j \in \mathbb{R})$.

The main result in this section is the following:

Theorem 2.1. *Let X be a quasiprojective variety and let $\bar{X} = X \cup D$ be a compactification such that $H^1(\bar{X},\mathbb{C}^*) = 0$. Let $S^{n,p}_{\rho,l}$ be the subset of $\mathcal{U}$ defined as follows.*

$$S^{n,p}_{\rho,l} = \{u \in \mathcal{U}|\dim Gr^p_F H^n(V_{\rho\otimes exp(u)}) \ge l\} \tag{4}$$

Then (4) *is a finite union of polytopes.*

Proof. Let $\pi : (\tilde{X},\tilde{D}) \to (\bar{X},D)$ be a log-resolution of $(\bar{X},D)$, i.e. the total transform $\tilde{D} = \bigcup_{k=1,\ldots,K}\tilde{D}_k$ of D is a normal crossing divisor. We claim that on $\tilde{X}$ there are only finitely many bundles $\mathcal{V}$ which are the Deligne's extensions of the local systems in the family $V \otimes L_\chi$ where $\chi \in \mathrm{Char}\pi_1(X)$. Moreover, the collection of unitary local systems in the torus $\mathrm{Char}^u(\pi_1(X))$ having a fixed Deligne's extension is the image via the exponential map of a polytope in the universal cover $\widetilde{\mathrm{Char}^u(\pi_1(X))}$. The above theorem clearly follows from this.

Let $||\omega_{i,j}(\rho)||$ (resp. ω^χ) be the connection matrix of a flat logarithmic connection (for a locally trivial bundle $\mathcal{V}$ on $\tilde{X}$) corresponding to the local system ρ (resp. χ), where $\omega_{i,j}(\rho), \omega^\chi \in \Gamma(U,\Omega^1_{\tilde{X}}(\log(\tilde{D}))$ are logarithmic 1-forms in a chart $U \subset \tilde{X}$ biholmorphic to a disk so that $U \cap \tilde{D}$ in U is given by $z_1 \cdots z_k = 0$. For a component $\tilde{D}_k$ of the normal crossing divisor $\tilde{D}$ let $R^{\tilde{D}_k}_{i,j}(\rho) = Res_{\tilde{D}_k}\omega_{i,j}(\rho)$ be the entries of the matrix of residues along $\tilde{D}_k$ (for the extension $\mathcal{V}$). Let $\beta^{\tilde{D}_k}_i(\rho)$ be the collection of eigenvalues of the matrix $R^{\tilde{D}_k}_{i,j}(\rho)$. The components of the matrix of connection corresponding to the local system $V \otimes L_\chi$ in a basis v_i, $i = 1,\ldots,N$ (resp. e) of V (resp. L_χ) are $\omega_{i,j}v_j \otimes e + \omega^\chi v_i \otimes e$ (the connection on the tensor product is given by $\nabla(v \otimes e) = \nabla(v) \otimes e + v \otimes \nabla(e)$). Hence the eigenvalues of $Res_{D_k}\nabla_{\rho\otimes\chi}$ can be computed as:

$$\beta^{D_k}_i(\rho) + Res_{\tilde{D}_k}\nabla_\chi \quad (i = 1,\ldots,N). \tag{5}$$

where $Res_{\tilde{D}_k}\nabla_\chi$ is the residue of the log-connection ∇_χ in the rank one bundle along $\tilde{D}_k$.

Let $\omega_1, \dots, \omega_r, \quad \omega_s \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log(\tilde{D})))$ be a basis of the space of log 1-forms. Notice that $\dim H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log(\tilde{D}))) = r$ since $\dim H^1(X, \mathbb{C}) = \dim H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log(D))) + \dim H^1(\bar{X}, \mathcal{O}_{\bar{X}})$ (cf. [7]), $H^1(\bar{X}, \mathbb{C}^*) = H^1(\tilde{X}, \mathbb{C}^*)$ and we assumed that $H^1(\tilde{X}, \mathbb{C}^*) = 0$. We shall select forms ω_s so that their cohomology classes all belong to $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{C})$. The set of $(\alpha_1, \dots, \alpha_r) \in \mathbb{C}^r$ such that for a fixed ρ the connection $\nabla_\rho^{\alpha_1, \dots, \alpha_r}$ with the matrix

$$||\omega_{i,j}^{\alpha_1, \dots, \alpha_r}|| = ||\omega_{i,j}|| + \left(\sum \alpha_s \omega_s\right) I \tag{6}$$

yields a unitary local system form a r-dimensional $\mathbb{R}$-subspace $\widetilde{\mathrm{Char}^u(\pi_1(X))}$ of $\mathbb{C}^r$. Indeed, all connections with matrix $||\omega_{i,j}^{\alpha_1, \dots, \alpha_r}||$ are flat since the connection corresponding to $||\omega_{i,j}||$ is flat and any holomorphic log-form is closed (cf. [7]). Moreover, the holonomy of a connection along a path is given by solutions of a system of first order ODEs and hence is the exponent of a matrix A depending on the each summand in (6) linearly. Hence those $(\alpha_1, \dots, \alpha_r)$ which yield a skew-hermitian A form a $\mathbb{R}$-linear subspace in the universal cover $\widetilde{\mathrm{Char}(\pi_1(X))} = \mathbb{C}^r$ of the space of characters. In addition, there is a compact connected fundamental domain $\tilde{\mathcal{U}} \subset \widetilde{\mathrm{Char}^u(\pi_1(X))}$ for the action of $H_1(X, \mathbb{Z})$ such that the family of unitary local systems corresponding to elements in $\tilde{\mathcal{U}}$ coincides with $\mathrm{Char}^u(\pi_1(X))$. One can take $\tilde{\mathcal{U}}$ so that it is an affine transform of $\mathcal{U}$ given by (3).

The eigenvalues of the residue matrix of the connection $\nabla_\rho^{\alpha_1, \dots, \alpha_r}$ along a component $\tilde{D}_k$ in an extension $\mathcal{M}$ (i.e. a locally trivial bundle on $\tilde{X}$) are given by:

$$N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r, \mathcal{M}) = \beta_i^{\tilde{D}_k}(\rho) + \sum \alpha_s Res_{\tilde{D}_k} \omega_s \quad i = 1, \dots, \mathrm{rk} V \tag{7}$$

In particular, for each $i = 1, \dots, rkV$ and $\tilde{D}_k$, the image of the map $\tilde{\mathcal{U}} \to \mathbb{R}$ given by $(\alpha_1, \dots, \alpha_r) \to N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r)$ is bounded and for any collection of integers $n_i^{\tilde{D}_k}, i = 1, \dots, rkV, k = 1, \dots, K$ the set $(\alpha_1, \dots, \alpha_r)$ such that

$$n_i^{\tilde{D}_k} \leq N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r) < n_i^{\tilde{D}_k} + 1 \tag{8}$$

is a (possibly empty) polytope in $\tilde{\mathcal{U}}$.

Now let us describe the bundles on $\tilde{X}$ which are the Deligne's extensions of the connections in the family given by (6). Let us call collection $n_i^{\tilde{D}_k}$ realizable if there is a connection corresponding to $(\alpha_1, \dots, \alpha_r)$ such that $n_i^{\tilde{D}_k} = [N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r)]$. Let $\mathcal{M}(n_i^{\tilde{D}_k})$ be the Deligne extension of this connection. We claim that all connections with matrices (6) and residues satisfying (8) have $\mathcal{M}(n_i^{\tilde{D}_k})$ as its Deligne extension. Consider a sufficiently fine cover of $\tilde{X}$ by open sets such that both bundles $\mathcal{M}$ and $\mathcal{M}(n_i^{\tilde{D}_k})$ can be trivilized. Over each open set U of such a cover one has the transition matrix $g_U \in GL_N(\Gamma(U, \mathcal{O}(*\tilde{D}))$ from a frame $e_1, \dots, e_N$

of $\mathcal{M}$ to a frame $e'_1 \dots e'_N$ of $\mathcal{M}(n_i^{\tilde{D}_k})$. The connection matrices $\Omega(e_1, \dots, e_N)$ and $\Omega(e'_1, \dots, e'_r)$ are related as follows:

$$\Omega(e_1, \dots, e_N) = g_U^{-1} dg_U + \Omega(e'_1, \dots, e'_N) \tag{9}$$

Since both $exp(2\pi i Res_{\tilde{D}_k} \Omega(e_1, \dots, e_N))$ and $exp(2\pi i Res \Omega(e'_1, \dots, e'_N))$ conside with the monodromy of the connection around $\tilde{D}_k$ (cf. [6, Prop. 3.11]) one sees that the eigenvalues of $Res_{\tilde{D}_k} g_U^{-1} dg_U$ are integers. Moreover, these residues are equal to $n_i^{\tilde{D}_k}$ since $\mathcal{M}(n_i^{\tilde{D}_k})$ is the Deligne extension and the eigenvalues of the residues of the connection $\Omega(e'_1, \dots, e'_N)$ are in interval $[0, 1)$. This also implies that in the frame $e'_1, \dots, e'_N$ the eigenvalues of the residues of the connections matrix of any connection satisfying (8) has the residues in the interval $[0, 1)$, i.e. has $\mathcal{M}(n_i^{\tilde{D}_k})$ as its Deligne extension.

In the case when $N = 1$ the relation between the extensions $\mathcal{M}$ and $\mathcal{M}(n_i^{\tilde{D}_k})$ is particularly simple. Indeed, if ∇ is a logarithmic connection in a bundle $\mathcal{M}$ on $\tilde{X}$ and $B = \sum b_i \tilde{D}_i$ is a divisor on $\tilde{X}$ having support on $\tilde{X} - X$ then ∇ induces a connection ∇^B on $\mathcal{M} \otimes \mathcal{O}_{\tilde{X}}(B)$. The residues $Res(\cdot) : \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_{\tilde{D}_j}$ of two connections related as follows:

$$Res_{\tilde{D}_j}(\nabla^B) = Res_{\tilde{D}_j}(\nabla) - b_j id|_{\tilde{D}_j} \tag{10}$$

(cf. [15, Lemma 2.7]). In particular, for each member of the variation of the connection given by (6) the corresponding Delinge's connection has the form $\mathcal{M} \otimes \mathcal{O}_{\tilde{X}}(B)$ where the coefficients b_j in $B = \sum b_j \tilde{D}_j$ chosen so that the residues of $\nabla_{\tilde{D}_j}$ will satisfy $0 \leq Res_{\tilde{D}_j}(\nabla^B) < 1$. In the case or arbitrary N the above argument shows that possible number of isomorphism classes of bundles on $\bar{X}$ which are the Deligne's extensions of the connections (6) does not exceed:

$$\sum_{\tilde{D}_k, i} [sup_{(\alpha_1, \dots, \alpha_r)} N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r) - inf_{(\alpha_1, \dots, \alpha_r)} N_i^{\tilde{D}_k}(\alpha_1, \dots, \alpha_r)] \tag{11}$$

([..] is the integer part).

From the degeneration of the spectral sequence (2), it follows that

$$Gr_F^p H^k(V_{\rho \otimes \chi}) = H^p(\tilde{X}, \Omega_{\tilde{X}}^{k-p}(\log \tilde{D}) \otimes \mathcal{V}_{\rho \otimes \chi})$$

Since the logarithms of characters χ yielding a given Deligne's extension form a polytope (8) in $\mathcal{U}$ and since the rank of $Gr_F^p H^k(V_{\rho \otimes \chi})$ depends only on the Deligne's extension (and hence has only finitely many values bounded by the number of Deligne's extensions of the local systems $V_{\rho \otimes \chi}$), we see that the collection of α_i having fixed values is a finite union of polytopes.

