1

ELLIPTIC GENERA, REAL ALGEBRAIC VARIETIES AND QUASI-JACOBI FORMS

ANATOLY LIBGOBER

Abstract. This paper surveys the push forward formula for elliptic class and various applications obtained in the papers by L.Borisov and the author. In the remaining part we discuss the ring of quasi-Jacobi forms which allows to characterize the functions which are the elliptic genera of almost complex manifolds and extension of Ochanine elliptic genus to certain singular real algebraic varieties.

Introduction

Interest in elliptic genus of complex manifolds stems from its appearance in a wide variety of geometric and topological problems. At the first glance, this is an invariant of complex cobordism class modulo torsion and hence depends only on the Chern numbers of manifold. On the other hand elliptic genus is a holomorphic function of $\mathbb{C} \times \mathbb{H}$ where $\mathbb{H}$ is the upper half plane. In one of heuristic approaches, elliptic genus is an index of an operator on the loop space ([53]) and as such it has counterparts defined for C^∞, oriented or Spin manifolds (studied prior to the study of complex case (cf. [42]). It comes up in the study of geometry and topology of loop spaces and, more specifically, the chiral deRham complex (cf. [40]), in the study of invariants of singular algebraic varieties ([8]), in particular orbifolds, and more recently in the study of Gopakumar-Vafa and Nekrasov conjectures (cf. [37] [24]). It is closely related to the fast developing subject of elliptic cohomology (cf. [45]). There are various versions of elliptic genus: equivariant, higher elliptic genus obtained by twisting by cohomology classes of the fundamental group etc, elliptic genus of pairs and orbifold elliptic genus. There is interesting connection with singularities of weighted homogeneous polynomials (so called Landau-Ginzburg models).

In present note we shall review several recent developments in the study of elliptic genus and refer a reader to review [7] for additional details on earlier results. Then we shall focus on its aspects of elliptic genus: its extension to real singular varieties and its modularity property (or the lack of it). Extension of elliptic genus to real singular varieties was suggested by B.Totaro ([47]) and our approach is based on the push forward formula for the elliptic class used in [8] to extend elliptic genus from smooth to certain singular complex projective varieties.

In section 1.1 we shall discuss this push forward formula. It appears as the main technical tool in many applications mentioned later in the paper. The rest of section 1 discusses the relation with other invariants and series of applications based on the material of works [9],[8] and [6]. It includes a discussion of a relation between elliptic genus and E-function, application to McKay correspondence, elliptic genera of non-simply-connected manifolds (higher elliptic genera) and generalizations of a formula of R.Dijkgraaf, Moore, E.Verlinde, H.Verlinde. Other applications in equivariant

[1]author was partially supported by NSF grant

context are discussed in R.Waelder paper in this volume. The proof of independence of resolutions of our definition of elliptic genus for certain real algebraic varieties is given in section 3.2.

Section 2 deals with modularity properties of elliptic genus. In the Calabi Yau cases (of pairs, orbifolds etc.) elliptic genus is a weak Jacobi form (cf. definition below). Also it is important to have description of the elliptic genus in non Calabi Yau situations not just as a function on $\mathbb{C} \times \mathbb{H}$ but as an element of a finite dimensional algebra of functions. It turns out that in the absence of Calabi Yau condition the elliptic genus belong to a very interesting algebra of functions on $\mathbb{C} \times \mathbb{H}$, which we call the algebra of quasi-Jacobi forms, and which is only slightly bigger than the algebra of weak Jacobi forms. This algebra of quasi-Jacobi forms is a counterpart of quasi-modular forms (cf. [30]) and is related to elliptic genus in the same way as quasi-modular forms are related to the Witten genus (cf. [55]). The algebra of quasi-Jacobi forms is generated by certain two variable Eisenstein series (masterfully reviewed by A.Weil in [51]) and has many properties parallel to the properties in quasi-modular case. A detailed description of properties of quasi-Jacobi forms appears to be absent in the literature and we discuss the algebra of these forms in section 2 (for more details see its introduction). We end this section with discussion of differential operators Rankin-Cohen brackets on the space of Jacobi forms.

