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Development of the theory of Alexander invariants in Algebraic Geometry

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Abstract. This is an informal overview of the origins and history of the study of Alexander invariants in algebraic geometry.

> As I am neither a historian nor an archeologist of mathematics, I shall merely present my idea of major contributions in the subject..
>
> S.Lefschetz[1]

1. Introduction.

The study of Alexander polynomials is a very active area of mathematics playing an important role in many topological problems. Starting with the seminal paper of James Wadell Alexander [**2**], as of the date of this writing, a MatSciNet search generates 826 entries. There has been extensive research of the history of Alexander invariants in topology (cf. [**28, 41, 91**]). In this paper I will try to review the role of the Alexander polynomial and their generalizations in algebraic geometry.

In the latter, the Alexander polynomial is primarily a tool for understanding the topology of the complements to plane algebraic curves. Describing the origins of the studies of the complements one should probably start with the work of Enriques (cf. [**27**]) on multi-valued algebraic functions of several variables (or equivalently the finite covering spaces) since, it seems, they contain the earliest results on their fundamental groups of the complements. Enriques work was prompted by the search for a higher dimensional extension of the Riemann existence theorem. In the one variable case, Riemann showed the existence of a unique algebraic multivalued function on a projective line with a given branching locus and an arbitrary assignment of permutations σ_i of branches, except for being constrained by universal relation $\sigma_1 \cdot .... \cdot \sigma_n = 1$. In modern language, the reason for this single constraint is that the fundamental group of the complement to n points in $\mathbb{P}^1_{\mathbb{C}}$ is a free group with presentation containing n generators and a *single* relation

[1] cf. [**45**]

stating that product of generators is the identity. Enriques realized that in order to define a multivalued function of two variables branched along a curve $C \subset \mathbb{P}^2$, similarly to Riemann's case, one also need to assign permutations to intersections of C and the generic line. Now, however, besides the triviality of the product one has to satisfy additional relations among these permutations. Today we recognize Enriques relations among the permutations as the same as the relations satisfied by the generators of the fundamental group $\pi_1(\mathbb{P}^2 - C)$ which are also the generators of $\pi_1(\mathbb{P}^1 - C \cap \mathbb{P}^1)$ where $\mathbb{P}^1$ is a generic line in $\mathbb{P}^2$. These relations correspond to relation resulting from moving this generic line within a generic pencil along loops surrounding the lines in the pencil which are either tangent to C or contain singular points of C. Enriques work showed the subtlety of fundamental groups of the complements in algebraic geometry, albeit "algebraic" fundamental groups in the sense of Grothendieck, since Enriques was concern only with existence of *finite* coverings.

In his 1929 paper [**93**] O.Zariski, pursuing the program to study finite *cyclic* covers, showed the vanishing of irregularity, i.e., the absense of holomorphic 1-forms, of an n-fold cover of a projective plane branched over a singular curve in the case when n is a power of a prime. This work used constructs clearly bearing the appearance of knot polynomials. It is possible that he anticipated, and was occasionally ahead of topologists in sensing the role of polynomial invariants in the study of the complements (the idea of the formula for the first Betti number of a branched cyclic covering of a sphere should be credited to Zariski) but Alexander polynomials and their relation with the fundamental groups of complements did not enter his work. Though Zariski was under the strong influence of Lefschetz and met Alexander in November 1927 (cf. [**77**] chapter 6), he either learned about the Alexander polynomial only in March 1932 [2] from a rather roundabout source or preferred to avoid its use. In the footnote to [**95**] he writes:

After having communicated this theorem [3] at a meeting of American Mathematical Society in March 1932, I learned from Professor Lefschetz that the same theorem is proved by Werner Burau and that Burau's paper entitled "Kennzeichnung der Schlauchknoten" is being published in the forthcoming issue of the Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. The polynomial $F(x)$ in section (5) of this paper also obtained by Burau as the Alexander polynomial of of the knot (see. J.W.Alexander, "Topological invariants of Knots and Links", Transactions of American Mathematical Society, vol 30 (1928) p.273- 306.) This was to be expected since the connection between the homology characters of the k-sheeted manifolds with the knot as branch curve and the Alexander polynomial of the knot was clearly pointed out by the Alexander in his quoted paper.