It follows from (7) and (8) that each polytope is given by inequalities:

$$\begin{gathered} N_i^{\tilde{D}_k}(\mathcal{V}) \leq \beta_i^{\tilde{D}_k}(\rho) + \sum \alpha_s Res_{\tilde{D}_k} \omega_s < N_i^{\tilde{D}_k}(\mathcal{V}) + 1 \\ i = 1, \dots, \mathrm{rk} V, \quad k = 1, \dots, K \end{gathered} \tag{12}$$

where $N_i^{\tilde{D}_k}(\mathcal{V})$ are integers depending on the Chern classes of bundle $\mathcal{V}$ on $\tilde{X}$. Since classes ω_s belong to $H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{C}^*)$ one has $Res_{\tilde{D}_k}\omega_s \in \mathbb{Q}$ for any k. It follows that the linear subspaces in $\widetilde{\mathrm{Char}^u(\pi_1}(X))$ spanned by the sets of solutions of $\mathrm{rk}V \cdot K$ inequalities (12) for fixed $\mathcal{V}$ are given by equations with integer relatively prime coefficients and constant terms having the form $\gamma_1\beta_i^{\tilde{D}_k} + \gamma_2$ $(\gamma_1, \gamma_2 \in \mathbb{Q})$. $\square$

Corollary 2.2. *Let ρ be a representation with abelian image and let $\rho_1, \ldots, \rho_N \in \mathrm{Char}^u\pi_1(X)$ be its irreducible components. The isolated characters $\chi \in \mathrm{Char}^u(\pi_1(X))$ for which $\dim H^k(X, \rho \otimes \chi) \geq l$ generate a subgroup of $\mathrm{Char}^u\pi_1(X)$ such that its subgroup generated by ρ_i $(i = 1, \ldots, N)$ has finite index.*

Proof. Since the system of inequalities (12) has only isolated solutions, the logarithm of an isolated character χ as above, is the solution to a finite system of equations. As was remarked above, each linear subspace spanned by the sets of solutions is given by equations with integer relatively prime coefficients and constant terms which can be written as $\gamma_1\beta + \gamma_2$, $(\gamma_1, \gamma_2 \in \mathbb{Q})$. Such subspace contains a point $(\alpha_1, \ldots, \alpha_r)$ with coordinates having the form $\gamma_1\beta + \gamma_2$, $(\gamma_1, \gamma_2 \in \mathbb{Q})$. The exponential map takes such a point in $\widetilde{\mathrm{Char}^u(\pi_1}(X))$ to a point in $\mathrm{Char}^u(\pi_1(X))$ with coordinates $[e^{2\pi i\beta}]^{\gamma_1}e^{2\pi i\gamma_2}$. If the representation ρ has an abelian image then the subgroup in $\mathrm{Char}^u(\pi_1(X))$ generated by characters which are the irreducible component of ρ is the subgroup of the points with the coordinates $e^{2\pi i\beta}$ and the claim follows. $\square$

Corollary 2.3. *The subsets*

$$S^n_{\rho,l} = \{\chi \in \mathrm{Char}^u\pi_1(X) | \dim H^n(X, L_{\rho\otimes\chi}) \geq l\}$$

are unions of finite collections of translated subgroups. If ρ has an abelian image and if $\bar{\rho}_1, \ldots, \bar{\rho}_N \in \mathrm{Char}^u(\pi_1(X))$ are the irreducible components of ρ, then translations can be made by points generating a subgroup of $\mathrm{Char}^u(\pi_1(X))$ containing the subgroup generated by $\rho_1, \ldots, \rho_N$ as a subgroup of finite order.

Proof. The subsets $S^n_{\rho,l}$ clearly are algebraic subsets of $\mathrm{Char}^u\pi_1(X)$. On the other hand it follows from Theorem 2.1 that $S^n_{\rho,l} = \bigcup_{l_0+\cdots+l_n=l} \cap_{p=0,\ldots,n} S^{n,p}_{\rho,l_p}$ is a union of polytopes. The Zariski closure in $\mathrm{Char}^u\pi_1(X)$ of the image of a polytope is the intersection of algebraic subvarieties of $\mathrm{Char}^u\pi_1(X)$ containing this image. Since a polytope spans a linear subspace in $\widetilde{\mathrm{Char}^u\pi_1}(X)$, the Zariski closure is the image of the exponential map restricted on the spanning subspace i.e. it is a translated subgroup of $\mathrm{Char}^u\pi_1(X)$. The claim about the translation points follows by the argument identical with the one used in the proof of Corollary 2.2. $\square$

Remark 2.4. The proof of the Theorem 2.1 yields an alternative proof of the results in [1]. In [1] it is shown that (at least for trivial ρ) similarly defined subgroups of the full group of characters of $\pi_1(X)$ are unions of finite collections of translated subgroups. In particular the subgroups S^n_l in $\mathrm{Char}^u\pi_1(X)$ determine completely the subgroups in $\mathrm{Char}\pi_1(X)$. The Corollary 2.3 shows that translation is by points of finite order (rather than just the unitary ones cf. [1]). This also implies that the polytopes $S^{n,p}_{\rho,l}$ from the Theorem 2.1 for $\rho = 1$ determine the translated subgroups of $\mathrm{Char}\pi_1(X)$ completely. In [24], a procedure was outlined for calculation of the polytopes corresponding to $p = 0$ component of the Hodge filtration in some geometrically interesting cases.

Remark 2.5. Proofs of the Corollaries 2.2 and 2.3 use as the essential step the fact that the coordinates of translation characters belong to cyclotomic extension of the field generated by $e^{2\pi i \beta_i^{D_k}}$ for all D_k and i. This provides information about translation points also in the case when $\mathrm{Im}\rho$ is non-abelian.

Corollary 2.6. *Let $i : X_1 \to X_2$ be an embedding of quasiprojective submanifold X_1 into a quasi-projective submanifold X_2. Let ρ be a unitary representation of $\pi_1(X_2)$. Then either*

$$\{\chi \in \mathrm{Char}^u \pi_1(X_2) | \dim \mathrm{Ker} H^n(X_2, \mathcal{L}_{\rho\otimes\chi}) \to H^n(X_1, \mathcal{L}_{i*(\rho\otimes\chi)}) = l\}$$

or

$$\{\chi \in \mathrm{Char}^u \pi_1(X_2) | \dim \mathrm{Im} H^n(X_2, \mathcal{L}_{\rho\otimes\chi}) \to H^n(X_1, \mathcal{L}_{i*(\rho\otimes\chi)}) = l\}$$

are the images of a union of polytopes in the universal cover of U.

Proof. Let us select a resolution of a pair (X_1, X_2) (i.e., the compactifying divisor D_2 of X_2 is a normal crossings divisor and the closure of X_1 in this compactification has at infinity the divisor D_1 which also is a normal crossings divisor). The Deligne extension for the flat connection corresponding to the local system $i^*(\rho \otimes \chi)$ is restriction of the Deligne extension of connection of $\rho \otimes \chi$. Hence the characters in the corollary are the characters giving the Deligne extension for which the corresponding map $H^q(X_2, \Omega^p(\log D_2) \otimes \mathcal{V}) \to H^q(X_1, \Omega^p(\log D_1) \otimes \mathcal{V})$ has a fixed dimension of the kernel (resp. image). □

3. Local translated subgroup theorem for complement to germs of singularities and Hodge structure on cohomology of local systems

In this section we prove the local translated subgroups Theorem 1.1 and the local counterpart of the Theorem 2.1.

We shall use the same notations as in introduction: $\mathcal{X}$ is a germ of a complex space with an isolated singularity, $\mathcal{D}$ is a divisor on $\mathcal{X}$ with arbitrary singularities having r irreducible components. Assume that the link of $\mathcal{X}$, i.e. the intersection with a small sphere about the singular point, is simply connected (cf. [11]). This implies that:

$$H_1(\mathcal{X} - \mathcal{D}, \mathbb{Z}) = \mathbb{Z}^r \tag{13}$$

(cf. [11,23]). As generators for $H_1(\mathcal{X} - \mathcal{D}, \mathbb{Z})$ one can take the classes of 1-cycles each being the boundary of small disks transversal to $\mathcal{D}$ at a non-singular point of each component. In particular $H^1(\mathcal{X}-\mathcal{D},\mathbb{C}^*)=\mathbb{C}^{*r}$ and hence the rank one local systems are parameterized by the torus with a fixed coordinate system.

The main results of this section are the following two theorems (cf. Theorem 1.1 in the Introduction).

Theorem 3.1. *Let $(\mathcal{X},\mathcal{D})$ be a germ of a pair where $\mathcal{X}$ has an isolated normal singularity with a simply connected link and $\mathcal{D}$ is divisor with r irreducible components D_i $(i=1,\dots,r)$. Let*

$$\mathcal{S}^n_l=\{\chi\in \mathrm{Char}\pi_1(\mathcal{X}-\mathcal{D})|H^n(\mathcal{X}-\mathcal{D},\mathcal{L}_\chi)\geq l\}\ \ (1\leq n\leq \dim\mathcal{X})$$

where $\mathcal{L}_\chi$ is the local system corresponding to the character χ. Then $\mathcal{S}^n_l$ is a union of a finite collection of translated subgroups for any n and l. More precisely, there are (possibly trivial) subgroups $T_i\subset \mathrm{Char}\pi_1(\mathcal{X}-\mathcal{D})$, $(i\in\mathcal{I},\mathrm{Card}\,\mathcal{I}<\infty)$, and torsion characters ρ_i such that

$$\mathcal{S}^n_l=\bigcup_i \rho_i T_i$$

The arguments we are using in the proof of the Theorem 3.1 also yield the existence of the mixed Hodge structure on the cohomology of the unitary local systems extending the results of [1,31] to the local case:

Theorem 3.2. *Let $\mathcal{X}$ be a germ of an analytic space having an isolated normal singularity and let $\mathcal{D}$ be a divisor on $\mathcal{X}$. Denote by ρ a unitary representation of $\pi_1(\mathcal{X}-\mathcal{D})$ and let $\mathcal{L}_\rho$ be the corresponding local system. Then the cohomology groups $H^i(\mathcal{X}-\mathcal{D},\mathcal{L}_\rho)$ support the canonical $(\mathbb{C})$-mixed Hodge structure compatible with the holomorphic maps of pairs $(\mathcal{X},\mathcal{D})$ endowed with a local system on the complement $\mathcal{X}-\mathcal{D}$.*

We refer to [1,2] for a discussion of $\mathbb{C}$-mixed Hodge structures. Before proving these results, let us calculate the cohomology of local systems on the total space of a fibration with the fibers homotopy equivalent to $F=(\mathbb{C}^*)^k$. Let $\pi:T^*\to E$ be such a locally trivial fibration over a manifold E for which the associated $\mathbb{C}^k$ bundle $T\to E$ is a direct sum of k line bundles $L_1,\dots,L_k$. Denote by $c_1^1,\dots,c_1^k\in H^2(E)$ the first Chern classes of $L_1,\dots,L_k$, respectively. Let ρ_E be a unitary local system on E. Consider the homomorphisms:

$$\kappa^{a,b}:H^a(E,\mathcal{L}_{\rho_E})\otimes\Lambda^b(H_1(\mathbb{C}^*))\to H^{a+2}(E,\mathcal{L}_{\rho_E})\otimes\Lambda^{b-1}(H_1(\mathbb{C}^*))\tag{14}$$

given by:

$$\kappa^{a,b}(\beta\otimes\alpha_1\wedge\cdots\wedge\alpha_k)=\sum_i \beta\cup c_1^j\otimes\alpha_1\wedge\cdots\hat{\alpha}_j\cdots\wedge\alpha_k\tag{15}$$

and let

$$K^{a,b}=\mathrm{Ker}\kappa^{a,b}/\mathrm{Im}\kappa^{a-2,b+1}\tag{16}$$

Denote by $Im\pi_1(F)$ the image of the homomorphism $\pi_1(F) \to \pi_1(T^*)$ and let $\rho : \pi_1(T^*) \to \mathbb{C}^*$ be a unitary representation of the fundamental group of T^*. The exact sequence:

$$0 \to Im\pi_1(F) \to \pi_1(T^*) \to \pi_1(E) \to 0 \tag{17}$$

shows that a representation ρ has trivial restriction on $Im\pi_1(F)$ if and only if $\rho = \pi^*(\rho_E)$ for some unitary representation ρ_E of $\pi_1(E)$.