Finally in section 3 construct extension of Ochanine genus to real algebraic varieties with certain class of singularities. This extends some results of B.Totaro in [47].

For readers convenience we give ample references to prior work on elliptic genus where more detailed information on the subject can be obtained. Part 2 dealing with quasi-Jacobi forms can be read independently of the rest of the paper.

The author wants to express his gratitude to Lev Borisov. The material in section 1 is a survey of joint papers with him and results of section 2 are based on discussions with him several years ago.

1. Elliptic genus.

1.1. **Elliptic genus of singular varieties and push-forward formulas.** Let X be a projective manifold. We shall use the Chow groups $A_*(X)$ with complex coefficients (cf. [22]). Let F the ring of functions on $\mathbb{C} \times \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane. The elliptic class of X is an element in $A_*(X) \otimes_{\mathbb{C}} F$ given by:

$$\mathcal{E}LL(X) = \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)}[X] \tag{1}$$

where

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin \pi z) \prod_{l=1}^{l=\infty}(1-q^l) \prod_{l=1}^{l=\infty}(1-q^l \mathrm{e}^{2\pi \mathrm{i} z})(1-q^l \mathrm{e}^{-2\pi \mathrm{i} z}) \tag{2}$$

is the Jacobi theta function considered as an element in F with $q = e^{2\pi i \tau}$ (cf. [12]), x_i are the Chern roots of the tangent bundle of X and $[X]$ is the fundamental class of X. The component $Ell(X)$ in $A_0(X) = F$ is the elliptic genus of X.

Components of each degree of (1), evaluated on a class in $A^*(X)$, are linear combination of symmetric functions in c_i i.e. the Chern classes of X. In particular $Ell(X)$ depends only on the class of X in the ring $\Omega^U \otimes \mathbb{Q}$ of unitary cobordisms.

The homomorphism $\Omega^U \otimes \mathbb{Q} \to F$ taking X to $Ell(X)$ can be described without reference to theta functions as in (1). Let $M_{A_1,3}$ be the class of complex analytic space "having only A_1-singularities in codimension 3" i.e. having the singularities of the following type: the singular set $SingX$ of $X \in M_{A_1,3}$ is *a manifold* such that $\dim_{\mathbb{C}} SingX = \dim X - 3$ and for an embedding $X \to Y$ where Y is a manifold and a transversal H to $SingX$ in Y, the pair $(H \cap X, H \cap SingX)$ is analytically equivalent to the pair $(\mathbb{C}^4, H_0)$ where H_0 is given by $x^2 + y^2 + z^2 + w^2 = 0$. Each $X \in M_{A_1,3}$ admits two small resolutions $\tilde{X}_1 \to X$ and $\tilde{X}_2 \to X$ in which the exceptional set is fibration over $SingX$ with the fiber $\mathbb{P}^1$. One says that manifolds underlying the resolutions obtained from each other by classical flop.

Theorem 1.1. *(cf. [46]) The kernel of the homomorphism $Ell : \Omega^U \otimes \mathbb{Q} \to F$ taking an almost complex manifold X to its elliptic genus $Ell(X)$ is the ideal generated by classes of differences $\tilde{X}_1 - \tilde{X}_2$ of two small resolutions of a variety in $M_{A_1,3}$.*

More generally one can fix a class of singular spaces and a type of resolutions and consider the quotient of $\Omega^U \otimes \mathbb{Q}$ by the ideal generated by differences of manifolds underlying resolutions of the same analytic space. The quotient map by this ideal $\Omega^U \otimes \mathbb{Q} \to R$ provides a genus and hence the collection of Chern numbers (linear combination of Chern monomials $c_{i_1} \cdot .... \cdot c_{i_k}[X]$ ($\sum i_s = \dim X$) which can be made explicit via Hirzebruch procedure with generating series (cf. [27]). These are the Chern numbers which can be defined for the chosen class of singular varieties and chosen class of resolutions. The ideal in theorem 1.1, it turns out, corresponds to a much larger classes of singular spaces and resolutions. This method of defining Chern classes of singular varieties is an extension of the philosophy underlying the question of Goresky and McPherson (cf. [25]): which Chern numbers can be defined via resolutions independently of resolution.