Conceding priority to Mr. Burau, I publish only outline of my proof. I do this especially because of the additional result arrived at in the course of the proof of the effect that the Betti number R_1 of the k-sheeted Riemann manifold is the number of roots of the polynomial $F(x)$ which are also roots of the equation $x^k = 1$. Whether this is a general property of the Alexander polynomial or holds only

[2] i.e. this was four years after the appearance of Alexander's paper and after Zariski most fundamental papers on the topology of singular curves.

[3] Stating that the fundamental group of a plane curve singularity determines the Puiseux pairs.

for particular knots connected with algebraic singularities- is a question which is worthwhile considering. [4]

Papers [**93, 94, 95**] reveal that Zariski viewed the tower of cyclic covers with a given branch locus and its elements as the topological invariant of the branching locus. This point of view is rather natural for a practitioner of algebraic fundamental groups [5]. He saw the existence of covering spaces with a non-vanishing first Betti number as a manifestation of non-commutativity of the fundamental group of the complement but the connection with the structure of the fundamental group and the role of the Alexander polynomial as an intermidiary between covers and the group of the complement were not realized in his work.

D.Mumford [**67**] and B.Mazur [**10**] in the mid 60s were pointing out the similarities between the Alexander polynomial and Zariski constructs. In particular, Mumford posed a question about the role of the Alexander polynomial in his appendix to 1972 edition of Zariksi's Algebraic surfaces (cf. [**67**]) and B.Mazur asked about the relation between Alexander polynomials of links of singularities of a plane curve and the topology of the complement. These questions contributed to the development of the theory of Alexander invariants in algebraic geometry [6].

2. Alexander polynomials

The Alexander polynomial in its explicit form made first appearance in algebraic geometry in the paper [**46**].[7] The study of the fundamental groups of the complements to projective plane singular curves was largely stagnant for about 40 years between 1935 and the late 70s. Zariski's deep and fundamental discoveries prior to mid 30s included finding the presentation of the fundamental group of a plane curve in terms of generators and relators which now is known as the Zariski-van Kampen theorem, (the result which is parallel to the much earlier Wirtinger's knot groups presentation), finding examples of curves with non-abelian fundamental groups, namely the curve $f_3^2(x,y,z)+f_2^3(x,y,z)=0$ where $f_n(x,y,z)$ is generic form of degree n, dual curves to rational nodal curves, [8] the calculation of the irregularity of resolutions of singularities of curves branched over plane curves *with nodes and cusps* and finally initiating the local study of singularities of plane curves (see detailed analysis of Zariksi work in [**10**]). As was already mentioned above, Zariski saw one of the main merits of his remarkable paper [**94**] in its geometric method (i.e. using the position of cusps as an input) to show the non vanishing

[4]This problem posed by Zariski was addressed shortly thereafter by Goeritz in his paper in Amer. Journ. of Math. 56 (1934) no.1-4, 194-198 (who, following Reidemeister, calls $F(x)$ an L-polynomial rather than the Alexander polynomial) and was refined further by R.Fox, Hosokawa-Kinoshita and De Witt Sumners (for higher dimensional links). On interactions between Princeton or Viennese schools including Reidemeister, Wirtinger, and Goeritz cf. [**28**].

[5]A similar viewpoint is expressed by Alexander in footnote on p.303 in [**2**].

[6]Another tantalizing suggestion by Mazur on realtion between the Alexander polynomials and Iwasawa theory was not to date successfully pursued.

[7]Here we mean, of course, the *global* Alexander polynomial. As it was mentioned earlier, the Alexander polynomials of links of algebraic singularities were extensively studied starting from works of Burau and Goertz. Some of these developments are mentioned below.

[8]If C is dual to rational nodal curve of degree d then $\pi(\mathbb{P}^2-C)$ is isomorphic to the braid group of sphere on d-strings; Zariski argument extremely beautiful and elementary. This was the first appearance of braid groups in an essentially algebro-geometric problem. 50 year later, B.Moishezon discovered how pervasive braid groups are in these problems [**66**].

of irregularity of cyclic branched coverings and by this detecting that the fundamental group of the complement to branching locus is non-abelian. The precise relation between irregularity and the fundamental group, or rather its Alexander polynomial, was worked out in [**46**].