Lemma 3.3. *For any non-negative i one has the following*:

$$\begin{aligned} &\dim H^i(T^*, \mathcal{L}_\rho) \\ &\quad = \begin{cases} 0 & \text{if } \rho|_{\mathrm{Im}\pi_1(F)} \neq 1 \\ \sum_{a+b=i} \dim K^{a,b} & \text{if } \rho|_{Im\pi_1(F)} = 1 \text{ or equivalently } \rho = \pi^*(\rho_E). \end{cases} \end{aligned} \tag{18}$$

Proof. (of Lemma 3.3) (18) is clearly the case when E is a point which also yields that the fiber of $R^q\pi_*(L_\rho)$ for arbitrary E is either $\Lambda^q(H^1(F, \mathbb{C}))$ (if ρ_E is trivial) or zero (otherwise). For a more general E, the lemma follows from the Leray spectral sequence: $H^p(E, R^q\pi_*(\mathcal{L}_\rho)) \Rightarrow H^{p+q}(T^*, \mathcal{L}_\rho)$ which degenerates in the term E_3. Indeed if ρ is not induced from E then $E_2^{p,q} = 0$, $p, q \geq 0$ by (18). If $\rho = \pi^*(\rho_E)$ for a local system ρ_E on E then, since the local systems $R^q\pi_*(\mathbb{C})$ are trivial i.e. $E_2^{p,q} = H^p(E, \rho_E) \otimes \Lambda^q(H_1(F, \mathbb{C}))$, the degeneration can be seen using the compatibility property of differentials of a spectral sequence with products and vanishing of d_i, $i > 2$ on $E_*^{0.1}$. One arrives at the formula (18) for $E_3 = E_\infty$ since $d_2^{a,b} = \kappa^{a,b}$ as one can see using multiplicativity and the well-known identification of d_2 in the Leray spectral sequence of a circle bundle with the cup product with the first Chern class of the associated line bundle. □

Remark 3.4. Local systems on $\mathbb{C}^*$-fibrations over quasiprojective manifolds. Let T^* be a $(\mathbb{C}^*)^r$-fibration over E such that the associated $\mathbb{C}^r$-bundle T is a direct sum of holomorphic line bundles and let:

$$\mathcal{S}_l^n(T^*) = \{\chi \in \mathrm{Char}\pi_1(T^*) | H^n(T^*, \mathcal{L}_\chi) \geq l\} \tag{19}$$

If the base E is quasiprojective, then it follows from Lemma 3.3 and [1] that $\mathcal{S}_l^n(T^*)$ is a finite union of translated subgroups. Indeed, $\mathcal{S}_l^n(T^*)$ is an invariant of the homotopy type and, since T^* is homotopy equivalent to the complement in the total space of T to union of total spaces of rank $r-1$ subbundle of T which is also quasiprojective, one can apply [1].

Similarly, let ρ be a unitary local system on a the total space of a fibration T^*. Assume again that the associated $\mathbb{C}^r$ bundle T is a direct sum of line bundles. Then the cohomology $H^p(T^*, \mathcal{L}_\rho)$ support the canonical $\mathbb{C}$-mixed Hodge structure defined has follows. By Lemma 3.3 we can assume that $\rho = \pi^*(\rho_E)$. Consider the total space of bundle $T = \oplus_{i=1}^{i=r} L_i$. The complement to the union of divisors $\oplus_{i \neq j} L_i$ $(j = 1, \ldots, r)$ has the homotopy type of T^* and the mixed Hodge structure on $H^p(T^*, \rho_E)$ is obtained from the mixed Hodge structure constructed in [31], [1] via the identification

$$H^p(T^*, \pi^*(\rho_E)) = H^p(T - \cup_j \oplus_{i \neq j} L_i, \pi^*(\rho_E)) \tag{20}$$

In the case when ρ is trivial this is compatible with the standard mixed Hodge structure of a punctured neighbourhood.

The proofs of the theorems 3.1 and 3.2 use the following result on degeneration of Mayer Vietoris spectral sequence generalizing the case of projective manifolds discussed in [5] to quasiprojective situation with local systems.

Lemma 3.5. *Let $\bar{X} = \cup \bar{X}_i$ be a union of projective manifolds having only normal crossings and let $\mathbf{D}_i = \cup_j D_{i,j} \subset \bar{X}_i$ be a normal crossing divisor such that $D_{i,j} \cup \bar{X}_i \cap_{k \neq i} X_k$ is a normal crossing divisor on $\bar{X}_i$ as well. Denote by $X_i = \bar{X}_i - \cup D_{i,j}$, $X_{i_0,\dots,i_q} = X_{i_0} \cap \cdots \cap X_{i_q}$ and $X^{[q]} = \coprod X_{i_0,\dots,i_q}$. Let $\rho : \pi_1(\cup X_i) \to \mathbb{C}^*$ be a local system on X and $\rho_{i_0,\dots,i_q}$ be the local system on $X_{i_0,\dots,i_q}$ induced by ρ. Let $H^p(X^{[q]}, \mathcal{L}^{[q]}) = \oplus_{i_0<\dots<i_q} H^p(X_{i_0,\dots,i_q}, \mathcal{L}_{i_0,\dots,i_q})$. This data determines a spectral sequence*:

$$E_1^{p,q} = H^p(X^{[q]}, \mathcal{L}^{[q]}) \Rightarrow H^{p+q}(X, \mathcal{L}) \tag{21}$$

degenerating in term E_2.

Proof. Let $\mathcal{M}_{i_0,\dots,i_q}$ be the Deligne's extension (cf. [6]) corresponding to the local system $\mathcal{L}_{i_0,\dots,i_q}$ and $\mathcal{M}^{[q]}$ be the corresponding bundle on $X^{[q]}$. The spectral sequence in this lemma is the spectral sequence of the double complex

$$A^{\bullet,\bullet} = A^p(X^{[q]}(log), \mathcal{M}^{[q]}, (\nabla, \delta)) \tag{22}$$

with components being C^∞ logarithmic forms on $\bar{X}^{[q]} (= \coprod \bar{X}_{i_0,\dots,i_q})$ with the poles along the intersections of components of $\mathbf{D}_i$ with $\bar{X}_{i_0,\dots,i_q}$ (twisted deRham complex for the Deligne's extension $\mathcal{M}^{[q]}$). The differentials respectively are the differential $\nabla : A^{p,q} \to A^{p+1,q}$ of the connection and the differential $\delta : A^{p,q} \to A^{p,q+1}$ which takes $\omega \in A^{p,q}$ having components $\omega(i_0, \dots, i_q) \in A^p(X_{i_0,\dots,i_q}(log), \mathcal{L}|_{X_{i_0,\dots,i_q}})$ to $\delta(\omega)(i_0, \dots, i_{q+1}) = \sum (-1)^k \omega(i_0, \dots \hat{i}_k, \dots i_q)|_{X_{i_0,\dots,i_{q+1}}}$.

Clearly the term $E_1^{p,q}$ of the spectral sequence of the double complex (22) coincides with expression in (21). The abutment of this spectral sequence is $H^*(X, \mathcal{L})$ since the single complex $(A^{\bullet,\bullet}, \nabla + \delta)$ is a locally free *acylic* resolution of locally constant vector bundle of the local system $\mathcal{L}$. Acyclicity follows from the local calculation which identifies for a sufficiently small open subset $U \subset X$ the group $H^n(U, \mathcal{L})$ with the abutement of the spectral sequence having $E_2^{q,p} = H_\delta^q H_\nabla^p$ and vanishing of the latter: indeed for U over which $\mathcal{L}$ is not trivial one has $H_\nabla^p = 0$ and for U over which $\mathcal{L}$ is trivial $E_2^{0,q}$ is the cohomology of the simplex (cf. [5]).

To show the E_2-degeneration of (21) first let us consider the case when none of the eigenvalues of the holonomy of the connection about the components of $\mathbf{D}_i \cap X^{[q]}$ is equal to one. In this case, for each component $X_{i_0,\dots,i_q}$ one has:

$$H^i(X_{i_0,\dots,i_q}, \mathcal{L}_{i_0,\dots,i_q}) = H^i(\bar{X}_{i_0,\dots,i_q}, j_* \mathcal{L}_{i_0,\dots,i_q}) = H^i_{(2)}(X_{i_0,\dots,i_q}, \mathcal{L}_{i_0,\dots,i_q})$$

(cf. [31,32]; here j is the embedding $X_{i_0,\dots,i_q} \to \bar{X}_{i_0,\dots,i_q}$ and L^2-cohomology are with respect to a complete metric on $X_{i_0,\dots,i_q}$ asymptotic to the Poincare metric).

Using the twisted analog of $\partial\bar{\partial}$ lemma (cf. [33]) as in compact case [5] one obtains degeneration in ths case when all monodormies of $\mathcal{L}_{i_0,\ldots,i_q}$ are non trivial.

General case now can be deduced by induction over the number of components $\bigcup_i \mathbf{D}_i$ for which $\mathcal{L}$ has non trivial monodormy. Let $X' = (X_1 - D) \cup \bigcup_{i \geq 2} X_i$ i.e. X' is a quasiprojective normal crossing having one more compactifying component than does X. Let $\mathcal{L}$ be a local system on X' such that the holonomy about D is trivial. The connection matrix of the flat connection corresponding to the local system $\mathcal{L}|_{X_{1,i_1,\ldots,i_q}}$ has an integer residue along D since the holonomy is trivial along D. Hence in the Deligne's extension this residue is equal to zero along D. In particular we have a well defined holomorphic connection on X and its restriction to D. Then one has

$$0 \to A^p(log)(X^{[q]}, \mathcal{M}) \to A^p(log)(X'^{[q]}, \mathcal{M}) \to A^{p-1}(log)(D^{[q]}) \to 0 \quad (23)$$

where the last term is the log complex of $D^{[q]} = D \cap X^{[q-1]}_{\neq 1}$ with $X^{[q-1]}_{\neq 1}$ is disjoint union of intersections of irreducible components of X different from X_1 and the last map is the residue map of log-forms along D.

Recall that the degeneration of a spectral sequence of a differential graded algebra $A^\bullet$ filtered by a decreasing filtration $F^p(A)$ in term E_k is equivalent to condition:

$$F^p(A^\bullet) \cap dA \subset dF^{p-k+1}(A^\bullet) \quad (24)$$

(cf. [26] Lemma (1.5) p.144 and [7], (1.3.2) and (1.3.4)) One sees directly, that condition (24) for the endterms of (23) yields it for the middle terms as well. □

Proof. (of Theorem 3.1) *Step 1. Identification of $\mathcal{X} - \mathcal{D}$ with a union of torus bundles.* Let $\bigcup_{i \in I} E_i$ be the exceptional locus of a log resolution of the pair:

$$\left(\tilde{\mathcal{X}}, \bigcup_{i \in I} E_i \cup \bigcup_{j=1}^{j=r} \tilde{\mathcal{D}}_j\right) \to (\mathcal{X}, \cup D_j)$$

where $\tilde{D}_j$ are the proper preimages of the components D_j. Recall that a resolution of a pair (X, D) where X is a normal variety is a morphism $f : Y \to X$ such that the union of the exceptional locus of f and the proper preimage of D in Y is a normal crossing divisor. Given (X, D) such a morphism f always exist (cf. [18]). Let $\partial T(\cup E_i)$ be the boundary of a regular neighborhood $T(\cup E_i)$ of $\cup E_i$ in $\tilde{\mathcal{X}}$. We have the identification (homotopy equivalence)

$$\partial T(\cup E_i) - \partial T(\cup E_i) \cap \cup \tilde{D}_j = \mathcal{X} - \cup D_j$$

The (open) manifold $\partial T(\cup E_i) - \partial T(\cup E_i) \cap \cup \tilde{D}_j$ can be constructed inductively as a union of tori bundles over quasiprojective varieties intersecting along unions of tori bundles lower dimension. More precisely, consider the stratification of $\bigcup E_i$ in which each stratum is a connected component of a set consisting of points belonging to exactly l components of this union. Complement to $\cup D_j$ in an intersection of the boundaries $\partial T(E_i)$ of small regular neighborhoods in $\tilde{\mathcal{X}}$ of E_i containing this stratum is a fibration with the fiber being the torus $(S^1)^l$. Here l is the number of components $E_i, i \in I$ containing this stratum. Each such torus fibrations has a compact base if and only if the corresponding component E_i of the exceptional locus has empty intersection with the proper preimage of $D \subset \mathcal{X}$. One has a locally trivial torus fibrations over the complement in an intersection of components E having non empty intersection with the proper preimage of D due to normal crossing condition on union of the exceptional locus and the proper preimage of E (note that we do not need assumption that $\mathcal{X}$ has an isolated singularity).