Definition 1.2. An analytic space X is called **Q**-Gorenstein if the divisor D of a meromorphic form $df_1 \wedge .... \wedge df_{\dim X}$ is such that for some $n \in \mathbb{Z}$ the divisor nD in locally principal (i.e. K_X is $\mathbb{Q}$-Cartier). In particular for any codimension one component E of the exceptional divisor of a map $\pi : \tilde{X} \to X$ the multiplicity $a_E = \text{mult}_E \pi^*(K_X)$ is well defined and a singularity is called log-terminal if there is resolution π such that $K_{\tilde{X}} = \pi^*(K_X) + \sum a_E E$ and $a_E > -1$. A resolution is called crepant if $a_E = 0$.

Theorem 1.3. *([8]) The kernel of the elliptic genus $\Omega^U \otimes \mathbb{Q} \to F$ is generated by the differences of $\tilde{X}_1 - \tilde{X}_2$ of manifolds underlying crepant resolutions of the singular spaces with $\mathbb{Q}$-Gorenstein singularities admitting crepant resolutions.*

The proof of theorem (1.3) is based on extension of elliptic genus $Ell(X)$ of manifolds to the elliptic genus of pairs $Ell(X, D)$ where D is a divisor on X having the normal crossings as the only singularities. This is similar to the situation in the study of motivic E-functions of quasi-projective varieties (cf.[2],[39]). In fact, motivation from other problems (e.g. study of McKay correspondence cf. [9]) suggests looking at the triples (X, D, G) where X is a normal variety, G is a finite group acting on X and to introduce the elliptic class $\mathcal{E}LL(X, D, G)$ (cf. [9]). More precisely, let D be a $\mathbb{Q}$-divisor i.e. $D = \sum a_i D_i$ with D_i being irreducible and $a_i \in \mathbb{Q}$. The pair (X, D) is called Kawamata log-terminal (cf. [34]) if $K_X + D$ is $\mathbb{Q}$-Cartier and there is a birational morphism $f : Y \to X$ where Y is smooth and the union of the proper preimages of components of D and the components of exceptional set $E = \bigcup_{\mathcal{J}} E_j$ form a normal crossing divisor such that $K_Y = f^*(K_X + \sum a_i D_i) + \sum \alpha_j E_j$ where $\alpha_j > -1$ (here K_X, K_Y are the canonical classes of X and Y respectively). The triple (X, D, G) where X is a non-singular variety, D is a divisor and G is a finite group of biholomorphic automorphisms is called G-normal (cf. [2], [9]) if the components of D form a normal crossings divisor and the isotropy group of any point acts trivially on the components of D containing this point.

Definition 1.4. (cf. [9] definition 3.2) Let (X, E) be a Kawamata log terminal G-normal pair (i.e. in particular X is smooth with D being a normal crossing divisor) and $E = -\sum_{k \in \mathcal{K}} \delta_k E_k$. The *orbifold* elliptic class of (X, E, G) is the class in $A_*(X, \mathbb{Q})$ given by:

$$\mathcal{E}ll_{orb}(X,E,G;z,\tau) := \frac{1}{|G|} \sum_{g,h,gh=hg} \sum_{X^{g,h}} [X^{g,h}] \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \tag{3}$$

$$\times \prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i}\lambda(h)z}$$

$$\times \prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} + \epsilon_k(g) - \epsilon_k(h)\tau - (\delta_k+1)z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} + \epsilon_k(g) - \epsilon_k(h)\tau - z)} \frac{\theta(-z)}{\theta(-(\delta_k+1)z)} \mathrm{e}^{2\pi \mathrm{i}\delta_k \epsilon_k(h) z}.$$

where $X^{g,h}$ denotes an irreducible component of the fixed set of the commuting elements g and h and $[X^{g,h}]$ denotes the image of the fundamental class in $A_*(X)$. The restriction of TX to $X^{g,h}$ splits into linearized bundles according to the ($[0,1)$-valued) characters λ of $\langle g, h \rangle$ (sometimes denoted λ_W where W is a component of the fixed point set). Moreover, $e_k = c_1(E_k)$ and ϵ_k is the character of $\mathcal{O}(E_k)$ restricted to $X^{g,h}$ if E_k contains $X^{g,h}$ and is zero otherwise.