Important results between Zariski and the revival of interest in the study of plane singular curves before the 80s include [**62**], [**88**] and calculation of the fundamental group of the complement to $f_p^q + f_q^p = 0$ (cf. [**71**]). Often overlooked is the important work of Chisini and his school which, however, more or less dissipated without much follow up in the mid 50s. Very important was the vast development of local theory of singularties and in particular singularities of plane curves in 60s and 70s (which cannot be overviewed in this note). It provided one of the most beautiful interactions between algebraic geometry and topology with inspiration coming from works of Brieskorn on consruction of exotic spheres using singularities as well as from Milnor's book [**65**]. Much of it is summarized in the influential collection [**76**], and the book [**5**] with the case of curves discussed in [**13**] and a little later in [**26**]. Among many other results, the Alexander polynomial of a link of a singularity was identified with the characteristic polynomial of the monodromy, ([**65**]) the later already presented itself as the central tool in the local study of the topology of singularities. This allowed one to better understand many of Zariski's local results. For example in (cf. [**61**]) the results of Brauner, Zariski and Burau were strengthened by showing that not just the fundamental group but the Alexander polynomial of the knot alone determines the topological type of an algebraic knot.

Revival of interest in the *global* study of plane singular curves in [**46**] is due in large extent to Fulton-Deligne solution to the so-called Zariski problem.[9] In the late 70s this problem was perceived as the main question in the theory of plane singular curves: the question asked if the fundamental group of the complement to a plane curve, having ordinary nodes as the only singularities, is abelian. Zariski derived this statement from his calculation of the fundamental group of the complement to a union of r mutually transversal lines in $\mathbb{P}^2$ (which is $\mathbb{Z}^{r-1}$), and the Severi assertion in [**83**], that the system of plane curves of fixed degree with nodes as the only singularities is irreducible. Severi's reasoning was incomplete but an actively sought-after proof of the missing point was finally found in [**37**]. Fulton's ([**34**]) and Deligne's ([**21**]) proofs follow different lines and their arguments deal directly with fundamental groups.

After Fulton and Deligne's work it was quite natural to turn to non abelian cases. Zariski-van Kampen theorem, proved in the 30s (cf. [**89**], [**16**]), though a very beautiful and important fact, sometimes gives the wrong impression that the problem of fundamental groups of the complements is solved. In fact, it says very little about the properties or structure of fundamental groups and even for such natural problems as commutativity for curves with mild singularities by itself it is not too helpful. Even establishing if Enriques result mentioned above ([**27**]) gives complete information of the fundamental group depends on whether the topological

[9]Perhaps one should mention another "missed opportunity" for a systematic study of the topology of the complements. There was much interest among physicists in a study of ramification of integrals in connection with Feynmann integrals and Landau singularities of S-matrix which generated many interesting calculations (Fotiadi, Froissart, Regge) but not a systematic theory cf. [**78**] (there were, of course, many very fruitful interactions with singularity theory)

fundamental group is residually finite [10] : if not, then monodromy groups of finite covers branched over singular curves described by Enriques miss information in the intersection of the subgroup of finite index in the topological π_1.

With this state of affairs, it appeared reasonable to try to find "manageable" invariants of fundamental group of complements to algebraic curves. The definition of Alexander modules ([**38**]) which in 70s were mainly invariants of knot or link groups, worked for any group which is finitely generated, finitely presented and has surjective homomorphisms $\phi : G \to \mathbb{Z}$. In this case, the Alexander module is defined as the quotient $Ker(\phi)/Ker(\phi)'$ (i.e. of a kernel by its commutator) considered as the module over the group ring of $\mathbb{Z}$ i.e. the ring of Laurent polynomials. Given a presentation of the fundamental group by generators and relators, one can find the presentation of $Ker\phi$ using the Reidemeister-Schreier method and hence calculate the abelianization (with the action of $G/Ker\phi$ on it).