Step 2. Translated subgroup property for each stratum. For each stratum S, the total space $T^*(S)$ of this $(S^1)^l$-fibration is a subset of $\partial T(\cup E_i) - \partial T(\cup E_i) \cap \cup \tilde{D}_j$. The collection of characters in $\mathbf{T} = \mathrm{Char}(\pi_1(\partial T(\cup E_i) - \partial T(\cup E_i) \cap \cup \tilde{D}_j))$, which when restricted on $T^*(S)$ yields a character with the corresponding local system having $H^n(T^*(S), \mathcal{L}_\chi) \geq l$, is a union of translated subgroups in $\mathbf{T}$ (cf. (19) or, since S is quasiprojective, apply the remark after proof of lemma 3.3).

Step 3. Degeneration of Mayer-Vietoris spectral sequence for torus bundles. The cohomology $H^n(\partial T(\cup E_i) - \partial T(\cup E_i) \cap \cup \tilde{D}_j, \mathcal{L}_\chi)$ is the abutment of a Mayer Vietoris spectral sequence:

$$E_1^{p,q} = H^p(A^{[q]}, \mathcal{L}_{\chi|_{A^{[q]}}}) \Rightarrow H^{p+q}\left(\bigcup A_i \mathcal{L}_\chi\right) \tag{25}$$

where $A^{[q]}$ is a torus bundle over a stratum of the above stratification. This spectral sequence has trivial differentials d_i for $i \geq 2$. Indeed, the degeneration of the spectral sequence (25) in the case when $A^{[0]}$ are quasiprojective was shown in Lemma 3.5. The case when $A^{[q]}$ are tori bundles over quasiprojective manifolds follows from formulas in Lemma 3.3.

Step 4. End of the proof. Now let us consider the collection of quasi-affine subsets $\mathcal{T}$ of the torus $\mathbf{T}$ with each quasi-affine subset being a finite union of subgroups of $\mathbf{T}$ with a removed collection of translated subgroup of $\mathbf{T}$ (possibly empty). Each collection of local systems with fixed dimension $H^n(A, L_\chi)$, in the case when A is quasiprojective, belongs to $\mathcal{T}$ and hence the collection χ for which each $E_2^{p,q}$ term in the spectral sequence (25) has a fixed dimension also belongs to $\mathcal{T}$ as follows from Corollary 5.7 (since d_1 in (25) is the restriction map). This proves the theorem. □

Proof. (of theorem 3.2) The main point is that, while making the calculation of the cohomology of local system on resolution of $\mathcal{X}, \mathcal{D}$ using the isomorphism $H^i(\mathcal{X} - \mathcal{D}, \mathcal{L}_\chi) = H^i(\tilde{\mathcal{X}} - (\bigcup_{i \in I} E_i \cup \tilde{\mathcal{D}}))$ where $\tilde{\mathcal{X}}, \cup_{i \in I} E_i \cup \tilde{D}$ is a resolution of pair $\mathcal{X}, \mathcal{D}$, one can replace $\bigcup_{i \in I} E_i \cup \tilde{\mathcal{D}}$ by the union of components E_i such that the restriction of χ of $\pi_1(\partial T(E_i^\circ) - \tilde{\mathcal{D}})$ is a pull back of a character of $\pi_1(E_i^\circ - \tilde{\mathcal{D}})$ with respect to projection $\partial T(E_i^\circ) - \tilde{\mathcal{D}} \to E_i^\circ - \tilde{\mathcal{D}}$ (here E_i° is the locus of points of E_i which are nonsingular points of $\bigcup E_i$; the set of indices of E_i's labeling these components will be denoted I_χ). In quasiprojective case this corresponds to isolating components with trivial holonomy of the connection. (cf. [1,31]). Moreover, in the case of $\mathbb{C}^{*r}$-bundles we have the mixed Hodge structure as described in the Remark 3.4. Then all steps used in [31] to derive the extension of [7] with modifications as in local case of [13], i.e. constructing the Mixed Hodge complex yielding the Mayer Vietoris spectral sequence go through in our local case as well.

More precisely, let $E^{\circ}_{i_1,\dots,i_k}$ be the locus of points of $\bigcup_{i\in I} E_i$ which belong to the components $E_{i_1},\dots,E_{i_k}$ but do not belong to any other components of E (recall that I denotes the set of indices of components of exceptional set). Let $\partial T(E^{\circ}_{i_1,\dots,i_k})-\tilde{\mathcal{D}}$ be $T(E_{i_1}\cap\dots\cap E_{i_k})-\bigcup E_i$ where $T(E_{i_1}\cap\dots\cap E_{i_k})$ is a tubular neighborhood in $\mathcal{X}$. One has

$$\partial T\left(\bigcup E_i\right)-\mathcal{D}=\bigcup_{(i_1,\dots,i_k)\subset I}\partial T(E^{\circ}_{i_1,\dots,i_k})-\tilde{\mathcal{D}} \tag{26}$$

For each character $\chi\in \mathrm{Char}\pi_1(\mathcal{X}-\mathcal{D},\mathcal{L})$, let $\chi_{\partial T(E^{\circ}_{i_1,\dots,i_k})-\tilde{\mathcal{D}}}$ be the induced character of $\pi_1(\partial T(E^{\circ}_{i_1,\dots,i_k})-\tilde{\mathcal{D}})$. Let, as above, I_χ be the collection of components of exceptional divisor for which $\chi_{\partial T(E_i)}$ is a pull back of a character $\chi_{E_i^{\circ}-\mathcal{D}\cap E_i^{\circ}}$ (i.e., which restriction of the boundary of transversal to E_i circle is the trivial character: cf. (3.3); we shall call such components *fiber χ-trivial*). The characters of $\pi_1(E^{\circ}_{i_1,\dots,i_k}-E^{\circ}_{i_1,\dots,i_k}\cap\mathcal{D})$, i.e., classes in $H^1(E^{\circ}_{i_1,\dots,i_k}-E^{\circ}_{i_1,\dots,i_k}\cap\mathcal{D},\mathbb{C}^*)$ obtained from the characters of $\tilde{\mathcal{X}}-\bigcup_{i\in I}E_i\cup\tilde{\mathcal{D}}$ (and hence compatible with restrictions), define as a result of degeneration of Mayer Vietoris spectral sequence, the characters of $\pi_1(\bigcup_{i\in I_\chi}E_i)$ and hence the local system on $T(\bigcup_{i\in I_\chi}E_i)$ which we also denote $\mathcal{L}_\chi$.

Two Mayer Vietoris spectral sequences:

$$\begin{aligned}E_1^{p,q} &= H^p\left(\bigcup_{(i_1,\dots,i_{q-1})\in I}\partial T(E^{\circ}_{i_1,\dots,i_{q-1}})-\tilde{\mathcal{D}},\mathcal{L}_{\chi_{T(E^{\circ}_{i_1,\dots,i_{q-1}})-\tilde{\mathcal{D}}}}\right)\\ &\Rightarrow H^{p+q}\left(\partial T\left(\bigcup E_i\right)-\partial T\left(\bigcup E_i\right)\cap\mathcal{D},\mathcal{L}_\chi\right)\end{aligned} \tag{27}$$

and

$$\begin{aligned}E_1^{p,q} &= H^p\left(\bigcup_{(i_1,\dots,i_{q-1})\in I_\chi}\partial T(E^{\circ}_{i_1,\dots,i_{q-1}})-\tilde{\mathcal{D}},\mathcal{L}_{\chi_{T(E^{\circ}_{i_1,\dots,i_{q-1}})-\tilde{\mathcal{D}}}}\right)\\ &\Rightarrow H^{p+q}\left(\partial T\left(\bigcup_{i\in I_\chi E_i}\right)-\partial T\left(\bigcup E_i\right)\cap\mathcal{D},\mathcal{L}_{\chi_{\partial T(\bigcup_{i\in I_\chi}E_i)-\partial T\left(\bigcup E_{i\in I_\chi}E_i\right)\cap\mathcal{D}}}\right)\end{aligned} \tag{28}$$

yield that it is enough to show the existence of the Mixed Hodge structure on the cohomology of local systems of the boundary of the neighborhood only for those components for which the restriction of χ on the boundary is the pullback from this component.

In order to describe the cohomological $\mathbb{C}$-Hodge complex $(A,F,\bar{F},W)$ on $\mathcal{X}-\mathcal{D}$ with cohomology being the cohomology of the abutment of the spectral sequence (28) first let us describe the complex for each intersection $\partial T(E^{\circ}_{i_1,\dots,i_{q-1}})-\tilde{\mathcal{D}}$

and the local system $\mathcal{L}_{\pi^*(\chi)}$ pulled back to it from the base. If $\mathbb{T}(E_{i_1,\dots,i_{q-1}})$ is defined as the projectivization of the total space of the direct sum of the normal bundle to $E_{i_1,\dots,i_{q-1}}$ in $\mathcal{X}$ with $\mathcal{O}_{E_{i_1,\dots,i_{q-1}}}$ and $\iota_{i_1,\dots,i_{q-1}}$ is the embedding of the neighbourhood of $E_{i_1,\dots,i_{q-1}}$ in $\mathbb{T}(E_{i_1,\dots,i_{q-1}})$ then $\mathbb{T}(E_{i_1,\dots,i_{q-1}})$ is a compactification of the total space of the normal bundle of $E_{i_1,\dots,i_{q-1}}$ in $\mathcal{X}$. $\mathbb{T}(E_{i_1,\dots,i_{q-1}})$ is also a compactificaiton of $\mathbb{C}^*$-fibration over $E^{\circ}_{i_1,\dots,i_{q-1}} - E^{\circ}_{i_1,\dots,i_{q-1}} \cap \tilde{\mathcal{D}}$ and the complement in it to the total space of this fibration is a normal crossing divisor which we denote as $D_{i_1,\dots,i_{q-1}} \subset \mathbb{T}(E_{i_1,\dots,i_{q-1}})$. This fibration is homotopy equivalent to the punctured neighbourhood of $E^{\circ}_{i_1,\dots,i_{q-1}} - \tilde{\mathcal{D}}$ in $\mathcal{X}$. We can use the just described compactification to construct (following [1]) log-complex associated with the (Deligne's extension) $\bar{V}$ of the connection with the holonomy given by ρ. We consider trifiltered real analytic log-complex:

$$A^{\bullet}_{i_1,\dots,i_q} = (A^{\bullet}_{\mathbb{T}(E_{i_1,\dots,i_{q-1}})}(\log D_{i_1,\dots,i_{q-1}}) \otimes \bar{V}, F, \bar{F}, W) \tag{29}$$

used in [1] and generalizing the complex considered in [27] in the case $\bar{V}$ is trivial and providing supporting conjugation analog on the log-complex with connection constructed in [31]. The log-complex (29) is quiasiisomorphic to direct image of its restriction on punctured neighbourhood $T(E_{i_1,\dots,i_{q-1}})$ (i.e. with deleted zero-section) which we denote

$$(\tilde{A}^{\bullet}_{i_1,\dots,i_q} \otimes \bar{V}, F, \bar{F}, W) \tag{30}$$

Here $F, \bar{F}, W$ are the filtrations induced by corresponding filtrations of (29) One has the maps $\delta : \tilde{A}^{\bullet}_{i_1,\dots,i_q} \to \tilde{A}^{\bullet}_{i_1,\dots,i_q,i_{q+1}}$ induced by inclusion of punctured neighbourhoods which allow to form differential graded complex

$$\tilde{A}^{\bullet,\bullet} = \oplus_{i_1,\dots,i_{q-1}} \tilde{A}^{\bullet}_{i_1,\dots,i_{q-1}} \tag{31}$$