One would like to define the elliptic genus of a Kawamata log-terminal pair (X_0, D_0) as (3) calculated for a G-equivariant resolution $(X, E) \to (X_0, D_0)$. Independence of (3) of resolution and the proof of (1.3) both depend on the following push-forward formula:

Theorem 1.5. *Let (X, E) be a Kawamata log-terminal G-normal pair and let Z be a smooth G-equivariant locus in X which is normal crossing to $Supp(E)$. Let $f : \hat{X} \to X$ denote the blowup of X along Z. We define $\hat{E}$ by $\hat{E} = -\sum_k \delta_k \hat{E}_k - \delta Exc(f)$ where $\hat{E}_k$ is the proper transform of E_k and δ is determined from $K_{\hat{X}} + \hat{E} = f^*(K_X + E)$. Then $(\hat{X}, \hat{E})$ is a Kawamata log-terminal G-normal pair and*

$$f_* \mathcal{E}ll_{orb}(\hat{X}, \hat{E}, G; z, \tau) = \mathcal{E}ll_{orb}(X, E, G; z, \tau). \tag{4}$$

Independence of resolution is a consequence of the weak factorization theorem (cf. [1]) and (1.5), while theorem (1.3) follows since both $\mathcal{E}ll(X_1)$ and $\mathcal{E}ll(X_2)$ coincide with the elliptic genus of the pair $(\tilde{X}, \tilde{D})$ where $\tilde{X}$ is a resolution of X dominating both X_1 and X_2 (where $D = K_{\tilde{X}/X}$ cf. [8] Prop. 3.5 and also [52]). For discussion of orbifold elliptic genus on orbifolds more general than just global quotients see [19].

1.2. **Relation to other invariants.** V.Batyrev in [2] for a G-normal triple (X, D, G) defined E-function $E_{orb}(X, D, G)$ depending on the Hodge theoretical invariants (there is also motivic version cf.[2],[39]). Firstly for a quasi-projective algebraic variety W one put (cf. [2], Definition 2.10):

$$E(W, u, v) = (-1)^i \sum_{p,q} \dim Gr_F^p Gr_W^{p+q}(H_c^i(W, \mathbb{C}))u^p v^q \tag{5}$$

where F and W are Hodge and weight filtrations of Deligne's mixed Hodge structure ([16]). In particular $E(W, 1, 1)$ is the topological Euler characteristic of W (with compact support). If W is compact then one obtains Hirzebruch's χ_y-genus:

$$\chi_y(W) = \sum_{i,j} (-1)^q \dim H^q(\Omega_W^p)y^p \tag{6}$$

(cf. [27]) for $v = -1, u = y$ and hence the arithmetic genus, signature etc. are special values of (5). Secondly, for a G-normal pair as in (1.4) one stratifies $D = \bigcup_{k\in K} D_k$ by strata $D_J^\circ = \bigcap_{j\in J} D_j - \bigcup_{k\in K-J} D_k$, $(J \subset K,\ \bigcap_{j\in J} D_j = X$ for $J = \emptyset)$ and defines

$$E(X, D, G, u, v) = \tag{7}$$
$$\sum_{\{g\}, W\subset X^g} (uv)^{\sum \epsilon_{D_i}(g)(\delta_i+1)} \sum_{J\subset \mathcal{K}^g} \prod_{j\in J} \frac{uv-1}{(uv)^{\delta_j+1}-1} E(W \cap D_J^\circ/C(g, J))$$

where $C(g, J)$ is the subgroup of the centralizer of g leaving $\bigcap_{j\in J} D_j$ invariant.