For an algebraic curve in $\mathbb{P}^2$ with r components of degrees $d_1, ..., d_r$, one has

$$H_1(\mathbb{P}^2 - C, \mathbb{Z}) = \mathbb{Z}^r/(d_1, .., d_r) \quad (1)$$

(quotient by cyclic subgroup generated by the element $(d_1, ..., d_r)$) Hence if one of the components has a degree one, i.e., if we consider a curve in affine plane, then H_1 of the complement is free abelian and knot theory construction does apply. One even has preferred surjection $\pi_1(\mathbb{C}^2 - C) \to \mathbb{Z}$ given by the total linking number with C. As in knot theory, one can remedy the fact that $\mathbb{Z}[t, t^{-1}]$ is not a principal ideal domain by considering $Ker(\phi)/Ker(\phi)' \otimes \mathbb{C}$ as $\mathbb{C}[t, t^{-1}]$ module. The order of this $\mathbb{C}[t, t^{-1}]$-module is called the Alexander polynomial $\Delta_C(t)$. This *global* Alexander polynomial gives an answer to the above mentioned problem raised by Artin and Mazur in (cf. [**10**]): provide reasonable conditions for regularity (equivalently, the vanishing of the first Betti number) of cyclic multiple planes in terms of local Alexander polynomials. It was shown in [**46**] that a n-fold cyclic cover of $\mathbb{P}^2$ is regular if none of the roots of local Alexander polynomials of links of singularities of the branching locus is a root of cyclotomic polynomial $t^n - 1$. For example, in the case considered in Zariski paper [**94**] which was a motivation for [**10**], the irregularity (i.e., $\frac{b_1}{2}$) of non-singular birational model of a cyclic branched over of $\mathbb{P}^2$ branched over curves with nodes and cusps as the only singularities vanishes unless the degree is divisible by 6 and, of course, it is well-known that the local Alexander polynomial of a cusp $x^2 + y^3 = 0$ is $t^2 - t + 1$ vansihing at the primitive roots of unity of degree 6. The proof in [**94**] is very ingenious and based on the theory of adjoints with input from a topological result from his previous work [**93**] and in fact anticipated many distant developments including the theory of multiplier ideals (see below). Conditions for the regularity of cyclic multiple planes in [**46**] were obtained by purely topological means inspired by Deligne's arguments (cf [**21**]). Later the arguments were simplified and corrected using ideas in Nori's fundamental work on the complements to algebraic curves on projective surfaces (cf. [**70**]). The regularity of coverings of degrees which are powers of a prime, proven by Zariski prior to [**94**] [11] and played a crucial role in the argument in the latter paper, received in [**46, 47**] a proof similar to the proof of the vanishing of the first Betti number of branched coverings of S^3 of a degree equal to a power of a prime.

[10]This still remains to be open.

[11]This is one of the cases when Zariski was clearly ahead of developments in knot theory.

Method of [**94**] for calculation of irregularity of cyclic multiple planes was generalized in [**48**] so that it can be applicable to curves with arbitrary singularities. Work [**94**] uses $z^n = f(x, y)$ as a biregular model of cyclic multiple plane V_n branched over $f(x, y) = 0$ and then applied the theory of adjoints to calculate the geometric genus of this model. Since the holomorphic Euler characteristic depends only on combinatorial data of $f(x, y) = 0$ this gives the irregularity of V_n. By restricting adjoint surfaces of V_n to projective plane containing branching locus [**94**] obtains that irregularity of V_n is the sum of a linear functionof n and a periodic function having possibly a non zero value equal to the superabundance of curves of degree $d - 3 - \frac{d}{6}$ passing through the cusps of the branching curve only for $6|n$. Here the superabundance is the difference between the dimension of the space of curves of a given degree passing through the cusps and the dimension which is expected (i.e. the dimension of the space of curves of fixed degree minus the number of cusps). Since from [**93**] it was known already that the irregularity of multiple planes for n being a power of prime is zero the implication was that the linear term is absent.

Analysis of Zariski arguments, results of which were described in paper [**48**], yielded that if the branching curve has singularities more complicated than cusps, then one has to make the following generalization. The irregularity of a cyclic multiple plane can be expressed in terms of the superabundances of certain linear systems of plane curves specified by the degree and the condition that the local equations of the curves in the system must belong to the ideals depending on the local type of the singularities (rather than, as in the case of cusps, just passing through the singular points). In fact the calculation gives logarithms of roots of the global Alexander polynomial which, by the divisibility theorem of [**46**], are rational numbers κ such that each $exp(2\pi i\kappa)$ is a root of the local Alexander polynomial. These logarithms are among the members of collections of rational numbers associated with each singular point and defined as follows. For each ϕ in the local ring of the singular point, one considers the minimal $\kappa = \frac{i}{n}$ such that the 2-form $\frac{z^i\phi(x,y)dx\wedge dy}{z^{n-1}}$, defined on non-singular locus surface of $z^n = f$, extends on a resolution of singularities of the latter to a holomorphic form. Moreover, the set of ϕ's, yielding the value of $\kappa(\phi) \geq \kappa$ form the ideal in the local ring depending on κ. The rational number κ was called in [**48**] the *constant of quasi-adjunction* and the above ideal of corresponding ϕ was called the *ideal of quasi-adjunction*. Shortly after [**48**], in [**63**] the constants of quasi-adjunction were identified with the spectrum of singularity defined using the Hodge theory.