The total complex $T^n = \oplus_{i+j=n} \tilde{A}^{i,j}$ with the usual differential $d + (-1)^j \delta_j$ and filtrations $F^p T^n = \oplus_{i \geq p, i+j=n} \tilde{A}^{i,j}$, $\bar{F}^p = \oplus_{j \geq p, i+j=n}$, $W_k T = \oplus W_{k+i} \tilde{A}^{i,j}$ (cf. [2,27]) is a $\mathbb{C}$-mixed Hodge complex calculating the abuttment of the spectral sequence (28) and hence (27) i.e. yielding the Mixed Hodge structure on the cohomology of the abutment. Compatibility with holomorphic maps follows form the corresponding compatibility in quasiprojective case. □

Remark 3.6. Assume that one has an embedding $(\mathcal{X}, \mathcal{D}) \subseteq (\bar{\mathcal{X}}, \bar{\mathcal{D}})$ such that $(\bar{\mathcal{X}}, \bar{\mathcal{D}})$ is quasiprojective and such that χ is a pullback of a character of $\pi_1(\bar{\mathcal{X}} - \bar{\mathcal{D}})$. For a germ on an isolated non-normal crossing in $\mathbb{C}^n$, the extension of each irreducible component $\mathcal{D}$ to an irreducible hypersurface in $\mathbb{C}^n$ yields such $(\bar{\mathcal{X}}, \bar{\mathcal{D}})$. The mixed Hodge structure of the Theorem 3.2 on $H^i(\mathcal{L}_\chi)$ where $\mathcal{L}_\chi$ is the local system on the boundary of a punctured neighbourhood of union of fiber χ-trivial exceptional divisors minus $\mathcal{D}$ can be constructed using methods of [13] (by above, these cohomology coincide with $H^i(\mathcal{X} - \mathcal{D}, \mathcal{L}_\chi)$). Indeed, such a punctured neighbourhood is the intersection of regular neighbourhood of union of fiber χ-trivial divisors and $\bar{X} - \bar{D}$. Both spaces support the canonical ($\mathbb{C}$-) mixed Hodge structure. In particular, the cohomology of such a local system on the boundary of a neighborhood of union of fiber χ-trivial exceptional divisors can be calculated using hypercohomology of the mapping cone of the following bifiltered complexes of sheaves corresponding to presentation $T(\bigcup_{i\in I_\chi} E_i - \mathcal{D}\cap\bigcup_{i\in I_\chi} E_i)$ as an intersection of a quasiprojective variety containing $\tilde{\mathcal{X}}-\tilde{\mathcal{D}}$ (i.e., the resolution of singularities $\tilde{\bar{\mathcal{X}}}$, $\tilde{\bar{\mathcal{D}}}$ of $\bar{\mathcal{X}}$, $\bar{\mathcal{D}}$ in which preimage of $\mathcal{X}$, $\mathcal{D}$ is $\tilde{\mathcal{X}}$, $\tilde{D}$) and a regular neighborhood of $\bigcup_{i\in I_\chi} E_i - \mathcal{D}\cap\bigcup_{i\in I_\chi} E_i$ (this construction in a standard way can be upgraded to the level of mixed Hodge complexes). The first complex has as its hypercohomology the the cohomology of the regular neighborhood of $T(\bigcup_{i\in I_\chi} E_i) - \mathcal{D}$ and is given by:

$$\Omega^{\cdot}_{\bigcup_{i\in I_\chi} E_i}(\log\mathcal{D}) = \oplus_{p+q=n}(\pi_q)_*\Omega^p_{E^{[q]},i\in I_\chi)}(\log(\mathcal{D}\cap\cup E))\otimes\mathcal{V}_\chi \tag{32}$$

i.e., the single complex associated to the double complex with components being the push forward to $\bigcup_{i\in I_\chi} E_i$ of log forms with the values in the induced (from $\tilde{\bar{\mathcal{X}}}$) the Deligne's extension on all $q+1$ fold intersections $E^{[q]}$ of E_i, $i\in I_\chi$ relative to divisors on connected components of $E^{[q]}$ induced by intersections with $\mathcal{D}$ and with differential given by the log-connection.

The second complex, having as its hypercohomology the cohomology of the complement to $\bigcup_{i\in I_\chi} E_i\cup\mathcal{D}$ in the resolution of singularities of quasiprojective variety $\bar{X}$, is given by:

$$\Omega^{\cdot}_{\tilde{\bar{\mathcal{X}}}}(\log\cup_{i\in I_\chi} E_i\cup\mathcal{D})\otimes\mathcal{V}_\chi \tag{33}$$

The Hodge filtration and weight filtrations are given in the usual way (i.e., by truncation and by considering forms with vanishing m-residues, respectively).

The construction in the theorem yields:

Corollary 3.7. *The above mixed Hodge structure is functorial in the following sense. Consider the homomorphism*:

$$h^k(f): H^k(\widetilde{\mathcal{X}-\mathcal{D}},\mathcal{V})\to H^k(\mathcal{X}-\mathcal{D}, f_*(\mathcal{V}))$$

induced by an unbranched covering $f:\widetilde{\mathcal{X}-\mathcal{D}}\to\mathcal{X}-\mathcal{D}$ *(here* $f_*(\mathcal{V})$ *is the direct image of the local system, corresponding to the induced character of subgroup* $\pi_1(\widetilde{\mathcal{X}-\mathcal{D}})$ *of* $\pi_1(\mathcal{X}-\mathcal{D})$*). Then* $h^k(f)$ *is a morphism of mixed Hodge structures.*

The above Proof of Theorem 3.1 allows one to deduce the following property of the mixed Hodge structures described in Theorem 3.2:

Corollary 3.8. *Let* $\mathcal{X}$, $\mathcal{D}$ *be as above. Then the subset of a fundamental domain in the universal cover* $\mathcal{U}$ *of* $\mathrm{Char}^u(\pi_1(\mathcal{X}-\mathcal{D})$ *given by*:

$$\{u\in\mathcal{U}|\dim Gr^p_F H^n(\mathcal{X}-\mathcal{D}, V_{exp(u)})\geq l\} \tag{34}$$

is a finite union of polytopes.

Proof. It follows from the degeneration of the Mayer-Vietoris spectral sequence mentioned in the Proof of Theorem 3.1, the compatibility of d_1 in it with the mixed Hodge structures and the theorem in the previous section. □

This extends the results of [22] on Hodge decomposition of characteristic varieties of germs of plane curves where also examples of such "Hodge decomposition" of $S_l^{1,p}$ in this case are given.

Remark 3.9. The construction of the mixed Hodge structure in the local case also can be extended to the case of cohomology of a unitary local system, i.e. to the context of [31]. The Theorem 3.1 can be modified in an obvious way to include a twisting by a higher rank local system as in Theorem 2.1.

4. Isolated non-normal crossings

Now we shall apply the results of previous section to the case of isolated non-normal crossings. First recall the following, already mentioned in the introduction:

Definition 4.1. (cf. [11,23]) An isolated non-normal crossing (INNC) is a pair $(\mathcal{X}, \mathcal{D})$ where $\mathcal{X}$ is a germ of a complex space $\mathcal{X}$ having $\dim\mathcal{X} - 2$-connected link $\partial B \cap \mathcal{X}$ where B is a small ball about $P \in \mathcal{D}$[6] and where $\mathcal{D}$ is a divisor on $\mathcal{X}$ which has only normal crossings at any point of $\mathcal{X} - P$.

The global isolated-non normal crossings divisors were considered in [24] where the homotopy groups of the complement were related to the local invariants which are certain polytopes. The goal of this section is to relate them to the polytopes discussed in this paper.

For the local INNCs as in 4.1, one has the following homotopy vanishing theorem (cf. [23, Th.2.2], [11]):

Theorem 4.2. *If* $(\mathcal{X}, \mathcal{D})$ *is an INNC,* $\dim\mathcal{X} = n + 1$ *and* r *is the number of irreducible components in* $\mathcal{D}$ *then*:

$$\pi_1(\mathcal{X} - \mathcal{D}) = \mathbb{Z}^r, \quad \pi_i(\mathcal{X} - \mathcal{D}, x) = 0 \quad \text{for } 2 \le i \le n - 1$$

The main invariant of local INNCs is $\pi_n(\mathcal{X} - \mathcal{D})$ considered as a $\pi_1(\mathcal{X} - \mathcal{D})$-module. It follows from the Theorem 4.2 that this homotopy group is isomorphic as a $\mathbb{Z}[\pi_1(\mathcal{X} - \mathcal{D})]$-module to the homology of the infinite abelian cover $H_n(\widetilde{\mathcal{X} - \mathcal{D}}, \mathbb{Z})$. The $\pi_1(\mathcal{X} - \mathcal{D})$-module structure on the latter is given by the action of the fundamental group on the homology of the universal cover.

Definition 4.3. (cf. [20,21]) l-th characteristic variety of $(\mathcal{X}, \mathcal{D})$ is the reduced support of $\pi_1(\mathcal{X} - \mathcal{D})$ module $\Lambda^l(\pi_n(\mathcal{X} - \mathcal{D}) \otimes_{\mathbf{Z}} \mathbb{C})$. In other words:

$$S_l(\mathcal{X}, \mathcal{D}) = \{\wp \in \operatorname{Spec}\mathbb{C}[\pi_1(\mathcal{X} - \mathcal{D})] = \mathbb{C}^{*r} \mid (\Lambda^l(\pi_n(\mathcal{X} - \mathcal{D}) \otimes_{\mathbf{Z}} \mathbb{C}))_{\wp} \neq 0\} \tag{35}$$

(here subscript $\wp$ denotes the localization in the prime ideal $\wp$).

6 For example a germ of a complete intersection with isolated singularity.

INNCs are reducible analogs of isolated singularities corresponding to the case $r = 1$. The homotopy vanishing in Theorem 4.2 is equivalent to the Milnor's theorem asserting that the Milnor fiber of an isolated singularity of a hypersurface having dimension n is $(n-1)$ connected (cf. [25]). Indeed, by Milnor's fibration theorem one has a locally trivial fibration: $\phi : \mathcal{X} - \mathcal{D} \to S^1$. Clearly, the identities of the Theorem 4.2 are equivalent to the requirement that the fiber F of ϕ satisfies: $\pi_i(F) = 0 \ \ 0 \le i \le n-1$. Moreover, in this case (i.e. when $r = 1$) the variety $S_1 \subset \mathbb{C}^*$ is the collection of eigenvalues of the monodromy of Milnor fibration ϕ.

Note that since the universal cover $\widetilde{\mathcal{X} - \mathcal{D}}$ of $\mathcal{X} - \mathcal{D}$ is also a covering space of the Milnor fiber and since the latter has the homotopy type of an n-complex, it follows that the universal cover of $\mathcal{X} - \mathcal{D}$ has the homotopy type of a bouquet of n-spheres (cf. Remark 4.7 in [23]).

The characteristic varieties are equivalent to the loci considered in the Theorem 3.1:

Proposition 4.4. [11,23] *A local system* $\chi \in \mathrm{Char}(\pi_1(\mathcal{X} - \mathcal{D}))$, $\chi \neq \mathbf{1}$ *corresponds to a point* $S_l \subset \mathrm{Spec}\mathbb{C}[\pi_1(\mathcal{X} - \mathcal{D})]$ *after the canonical identification* $\mathrm{Char}(\pi_1(\mathcal{X} - \mathcal{D})) = \mathrm{Spec}\mathbb{C}[\pi_1(\mathcal{X} - \mathcal{D})]$ *if and only if*

$$\dim H^n(\mathcal{L}_\chi) \ge k$$

i.e. the jumping loci of the cohomology of local systems coincide with the supports of the homotopy group π_n.

As result we obtain the following:

Corollary 4.5. *The components of characteristic variety are translated subgroups by torsion points.*

This corollary can be viewed as a generalization of the classical monodromy theorem since in the case $r = 1$ this is equivalent to the claim that an eigenvalue of the monodromy operator is a root of unity.