The main result in [**48**] is that the root of unity $exp(2\pi\sqrt{-1}\kappa)$ is a root of the global Alexander polynomial if and only if $dim H^1(\mathbb{P}^2, \mathcal{J}_{Sing}(d - 3 - d\kappa) \neq 0$. Here $\mathcal{J}_{Sing} \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf which has a stalk different from the local ring only at the singular point and the stalk at such point is the ideal of ϕ's such that $\kappa(\phi) \geq \kappa$. The condition of extendability of ω_ϕ in [**48**] is obtained by using a resolution of the normalization of the fiber product $V_n \times_{\mathbb{C}^2} \tilde{\mathbb{C}}^2$, where V_n is the cyclic multiple plane and $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is an embedded resolution of germ of f near a singular point. If E_k are exceptional loci of such a resolution, $a_k = \text{ord}_{E_k}\pi^*(f), c_k = \text{ord}_{E_i}\pi^*(dx \wedge dy)$ and $e_k(\phi) = \text{ord}_{E_k}\pi^*(\phi)$, then the extendability condition is $c_k + 1 - e_k(\phi) > a_k(1 - \frac{i}{n})$. The constant of quasi-adjunction $\kappa(\phi)$ above is the minimal solution to the above inequality and the ideal of quasi-adjunction is $\pi_*(\mathcal{O}(K_{\tilde{\mathbb{C}}^2/\mathbb{C}^2} - \kappa(f)\mathcal{O}_{\tilde{\mathbb{C}}^2})$ which since mid 90s became known as *the multiplier ideal* of divisor κD (where

$D = (f)$ is the zero divisor of f). The collection of constants of quasi-adjunction is now also known as the collection of jumping numbers (cf. [**68**], [**43**]).

Besides paper [**46**], Zariski's result on the dependence of topology on the position of singularities was reconsidered in the work ([**29**]). This paper deals with the Milnor fibers of homogeneous polynomials which are the defining equations of plane singular curves and used the cyclic coverings trick as in [**90**]. The Alexander polynomial via such use of Milnor fiber was considered also in [**79**]. A little later an interpretation of Alexander polynomial in terms of differential forms were obtained in ([**42**]).

Since the late 70s or early 80s, B.Moishezon initiated his program of calculation of the fundamental groups of the complements to the branching loci of generic projections of surfaces in $\mathbb{P}^N$. He made explicit calculations for the case of generic projection of non-singular surfaces in $\mathbb{P}^3$. In [**66**] it was shown that $\pi_1(\mathbb{P}^2 - C_n) = B_n/(\Delta^2)$. Here C_n is the branching curve of generic projection of a non-singular surface in $\mathbb{P}^3$ having degree n ($degC_n = n(n-1)$, the number of cusps is $n(n-1)(n-2)$ and the number of nodes is $\frac{1}{2}n(n-1)(n-2)(n-3)$), B_n is the Artin braid group on n-strings and Δ^2 is the generator of its center. Moishezon's program subsequently was carried on also after Moishezon's death, (cf. [**86**] and references therein). Moreover, later the techniques of braid monodromy, i.e. the monodromy with the values in a braid group, turned out to be useful in the study of symplectic subvarieties (cf. [**11**]). In [**49**], it was found that braid monodromy determines the homotopy type of the complement to the plane singular curve and in [**50**] it was shown how to calculate Alexander polynomials in terms of braid monodromy. The latter suggested that there are polynomial invariants of the complements associated with any linear representation of braid over a principal ideal domain (the Alexander polynomial corresponds to a reduced Burau representation but the study of invariants corresponding to other representations was not pursued so far).

The Alexander polynomial continues to play a role in the classification of plane singular curves. The possibility that Alexander polynomial distinguishes connected components of equisingular families of plane curves was disproved in [**6**], though for sextics (with cusps on conic and in general position) the Alexander polynomial separates components (cf. also [**72, 84**]). The classification of Alexander polynomials of sextic curves was started in [**73, 31, 74**] and completed in [**19**]. Numerous examples of Alexander polynomials for curves with A_{2k} singularities were given in [**17**]. The dependence of Alexander polynomials on the line at infinity was studied in [**75**]. An extension to non abelian cases was made in [**44**].

3. Characteristic varieties

Characteristic varieties were introduced in [**53**] as zero sets of the Fitting ideals of $H_1(X_{ab}, \mathbb{C})$ where X_{ab} is the infinite abelian cover of a CW-complex X. As such, they appeared as affine algebraic subvarieties of the torus $Spec\mathbb{C}[H_1(X, \mathbb{Z})]$. Equivalently, they are the support loci of $\mathbb{C}[H_1(X, \mathbb{Z})]$-modules $\Lambda^i H_1(X_{ab}, \mathbb{C})$. The problem which at the time had been addressed by characteristic varieties was to find a multi-variable generalization of the Alexander polynomial, keeping in multi-variable setting to the extent possible the divisibility theorems and the relation with the homology of abelian covers i.e., to find an abelian generalization of the theory of Alexander polynomials. In topology, the study of multivariable Alexander polynomials goes back to R.Fox but earlier occurances are in works of Reidemester-Shumann and Burau (cf. references in [**87**]).

Complications relative to one variable case come from the fact that the group ring (even over $\mathbb{C}$) of the Galois group of a universal abelian covering, i.e. $\mathbb{C}[H_1(X,\mathbb{Z})]$, is not a principle ideal domain unless the latter covering is cyclic. In the case of the complement to links, the first Fitting ideal of the homology of an infinite abelian cover considered as a module over H_1, is non-principal, but has a closely related principal ideal (cf. [**33**]). This allows one to define (cf. [**38**]) non-trivial multivariable Alexander polynomial of links.

Higher Fitting ideals of links are very far from being principal and therefore there were no multivariable higher Alexander polynomials and no classical formulas in knot theory for the homology of finite abelian cyclic coverings. In the case of $\pi_1(\mathbb{P}^2 - C)$ where C is as in (1) even the first Fitting ideal is very far from being principal in general. The work [**53**] resolved these issues. The characteristic varieties, as the Fitting ideals, depend only on the fundamental group $G = \pi_1(X, x_0)$. Of course, replacement of Alexander module but its support results in a loss of a lot of important data about the module G'/G'' but what is retained is enough to keep track of substantial amount of information on the fundamental group. For example, characteristic varieties contain enough information to control Betti numbers of unbranched (cf. [**53**] for an explicit formula) and branched coverings (cf. [**82**]) and allows one to derive an analog of divisibility theorem of Alexander polynomial at infinity by the global Alexander polynomial (cf. [**53**]).

At the time, additional motivation was coming from the work of P.Sarnak who in his study of representations asked for the type of growth of Betti numbers in the tower of finite abelian covers. Together with Adams (cf. [**1**]) he had shown, using the so-called Lang conjecture, that rkH_1 growth is polynomially periodic, i.e. can be expressed as polynomial $\sum a_{i_1,..,i_k}(n_1,..,n_k)n_1^{i_1} \cdot .... \cdot n_k^{i_k}$ with coefficients being periodic in $n_1,...,n_k$. In the case of knots, one has ordinary periodicity since $rkH_1(V_k)$ (V_k is a cyclic branched covering) is the number of common roots of cyclotomic polynomial $t^k - 1$ and higher Alexander polynomials associated with the knot. Polynomial growth of Betti numbers for branched covering of $\mathbb{P}^2$ was proved by E.Hironaka ([**39**]). As already mentioned, the formula in [**82**] works for branched covering spaces in a much more general context than coverings of 3-spheres on $\mathbb{P}^2$. In the case of plane curves or links it expresses the homology of the abelian cover in terms of characteristic varieties of sublinks of branching locus or characteristic varieties of complements to reducible curves which formed by collections of components of a given reducible curve. It also plays a role in establishing polynomial periodicity of Betti numbers.