Next we shall interpret the results of previous sections in terms of mixed Hodge theory of abelian covers.

As above, for a group G acting on a vector space V and a character $\chi \in \mathrm{Char}G$ we shall denote by V_χ the subspace $\{v \in V | g \cdot v = \chi(g)v\}$. The following is a direct generalization of Prop. 4.5 and 4.6 in [23] on homology of branched and unbranched covers and the relation with cohomology of local systems in the case of curves discussed in [16,21].

Proposition 4.6. a) *Let* $X = (\mathcal{X} - \mathcal{D}) \cap \partial B$ *(where* ∂B *is a small ball about the point of non normal crossing of* $\mathcal{D}$*). Let* $U_{\mathbf{m}} \to X$ $(\mathbf{m} = (m_1, \ldots, m_r))$ *be the abelian covering corresponding to the homomorphism* $\pi_1(X) = \mathbb{Z}^r \to \oplus \mathbb{Z}/m_i\mathbb{Z}$. *For* $\bar{\omega} = (\ldots, \omega_i, \ldots) \in \mathbb{C}^{*r}$ *let*

$$f(\bar{\omega}, X) = \max\{l | \bar{\omega} \in S_l(X)\} \tag{36}$$

Then

$$H_k(U_{\mathbf{m}}, \mathbb{Z}) = \Lambda^k(\mathbb{Z}^r) \quad (1 \le k < n) \tag{37}$$

$$\mathrm{rk}\, H_n(U_{\mathbf{m}}, \mathbb{Q}) = \sum_{\bar{\omega}, \omega_i^{m_i} = 1} f(\bar{\omega}, X) \tag{38}$$

b) *Let $V_{\mathbf{m}}$ be the branched covering space of $\mathcal{X} \cap \partial B$ branched over $\mathcal{D} \cap \partial B$. For a character $\chi \in \mathrm{Char} H_1(\mathcal{X} - \mathcal{D}, \mathbb{Z})$ let*

$$I_\chi = \{i | 1 \le i \le r \text{ and } \chi(\gamma_i^{j_i}) \ne 1 \text{ for some } j_i, 1 \le j_i < m_i\} \tag{39}$$

where $\gamma_i \in H_1(\mathcal{X} - \mathcal{D}, \mathbb{Z})$ is represented by the boundary of a small disk in $\mathcal{X}$ transversal to the component D_i of $\mathcal{D}$. Then χ can be considered as the character of $H_1(\mathcal{X} - \cup_{i \in I_\chi} D_i, \mathbb{Z})$ and

$$H_i(V_{\mathbf{m}}, \mathbb{Z}) = 0 \quad 1 \le i \le n-1$$

$$H_n(V_{\mathbf{m}}, \mathbb{Z}) = \sum_{\chi \in \mathrm{Char}(\oplus_i \mathbb{Z}/m_i\mathbb{Z})} f(\chi, \mathcal{X} - \cup_{i \in I_\chi} D_i) \tag{40}$$

If the components D_i of $\mathcal{D}$ are zeros of holomorphic functions: f_i for $i = 1, \ldots, r$, respectively, and $\mathcal{X} \subset \mathbb{C}^N$ as above, then $V_{\mathbf{m}}$ has the realization as the link of complex space $\mathcal{X}_{\mathbf{m}} \subset \mathcal{X} \times \mathbb{C}^r \subset \mathbb{C}^N \times \mathbb{C}^r = \{(\mathbf{x}, z_1, \ldots, z_r) | \mathbf{x} \in \mathbb{C}^N, (z_1, \ldots, z_r) \in \mathbb{C}^r\}$ such that:

$$z_1^{m_1} = f_1(\mathbf{x}), \ldots, z_r^{m_r} = f_r(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X} \tag{41}$$

Proposition 4.7. *Let $G = \oplus \mathbb{Z}/m_i\mathbb{Z}$ and let $\chi \in \mathrm{Char} G \subset \mathrm{Char} \mathbb{Z}^r$. The group G acts on $H_n(U_{\mathbf{m}}, \mathbb{Q})$ and*

$$H_n(U_{\mathbf{m}}, \mathbb{Q})_\chi = H_n(\mathcal{L}_\chi) \tag{42}$$

If $\chi(\gamma_i^{j_i}) \ne 1$ for $1 \le i \le r$ where γ_i, j_i as in the Proposition 4.6(b) then there is canonical isomorphism:

$$H_n(V_{\mathbf{m}}, \mathbb{Q})_\chi = H_n(\mathcal{L}_\chi) \tag{43}$$

Next we have the following corollary of the functoriality of the mixed Hodge structure:

Proposition 4.8. *The Mixed Hodge structure on the cohomology of an abelian cover with a finite Galois group G of a complement $\mathcal{X} - \mathcal{D}$ to an INNC determines the mixed Hodge structure on the cohomology of essential* (*cf.* [23]) *local systems of finite order via*:

$$h^{p,q}(\tilde{\mathcal{X}}_G)_\chi = h_\chi^{p,q}((\mathcal{X} - \mathcal{D})_G) = h^{p,q}(\mathcal{L}_\chi)$$

Here $(\mathcal{X} - \mathcal{D})_G$ is the Galois cover with group G and $\tilde{\mathcal{X}}_G$ is the corresponding branched cover.

Proof. We have the isomorphism $H^n((\mathcal{X} - \mathcal{D})_G)_\chi = H^n(\mathcal{L}_\chi)$ (for example one can use a chain description of the cohomology of local systems cf.[21] or the next section for twisted version of it) which by Corollary 3.7 is compatible with both filtrations. The first equality follows from the assumption that χ is essential. □

This yields the following calculation of the Hodge numbers of abelian covers:

Proposition 4.9. $\dim Gr_F^p H^n(U_{\mathbf{m}})_\chi = \max\{ l \mid \chi = exp(u) \text{ where } u \in S_l^{n,p}\}$. *If* χ *is as in Proposition 4.7 then this is also equal to* $\dim Gr_F^p H^n(V_{\mathbf{m}})_\chi$.

In particular the function $f(m_1, \dots, m_r) = \dim Gr_F^p H^n(U_{\mathbf{m}})$ is polynomial periodic with the degrees of polynomial depending on the dimensions of the polytopes $S_l^{n,p}$. This result for the Hodge groups $H^{p,0}$ was obtained in [3].

5. Twisted characteristic varieties

In this section we describe a multivariable generalization (cf. Definition 5.1) of the twisted Alexander polynomials considered in [4] and relate it to the subvarieties $S_\rho^{n,l}$ described in Theorem 2.1. It is similar to the way the characteristic varieties generalize the Alexander polynomials (cf. [21]). The Theorem 5.4 extends the cyclotomic property of the roots of Alexander polynomials of plane algebraic curves (cf. [19]) to twisted case. We use Theorem 5.4 to describe a group theoretical property of the fundamental groups which may not be shared by other groups. We also use twisted characteristic variety to obtain information on the homology of non unitary local systems (cf. Proposition 5.3).

Let X be a finite CW complex and $\tilde{X}$ (resp. $\tilde{X}_{ab}$) denotes the universal cover of X, (resp. its universal abelian cover). As usual $\pi_1'(X) = \pi_1(\tilde{X}_{ab})$ be the commutator of the fundamental group of X. Let $C_*(\tilde{X})$ be a chain complex of $\tilde{X}$ with $\mathbb{C}$-coefficients. We shall assume that each graded component is *a right* free $\mathbb{C}[\pi_1(X)]$-module. For a $\mathbb{C}[\pi_1(X))]$-module M we shall denote by M_{ab} the $\mathbb{C}[\pi_1'(X)]$-module obtained by restricting the ring $\mathbb{C}[\pi_1(X)]$ to $\mathbb{C}[\pi_1'(X)]$. A unitary representation $\rho : \pi_1(X) \to U(V)$ yields the left $\mathbb{C}[\pi_1'(X)]$-module V_{ab} (here $U(V)$ is the unitary group of a hermitian vector space V). The modules $C_*(\tilde{X}) \otimes_{\pi_1'(X)} V_{ab}$ with the action of $g \in \pi_1(X, \mathbb{Z})$ given by $g(c \otimes v) = cg^{-1} \otimes g \cdot v$ form a complex of $\mathbb{C}[H_1(X, \mathbb{Z})]$ modules which results in the structure of a $\mathbb{C}[H_1(X, \mathbb{Z})]$-module on the homology of the abelian cover $H_i(\tilde{X}_{ab}, V_{ab})$ with coefficients in the local system V_{ab} (obtained by restriction of ρ on the commutator of the fundamental group). Recall that we assume that $H_1(X, \mathbb{Z})$ is torsion free and hence $\mathrm{Spec}\mathbb{C}[H_1(X, \mathbb{Z})]$ is connected: the general case requires only obvious modifications.

Definition 5.1. The twisted by ρ l-th characteristic variety (in degree n) (denoted as Ch_n^l) is the support of the module

$$\Lambda^l H_n(\tilde{X}_{ab}, V_{ab})$$

i.e., the subset of $\mathrm{Spec}\mathbb{C}[H_1(X, \mathbb{Z})]$ consisting of prime ideals $\wp$ in $\mathbb{C}[H_1(X, \mathbb{Z})]$ such that localization of $\Lambda^l H_n(\tilde{X}_{ab}, V)$ at $\wp$ is non zero.

Let us in addition fix a surjection $\epsilon : \pi_1(X) \to \mathbb{Z}$, and denote $\bar{\pi} = \mathrm{Ker}\, \epsilon$. Let $\rho : \pi_1(X) \to U(V)$ be a unitary representation. With such data one associates a twisted Alexander polynomial and the homology torsion which both were studied in [4, 17] (cf. references there for the prior work). The main geometric situations in which these invariants appear are the case when X is a complement to a knot (cf. [17]) and the case of the complement to a plane algebraic curve in $\mathbb{P}^2$ (cf. [4]).

Recall that the twisted Alexander polynomials and the torsion corresponding to the data (X, ϵ, ρ) are defined as follows. Consider the covering space $X_{\bar{\pi}}$ corresponding to the subgroup $\bar{\pi}$ and the local system $V_{\bar{\pi}}$ on the latter obtained by restricting ρ to $\bar{\pi}$. The k-th Alexander polynomial $\Delta_k(t)$ is the order of the $\mathbb{C}[\mathbb{Z}]$-module $\mathrm{Tor}\, H_k(\tilde{X}_{\bar{\pi}}, V_{\bar{\pi}})$ with the convention that the order of a trivial module is 1 (cf. [17, Def. 2.3]). The homology torsion is defined as:

$$\Delta_{\epsilon,\rho}(t) = \Pi_k \Delta_k(t)^{(-1)^{k+1}} \tag{44}$$

In other words, $\Delta_k = \Pi a_{k,i}(t)$ where $\mathrm{Tor}\, H_k(\tilde{X}_{\bar{\pi}}, V_{\bar{\pi}}) = \oplus_i (\mathbb{C}[\mathbb{Z}]/(a_{k,i}(t)))$ ($a_{k,i}(t)$ divides $a_{k,i+1}(t)$) is the cyclic decomposition.