The paper [**40**], relates the characteristic varieties to the cohomology of rank one local systems. The local systems of rank one are by definition just the characters of the fundamental groups and one can use them to define twisted cohomology $H^i(X,\chi)$ ($\chi \in Char\pi_1(X)$). This is very important since cohomology of local systems are amenable to the study using methods of deRham and Hodge theory (i.e., using deRham complex depending on the flat connection corresponding to the local systems). This theory originated by Deligne as part of his solution to the 21th Hilbert problem on Riemann-Hilbert correspondence (cf. [**20**]).

In particular one could apply results of [**4**], which first appeared very soon after characteristic varieties were introduced in [**53**], and which extended to the quasi-projective case a long series of results going back to deFranchis, Castelnuovo and more recently Beauville, Catanese, Green-Lazarsfeld, Deligne and Simpson. Notice that these authors were working in the projective case and with moduli space of topologically trivial holomorphic bundles corresponding to *unitary* connections. Some implications for the fundamental groups of projective manifolds were pointed out in these works but the goals and applicability of these studies were markedly different from the study of characteristic varieties (cf.[**85**]). For example, the problem of the dependence of the characteristic varieties of $\mathbb{P}^2 - C$ on the properties of C does not have a counterpart in the study of the jumping loci for cohomology of line bundles on projective varieties. The results of [**4**] were applied systematically to the study of characteristic varieties in [**55**] and played there a key role. Firstly, they elucidated the structure of the connected components by describing them as cosets of subgroups of $Spec\mathbb{C}[H_1(X,\mathbb{Z})] = Char\pi_1(X)$ and secondly by relating the components of characteristic varieties of positive dimension to holomorphic maps on hyperbolic curves (or irrational pencils in the projective case). More recently there were related to rational orbifold pencils (cf. [**9**]).

The work [**55**] provided an algorithmic procedure for calculation of the essential components of the characteristic varieties [12]. This procedure is a generalization of the method used to get results of [**48**] in turn generalizing [**94**]. The results in [**55**] describe the essential components of characteristic varieties in terms of local data which consists of new invariant of local singularities with several, say r, branches. Each component is obtained as a Zariski closure in $Char\pi_1(\mathbb{P}^2 - C)$ of the image of a polytope in the universal cover of the group of unitary characters of $Char\pi_1(\mathbb{P}^2 - C)$. These global polytopes are intersections of polytopes defined in terms of local data of a collection of singularities of C. The local data of singularities in turn is a set of polytopes belonging to the fundamental domain of $H_1(B - B \cap C, \mathbb{Z})$ (here B is a small ball about a singularity of C) acting on the universal cover of $Char_u\pi_1(B - B\cap C)$. One can think of this fundamental domain as the unit cube in $\mathbb{R}^r$ where r is the number of branches of a germ of C. In the case of irreducible germs, these polytopes are the elements of the spectrum of singularity of the germ which belong to $[0, 1) \subset \mathbb{R}$ (cf. [**63**]; see [**5**] for discussion of spectrum in other problems of local singularity theory). Once "a candidate" for global polytope is selected one still has to decide if the Zariski closure of the image in $Char\pi_1(\mathbb{P}^2 - C)$ will be an actual component of characteristic variety. For this one needs to check a global condition, similar to the condition in the case of irreducible curve, and which is non-vanishing of

$$H^1(\mathbb{P}^2, \mathcal{J}_{sing}(d - 3 - l)). \tag{2}$$

Here $\mathcal{J}_{sing} \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf having stalks non-equal to the full local ring only at the singular points of C and the ideals at singular points are the ideals of quasi-adjunction corresponding to the local polytopes used to generate the global candidate. Also, in (2) d is the total degree of C and $l \in \mathbb{Z}$ is given in terms of combinatorial data of local polytopes generating this candidate. The proof in [**55**] is an inductive one and represents essentially the calculation of p_g of the abelian

[12] They are essential in the sense that they do not come from a component of characteristic variety on a curves obtained by deleting some components of the given curve.

covers. An alternative argument for calculation of p_g of abelian covers using [**30**] was given recently in [**14**].

Applications of characteristic varieties in [**55**] and, shortly after that [**56**], included the study of topology of complex hyperplane arrangements. These works provide the relations between characteristic varieties and the purely combinatorial cohomology of Orlik-Solomon algebra. In paper [**56**] for the first time there started to emerge a possibility of classifying arrangements according to the type of characteristic varieties. This gave an impetus to further classification of pencils in [**32**] and [**92**].