If X has a homotopy type of a 2-dimensional complex such that $H_1(X, \mathbb{Z}) = \mathbb{Z}$ (which is the case for the complements to irreducible algebraic curves in $\mathbb{P}^2$ or knots in S^3) and ϵ is the abelianization: $\pi_1(X) \to H_1(X, \mathbb{Z})$ (i.e. $\bar{\pi}$ is the commutator π_1'), then the homology torsion is just $\frac{\Delta_1(t)}{\Delta_0(t)}$ since the $\mathbb{C}[\mathbb{Z}]$- module $H_k(\tilde{X}_{\pi_1'}, V_{\pi_1'})$ is free for $k = 2$ (and trivial for $k > 2$). To be more specific, the chain complex calculating $H_k(\tilde{X}_{\pi_1'}, V_{\pi_1'})$ is given by:

$$0 \to C_2(\tilde{X}) \otimes_{\pi_1'} V_{\pi'} \to C_1(\tilde{X}) \otimes_{\pi_1'} V_{\pi'} \to C_0(\tilde{X}) \otimes_{\pi_1'} V_{\pi'} \to 0 \tag{45}$$

Since the cell structure of $\tilde{X}$ can be chosen so that all $C_i(\tilde{X})$ are free $\pi_1(X)$-modules and since the structure of $\mathbb{C}[\mathbb{Z}]$-module on $\mathbb{C}[\pi_1(X)] \otimes_{\pi_1'} V_{\pi_1'}$ yields a free $\mathbb{C}[\mathbb{Z}]$ module this implies that $H_2(\tilde{X}_{\pi_1'}, V_{\pi_1'}) \subseteq C_2(\tilde{X}) \otimes_{\pi_1'} V_{\pi_1'}$ is also free. Hence $\Delta_2(t) = 1$. Moreover, $\Delta_0(t)$ can be calculated in terms of ρ as follows. Denoting by $\partial_i : C_i(\tilde{X}) \to C_{i-1}(\tilde{X})$ the boundary operator of the chain complex of the universal cover, one has $H_0(\tilde{X}) \otimes_{\pi_1'} V_{\pi_1'} = \mathrm{Coker}\partial_1 \otimes \mathrm{id}$, $C_0(\tilde{X}) = \mathbb{C}[\pi_1(X)]$, and $\mathrm{Im}\partial_1$ is the augmentation ideal of the group ring $\mathbb{C}[\pi_1(X)]$. Hence $H_0(\tilde{X}_{\pi_1'}) \otimes_{\pi_1'} V_{\pi_1'} = \mathbb{C} \otimes_{\pi_1'(X)} V_{\pi_1'}$ where the action of $\pi_1'(X)$ on $\mathbb{C}$ is trivial. This implies that the latter is isomorphic to $\pi_1'(X)$ covariants of V. The $\mathbb{C}[\mathbb{Z}]$-module structure is given by the action of the generator of π_1/π_1' on the π_1' covariants $V^{\pi_1'}$ of V. For example, if ρ has an abelian image, i.e. $\pi_1' \subseteq \mathrm{Ker}\rho$, then $V^{\pi_1'} = V$ and Δ_0 is the characteristic polynomial of the generator of $\mathbb{Z}$ acting on V (cf. [17]).

Returning back to the case of an arbitrary CW-complex X, the relation between the twisted Alexander polynomial and twisted characteristic varieties can be stated as follows:

Proposition 5.2. *Let X be a CW complex such that $H_1(X, \mathbb{Z}) = \mathbb{Z}$, $\rho : \pi_1(X) \to U(V)$ is a unitary local system and $\mathrm{Ch}_n^l \subseteq \mathrm{Spec}\mathbb{C}[\mathbb{Z}] = \mathbb{C}^*$ is the collection of characteristic varieties associated with (X, ρ) in Definition 5.1. For $\xi \in \mathbb{C}^*$ let $l_n(\xi) = \max\{l \mid \xi \in \mathrm{Ch}_n^l(X, \rho)\}$ and $b_n = \min\{l_n(\xi) | \xi \in \mathrm{Spec}\mathbb{C}[\mathbb{Z}]\}$. Then $\Delta_n(\xi) = 0$ if and only if $l_n(\xi) > b_n$.*

Proof. Let us consider the cyclic decomposition of $R = \mathbb{C}[t, t^{-1}]$-module $H_n(X_{ab}, V_{ab})$:

$$H_n(X_{ab}, V_{ab}) = R^{\oplus c_n} \oplus R/(f_n^s) \tag{46}$$

Then b_n is equal to the rank c_n. Moreover for $\xi \in \operatorname{Spec} R = \mathbb{C}^*$ the integer $l_n(\xi)$ is equal to c_n plus the number of torsion summands in (46) having ξ as its root. This yields the claim. □

Proposition 5.3. *If $\pi_i(X) = 0$ for $2 \le i < k$ or if $k = 1$ then the dimension of the homology group $H_k(X, V \otimes L_\chi)$ corresponding to a character $\chi \in \operatorname{Char}\pi_1(X)$ and a local system ρ is given by*:

$$\max\{i \,|\, \chi \in \mathrm{Ch}^i_k(X, \rho)\}$$

Proof. Using the identification $C_*(\tilde{X}) \otimes_\pi (V \otimes_\pi L_\chi) = (C_*(\tilde{X}) \otimes_{\pi_1'} V) \otimes_{H_1(X)} \otimes L_\chi$ (recall that $C_*(\tilde{X}) \otimes_{\pi_1'} V$ has the structure of $\mathbb{C}[H_1(X, \mathbb{Z})]$-module: $l(c \otimes v) = (c\bar{l}^{-1} \otimes \bar{l}v)$ where $l \in H_1(X, \mathbb{Z})$ and $\bar{l}$ is its lift to π_1), the homology $H_*(X, V \otimes L_\chi)$ can be obtained as the abutment of the spectral sequence:

$$H_p(H_1(X, \mathbb{Z}), H_q(C_*(\tilde{X}_a b) \otimes_{\pi'} V) \otimes_{\mathbb{C}} L_\chi) \Rightarrow H_{p+q}(X, V \otimes L_\chi) \tag{47}$$

Considering the lower degree terms we have:

$$\begin{aligned} H_{k+1}(H_1(X, \mathbb{Z}), L_\chi) &\to H_k(\tilde{X}_{ab}, V)_{\chi, H_1(X,\mathbb{Z})} \to H_k(X, V \otimes_{\pi_1} L_\chi \\ &\to H_k(H_1(X, \mathbb{Z}), L_\chi) \end{aligned} \tag{48}$$

Since for $\chi \neq 1$ one has $H_k(H_1(X, \mathbb{Z}), \chi) = 0$. Using the interpretation of the dimension of the middle term via Fitting ideals we obtain the result. □

Theorem 5.4. *Let $X = \mathbb{P}^2 - C - L$ where C is an irreducible curve and L is a line at infinity. Let ρ be a unitary representation of the fundamental group and let F be the extension of $\mathbb{Q}$ generated by the eigenvalues of $\rho(\gamma)$ where γ is a boundary of a small disk transversal to C at its non singular point. Then the roots of $\Delta_\rho(C)$ belong to a cyclotomic extension of F.*

Proof. Let $f = 0$ be the equation of the curve and let $\omega = A\frac{df}{f}$ be the matrix of flat (logarithmic for a log resolution of the pair $(\mathbb{P}^2, C \cup L)$) connection, i.e. in a neighbourhood of any point in $\mathbb{P}^2$, except for those which are singularities of C, the connection is given by $\nabla_\chi(\cdot) = d \cdot + \omega \wedge \cdot$. Assume first that for a loop γ having the linking number with C equal to one we have:

$$\chi(\gamma) = exp\ 2\pi\sqrt{-1}\alpha \tag{49}$$

where $\alpha \in \mathbb{R}$, i.e. χ is unitary. If $\pi : (X', E) \to (\mathbb{P}^2, C \cup L)$ is the log resolution (i.e., E contains the exceptional set and the proper preimages of C and L) and $m_i = \mathrm{mult}_{E_i}\pi^*(f)$ then the residue of log extension of flat connection on $V \otimes L_\chi$ is equal to $m_i(A(0) + \alpha I)$. Hence the jumping loci are

$$m_i(\xi_{i,j} + \alpha) = n_i \tag{50}$$

for some integers n_i. Hence $(e^{2\pi\sqrt{-1}\alpha})\zeta_{m_i} = e^{2\pi\sqrt{-1}\xi_{i,j}}$ where $\zeta_{m_i}^{m_i} = 1$ and the claim follows.

If the character χ is not unitary, i.e. $\mathrm{Im}\alpha \neq 0$, can one see that $dim H^1(V \otimes L_\chi) = 0$ as follows. First notice that the Deligne's Hodge deRham spectral sequence (2) has in E_1 all terms equal to zero except possibly for the following:

$$\begin{array}{lll} H^2(X',\mathcal{V}_\chi) & 0 & 0 \\ H^1(X',\mathcal{V}_\chi) & H^1(X',\Omega^1(\log E)\otimes\mathcal{V}_\chi) & 0 \\ H^0(X',\mathcal{V}_\chi) & H^0(X',\Omega^1(\log E)\otimes\mathcal{V}_\chi) & H^0(X',\Omega^2(\log E)\otimes\mathcal{V}_\chi) \end{array} \tag{51}$$

Indeed, the terms above the diagonal $i + j = 2$ are zeros which can be seen as follows. This spectral sequence, for variable χ, depends only on the locally trivial bundle supporting the Deligne's extension of the connection corresponding to local system $V \otimes L_\chi$. This extension in turn does not depend on imaginary part $\mathrm{Im}\alpha$ (but differential a priori may be dependent on the latter). Since in the case $\alpha \in \mathbb{R}$ these terms are trivial due to degeneration of this spectral sequence in term E_1 and vanishing of $H^i(X, V \otimes L_\chi)$ for $i > 2$ the claim follows. This spectral sequence degenerates in term E_1 even if χ is not unitary since the differential d_1^α in the spectral sequence corresponding to the character χ given by (49) is the map induced on cohomology by the connection $\nabla_\chi : \Omega^i(\log E)\otimes\mathcal{V} \to \Omega^{i+1}(\log E)\otimes\mathcal{V}$ and hence depends on α linearly. Since it is trivial for $\alpha \in \mathbb{R}$ (i.e. in the unitary case) it is trivial for all α. The transgression d_2^α depends on α linearly also as can be seen from the following description of d_2^α: since for a form $\eta \in H^1(X',\mathcal{V}_\chi)$ the differential d_1^α takes its class to zero and hence we have $\nabla_\chi(\eta) = d\bar{\eta}$ where $\bar{\eta}$ is a 0-form in $\Omega^1(\log E)\otimes\mathcal{V}_\chi$ and $\nabla_\chi(\bar{\eta})$ represents the transgression.

Remark 5.5. Hodge-deRham spectral sequence may not degenerate in term E_1 for non-unitary local system (cf. [28]).

Remark 5.6. If $\dim V = 1$ (or, more generally, if $Im(\rho)$ is abelian), the twisted characteristic varieties coincide with the ordinary ones. If $\xi = \rho(\gamma)$ then the roots of $\Delta_\rho(t)$ are $\xi^{-1}\epsilon$ where ϵ is a root of ordinary Alexander polynomial (cf. also [17]) and the claim of 5.4 is immediate.

One way to obtain a local system of higher rank is the following. Let X be a CW-complex and let H be a normal subgroup of $\pi = \pi_1(X)$ such that $\mathrm{rk} H/H' < \infty$ (H' is the commutator subgroup of H). The action of π on H by conjugation preserves H' and hence we obtain an action on H/H'. Since for such action of π the subgroup H' acts trivially we also have the action of π/H. We shall denote by ρ_H the corresponding representation of π (or π/H) on $V_H = (H/H') \otimes \mathbb{C}$.

Theorem 5.7. *Let H be a finitely generated normal subgroup of $\pi = \pi_1(\mathbb{C}^2 - C)$ and $\gamma \in \pi$ an element in the conjugacy class of a loop which is the boundary of a small disk transversal to C. The eigenvalues of the action of γ on $(\pi' \cap H/(\pi' \cap H)') \otimes \mathbb{C}$ belong to a cyclotomic extension of the field F generated by the eigenvalues of $\rho_H(\gamma)$ acting on $(H/H') \otimes \mathbb{C}$.*

Proof. Recall that if G is the covering group of a Galois cover $f: Y \to X$ and V is a local system on X then one has the Leray spectral sequence

$$H_p(G, H_q(Y, f^*(V))) \Rightarrow H_{p+q}(X, V) \tag{52}$$

We shall consider the exact sequence of low degree terms corresponding to the spectral sequence of $G = \pi'/\pi' \cap H$ acting on the covering space $X_{\pi' \cap H}$ of $X_{\pi'}$ corresponding to the subgroup $\pi' \cap H \subset \pi'$:

$$H_p(G, H_q(X_{\pi' \cap H}, f^*(V_H))) \Rightarrow H_{p+q}(X_\pi, V_H) \tag{53}$$

After the identification $H_1(X_{\pi' \cap H}, f^*(V_H)) = H_1(X_{\pi' \cap H}) \otimes_{\mathbb{C}} V_H$ and $H_0(X_{\pi' \cap H}) = \mathbb{C}$ (since the pullback of the local system V_H on $X_{\pi' \cap H}$ is trivial because the restriction of ρ_H on $\pi' \cap H$ is trivial) we have:

$$H_2(G, \mathbb{C}) \to (H_1(X_{\pi' \cap H}) \otimes_{\mathbb{C}} V_H)_G \to H_1(X_{\pi'}, V_H) \to H_1(G, \mathbb{C}) \tag{54}$$

The eigenvalues of $\rho(\gamma)$ belong to the cyclotomic extension of F and since the action of G is trivial on $H_1(X_{\pi' \cap H})$) we obtain the claim. □

Remark 5.8. In the case when π/H is finite this result is not new: if so, then H is the fundamental group of a quasiprojective variety and the Alexander polynomial is cyclotomic in this case [24]. On the other hand this theorem imposes restrictions on finitely generated subgroups of the fundamental group and this appears to be a non-trivial restriction on this class of groups for which there exist a plane curve C such that $\pi = \pi_1(\mathbb{P}^2 - C)$.

6. Examples

6.1. Points in $\mathbb{P}^1$

Let us consider the case when X is the complement to $r+1$ distinct points in $\mathbb{P}^1$. Then $\pi_1(X)$ the free group on r generators and $\mathrm{Char}\pi_1(X) = \mathbb{C}^{*r}$. Let $(\alpha_1, \dots, \alpha_r) \in \mathcal{U}$ (cf. (3)) and not all $\alpha_i = 0$. Let $[-(\sum_i \alpha_i)] = -k$ $(1 \le k \le r)$. The Deligne's extension then is $\mathcal{O}_{\mathbb{P}^1}(-k)$. We have:

$$\dim H^0(\Omega^1(\log D)(-k)) = \dim H^0(\mathcal{O}(-2 + r + 1 - k)) = r - k \tag{55}$$

$$\dim H^1(\mathcal{O}(-k)) = k - 1 \tag{56}$$

Hence the dimensions of the graded components of Hodge filtration take constant values within the polytopes:

$$k < \sum \alpha_i \le k + 1 \tag{57}$$

with $\dim H^1(\mathcal{L}_\chi) = r - 1$.

6.2. Generic arrangement in $\mathbb{P}^2$

The cohomology of a non-trivial local system on the complement X to generic arrangement of $r+1$ lines are given as follows:

$$\mathrm{rk} H^2(X, \mathcal{L}_\chi) = 1 + \frac{r^2-3r}{2} \quad \mathrm{rk} H^i(X, \mathcal{L}_\chi) = 0 \quad (i \neq 2)$$

Together with the degeneration of the Hodge spectral sequence (2) this yields that each of the bundles of this spectral sequence in this case has only one non-vanishing cohomology group, i.e.

$$H^p(\Omega^q(\log D) \otimes \mathcal{V}) = (-1)^p e(\Omega^q(\log D) \otimes \mathcal{V}) \tag{58}$$

where $e(\mathcal{A})$ denotes the holomorphic euler characteristic of a bundle $\mathcal{A}$. Let us consider a local system for which the residues along r lines are $\alpha_1, \ldots, \alpha_r$ and

$$k - 1 < \sum \alpha_i \leq k \tag{59}$$

Then the Deligne's extension is $\mathcal{O}(-k)$ and we have:

$$e(\mathbb{P}^2, \mathcal{O}(-k)) = \frac{(k-1)(k-2)}{2} \tag{60}$$

$$e(\mathbb{P}^2, \Omega^2(\log D)(-k)) = e(\mathbb{P}^2, \mathcal{O}(r-2-k)) = \frac{(r-k-1)(r-k)}{2} \tag{61}$$

For the calculation of $e(\mathbb{P}^2, \Omega^1(\log D)(-k))$ let us use the Riemann-Roch. The logarithmic bundle has the following Chern polynomial (where $h \in H^2(\mathbb{P}^2, \mathbb{Z})$ is the generator and $c(\mathcal{A}) = \sum c_i(\mathcal{A})t^i$ and $c_i(\mathcal{A})$ are the Chern classes):

$$c(\Omega^1(\log D)) = (1 - ht)^{-r+2} \tag{62}$$

(cf. [8]). Therefore we have the following expression for the Chern character:

$$ch(\Omega^2(\log D)(-k) = 2 + (r-2-2k)h + \left(k^2 - (r-2)k - \frac{r-2}{2}\right) h^2 \tag{63}$$

Hence, since the Todd class $td(\mathbb{P}^2) = 1 + \frac{3}{2}h + h^2$, the Riemann Roch yields:

$$e(\mathbb{P}^2, \Omega^1(\log D)(-k)) = k^2 + r - k - rk \tag{64}$$

Hence we obtain the following:

Theorem 6.1. *For the generic arrangement in $\mathbb{P}^2$ one has the following*:

$$\dim Gr_F^0 = \frac{(k-1)(k-2)}{2}, \quad \dim Gr_F^1 = rk + k - r - k^2,$$
$$\dim Gr_F^2 = \frac{(r-k-1)(r-k)}{2}$$

6.3. Cone over generic arrangements

In the case of the cone over r points in $\mathbb{P}^1$ we obtain an ordinary plane curve singularity of multiplicity r which is the case discussed already in [22]. Consider now the cone over the generic arrangement in $\mathbb{P}^2$ considered in the previous example, i.e. the arrangement $C\mathcal{A}$ of $r+2$ planes in $\mathbb{P}^3$ with $r+1$ planes $H_1, \dots, H_{r+1}$ forming an isolated non-normal crossing and the remaining plane H_∞ (at infinity) being transversal to the first $r+1$ planes. One can make calculations on the blow up of $\mathbb{P}^3$ at the non-normal crossing point of the arrangement but from

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{P}^3 - C\mathcal{A}) \to \pi_1(\mathbb{P}^2 - \mathcal{A}) \to 0$$

one has the identification of $\mathrm{Char}\pi_1(\mathbb{P}^2 - \mathcal{A})$ with the subgroup of $\mathrm{Char}\pi_1(\mathbb{P}^3 - C\mathcal{A})$ in the standard coordinates in the latter given by

$$x_1 \cdots x_{r+1} = 1$$

From the Gysin sequence associated with $H_\infty - \cup_{i=1}^{i=r} H_i \cap H_\infty \subset \mathbb{P}^3 - \cup_{i=1}^{i=r} H_i$ we have the identification of the Hodge structures:

$$H^3(\mathbb{P}^3 - C\mathcal{A}, \mathcal{L}_\chi) = H^2(H_\infty - H_\infty \cap \cup H_i, \mathcal{L}_\chi) \tag{65}$$

Hence the polytopes $\mathcal{S}^2_k$ are given by $\alpha_1 + \cdots + \alpha_{r+1} = k$ and in each of these polytopes we have the value of $\dim Gr^k_F$ given by the Theorem 6.1.

6.4. Arrangement from a pencil of plane quadrics

Consider the arrangement of six lines $L_1, \dots, L_6$ composed of the three sides of triangle and the three cevians in it. Let H_∞ be the line at infinity. $\mathrm{Char}\pi_1(\mathbb{P}^2 - \cup L_i \cup H_\infty) = \mathbb{C}^{*6}$. The characteristic variety is given by

$$x_{i_1} \cdot x_{i_2} \cdot x_{i_3} = 1 \tag{66}$$

where (i_1, i_2, i_3) runs through triples having a triple point in common (cf. [21]). This is a two dimensional sub-torus in the torus of characters. Let us consider the characters $exp(u)$ where $u = (\alpha_1, \dots, \alpha_6)$ such that $\sum \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} = p + 1$ $(p = 0, 1)$ (i_1, i_2, i_3) as in (66). Let $\tilde{\mathbb{P}}^2$ be the blow up of $\mathbb{P}^2$ at the vertices of the triangle. The Deligne's extension is $\mathcal{O}(\sum_i (p+1)E_i - 2pH_\infty)$. We have using $K_{\tilde{\mathbb{P}}^2} = \sum E_i - 3H_\infty$ and Serre duality

$$\begin{aligned} H^1\left(\tilde{\mathbb{P}}^2, \mathcal{O}\left(\sum_i 2E_i - 4H_\infty\right)\right) &= H^1\left(\tilde{\mathbb{P}}^2, \mathcal{O}\left(-\sum E_i + H_\infty\right)\right) \\ &= H^1(\mathbb{P}^2, \mathcal{J}_B(H_\infty)) = 1 \end{aligned}$$

where $\mathcal{J}_B$ is the ideal sheaf of the set of the vertices of the triangle (the last equality follows from the Cayley-Bacharach theorem, cf. [21]). Similarly one obtains ($\mathcal{V}$ is the Deligne's extension):

$$Gr^0_F = \dim H^0(\tilde{\mathbb{P}}^2, \Omega^1(\log \cup E_i \cup H_\infty) \otimes \mathcal{V}) = 1$$

6.5. Arrangement from a net of quadrics in $\mathbb{P}^3$

Consider the arrangement $D_{8,4} = \cup_{i=1,\dots,8} H_i$ of eight planes in $\mathbb{P}^3$ introduced in [24]. Recall that these eight planes split in four pairs forming 4 quadrics belonging to a net. This net has eight base points which are the *only* eight non-normal crossings in the arrangement. Let H_∞ be a generic hyperplane in $\mathbb{P}^3$. One has $\mathrm{Char}\pi_1(\mathbb{P}^3 - D_{8,4} \cup H_\infty) = \mathbb{C}^{*8}$. Consider the subset in $\mathcal{U}$ (cf. Sect. 2.1) given by

$$\bar{S}_3^2: \quad \alpha_{i_1} + \cdots + \alpha_{i_4} = 3 \tag{67}$$

where $(i_1, \dots, i_4)$ runs through eight unordered collections of indices such that $H_{i_1}, \dots, H_{i_4}$ form a quadruple of planes containing one of eight non-normal crossings points. Note that the dimension (over $\mathbb{R}$) of the set of solutions is equal to three and the Zariski closure of this set is support of the homotopy group $\mathrm{Supp}\pi_2(\mathbb{P}^3 - D_{8,4}) \otimes \mathbb{C}$ considered in [24]. We claim that for any $u = (\alpha_1, \dots, \alpha_8) \in \bar{S}_3^2$ and for the corresponding character $exp(u)$ one has

$$\dim Gr_F^3 H^2(\mathbb{P}^3 - D_{8,4} \cup H_\infty, \mathcal{L}_{exp(u)}) = 1 \tag{68}$$

Indeed let $\tilde{\mathbb{P}}^3$ be the blow up of $\mathbb{P}^3$ in eight base points and $E_1, \dots, E_8$ the corresponding exceptional components. For a character $exp(u)$, where u belongs to (67), the Deligne's extension is $\mathcal{O}(3E_1 + \cdots + 3E_8 - 6H_\infty)$, since each of the eight planes in the arrangement contains four base points. The canonical class of $\tilde{\mathbb{P}}^3$ is $2E_1 + \cdots + 2E_8 - 4H_\infty$ and therefore we have:

$$\dim Gr_F^3 H^2(\mathbb{P}^3 - D_{8,4} \cup H_\infty, \mathcal{L}_{exp(u)}) = \dim H^2(\mathcal{O}((3E_1 + \cdots + 3E_8 - 6H_\infty)) \tag{69}$$

(using Serre duality)

$$= \dim H^1(\mathcal{O}(-E_1 - \cdots - E_8 + 2H_\infty) = \dim H^1(\mathbb{P}^3, \mathcal{J}_B(2H))$$

where $\mathcal{J}_B$ is the ideal sheaf of the base locus. Since B is the complete intersection of three quadrics in $\mathbb{P}^3$, the claim (68) follows from the Cayley-Bacharach theorem. In fact $\dim Gr_F^p = 1$ for $p = 0, 1$ as well and the corresponding characters are the exponents of u for which $\alpha_{i_1} = \cdots = \alpha_{i_r} = p + 1$.

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