Study of characteristic varieties, since they were introduced in 1992, developed into a particularly vast subject in the context of arrangements. Numerous calculations were made, for example, using braid monodromy [**18, 81**]. Characteristic varieties of quasi-projective varieties and arrangements were compared with characteristic varieties of related CW-complexes and groups (cf. [**23**]) and eventually led to characterization of Kähler groups which are the fundamental groups of 3-manifolds in [**24**]. This was, to date, the culmination of the progress started with the use of cyclotomic property of Alexander polynomials to show that certain groups cannot appear as the fundamental groups of certain class of algebraic varieties (cf. [**51**]). Alexander invariants were used [**7**] to show that arrangements constructed in [**80**], though combinatorially equivalent, have non-isomorphic fundamental groups. Important progress was made in understanding the components of characteristic varieties which are cosets of subgroups not belonging to characteristic variety (cf. [**25**]). An outstanding remaining problem in the topology of arrangenents is whether characteristic varieties depend only on combinatorial information about arrangements (cf. [**59**]).

4. Alexander invariants and homotopy groups

Early 90s, saw the start of developments of higher dimensional, i.e. corresponding to higher homotopy groups, counterparts of Alexander invariants. In the work [**52**] it was shown that for hypersurfaces in $\mathbb{P}^{n+1}$ with isolated singularities for $n \geq 2$ one has the following:

$$\pi_1(\mathbb{P}^{n+1} - V) \text{ is abelian}, \quad \pi_i(\mathbb{P}^{n+1} - V) = 0 \quad 2 \leq i \leq n-1 \tag{3}$$

Moreover, the group $\pi_n(\mathbb{P}^{n+1} - V)$ considered as the module over $\pi_1(\mathbb{P}^{n+1} - V)$ plays the role of the Alexander module. Many essential features of theory of curves were generalized to higher dimensions (cf. [**54**]). They include the dependence of $\pi_n(\mathbb{P}^{n+1} - V)$ on the local type of singularities, the possibility of calculations based on an appropriate generalization of Zariski-van Kampen theorem, the relation of $\pi_n(\mathbb{P}^{n+1} - V)$ with the Hodge number $h^{n,0}$ of the cyclic branched over V coverings of $\mathbb{P}^{n+1}$ [13] etc. The role of the Zariski sextic is played by the hypersurfaces corresponding to $n+1$-tuples of pairwise integers $p_1, ..., p_{n+1}$ (here $P = \Pi_{i=1}^{n+1} p_i$ and f_k is a generic form of degree k in $n+2$ variables):

$$f_{\frac{P}{p_1}}^{p_1} + ... + f_{\frac{P}{p_{n+1}}}^{p_{n+1}} = 0 \tag{4}$$

Moreover there is an abelian generalization of these results (i.e. based on the study of abelian rather than cyclic coverings). The condition on a reducible hypersurface which assures properties (3) is that components of V will have transversal

[13] Irregularity is equal to $h^{0,n}$ for $n = 1$.

intersections everywhere except for a finite number of points (i.e., V has isolated non-normal crossings (INNC) cf. [**58**], [**60**]). The characteristic variety of interest is the support of $\pi_n(\mathbb{P}^{n+1} - V) \otimes \mathbb{C}$ and much of the theory of characteristic varieties can be extended to higher dimensions (cf [**58**]). In particular, there is close connection with the jumping loci of local systems which were studied in [**60**] and more recently in [**14**]. There is a local situation providing a counterpart to many results in the local theory of isolated singularities (cf. [**57, 22**]). The case where the degrees of all components of reducible hypersurfaces are equal to one yields a link with the theory of arrangements. The classification of INNC arrangements with characteristic varieties having components of a positive dimension can be achieved with results having their counterpart in the theory of line arrangements.

Realization problems provide the biggest challenge in development of this theory: which polynomials can appear as the Alexander polynomials of plane algebraic curves or orders of $\pi_n(\mathbb{P}^{n+1} - V)$ where V is a hypersurface with isolated singularities? Similar questions are wide open in the case of components of characteristic varieties of $\pi_n(\mathbb{P}^{n+1} - V)$ where V is reducible hypersurface with isolated non-normal crossings. More connections with other areas of singularity theory are very likely (cf. [**15**]) for a connection between the polytopes of quasi-adjunction and Bernstein-Sato ideals of germs of plane curves. Algebro-geometric theory of Alexander invariants continue to be an actively developing area.

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