One shows that for a Kawamata log-terminal (X_0, D_0) pair the E-function $E(X, D, G)$ of a resolution does not depend on the latter but only on (X_0, D_0, G). Hence (7) yields an invariant of Kawamata log-terminal G-pairs. The relation with $Ell(X, D, G)$ is the following (cf. [9], Prop. 3.14) :

$$\lim_{\tau\to i\infty} Ell(X, D, G, z, \tau) = y^{\frac{-\dim X}{2}} E(X, D, G, y, 1) \tag{8}$$

where $y = exp(2\pi iz)$. In particular in non equivariant smooth case the elliptic genus for $q \to 0$ specializes into the Hirzebruch χ_y genus (6).

On the other hand, in non singular case Hirzebruch ([28], [29]) and Witten [53] defined elliptic genera of complex manifolds which are given by modular forms for the subgroup $\Gamma_0(n)$ on level n in $SL_2(\mathbb{Z})$ provided the canonical class of the manifold in question is divisible by n. These genera are of course combinations of Chern numbers but for $n = 2$ one obtains a combination of Pontryagin classes i.e. the invariant which depends only on underlying smooth rather than (almost) complex structure (this genus was first introduced by S.Ochanine cf. [42] and section 3). These level n elliptic genera up to a dimensional factors coincide with specialization $z = \frac{\alpha\tau+b}{n}$, $\alpha, \beta \in \mathbb{Z}$ for appropriate α, β specifying particular Hirzebruch level n elliptic genus (cf. [6], Prop. 3.4).

1.3. **Application I: McKay correspondence for elliptic genus.** The classical McKay correspondence is a relation between the representations of the binary dihedral groups $G \subset SU(2)$ (which are classified according to the root systems of type A_n, D_n, E_6, E_7, E_8) and the irreducible components of the exceptional set of the minimal resolution of $\mathbb{C}^2/G$. In particular, the number of conjugacy classes in G is the same as the number of irreducible components of the minimal resolution. The latter is a special case of the relation between the Euler characteristic $e(\widetilde{X/G})$

of a crepant resolution of the quotient X/G of a complex manifold X by an action of a finite group G and the data of the action on X:

$$e(\widetilde{X/G}) = \sum_{g,h,gh=hg} e(X^{g,h}) \tag{9}$$

A refinement of the relation (9) for the Hodge numbers and motives is given in [2], [18] [39]. In the case when X is projective one has refinement in which the Euler characteristic of manifold in (9) is replaced by elliptic genus of Kawamata log-terminal pairs. In fact, more generally, one has the following push forward formula:

Theorem 1.6. *Let $(X; D_X)$ be a Kawamata log-terminal pair which is invariant under an effective action of a finite group G on X. Let $\psi\colon X \to X/G$ be the quotient morphism. Let $(X/G; D_{X/G})$ be the quotient pair in the sense that $D_{X/G}$ is the unique divisor on X/G such that $\psi^*(K_{X/G} + D_{X/G}) = K_X + D_X$ (cf. [9]. Def. 2.7) Then*

$$\psi_* \mathcal{E}ll_{orb}(X, D_X, G; z, \tau) = \mathcal{E}ll(X/G, D_{X/G}; z, \tau).$$

In particular, for the components of degree zero one obtains:

$$Ell_{orb}(X, D_X, G, z, \tau) = Ell(X/G, D_{X/G}, z, \tau) \tag{10}$$

In the case when X is non-singular and X/G admits a crepant resolution $\widetilde{X/G} \to X/G$, for $q = 0$ one obtains $\chi_y(\widetilde{X/G}) = {\chi_y}^{orb}(X, G)$ and hence for $y = 1$ one recovers (9).

1.4. **Application II:Higher elliptic genera and K-equivalences.** Other applications of the push forward formula (1.5) is invariance of higher elliptic genus for the K-equivalences. A question posed in ([43]) (and answered in [3]) concerns the higher arithmetic genus $\chi_\alpha(X)$ of a complex manifold X corresponding to a cohomology class $\alpha \in H^*(\pi_1(X), \mathbb{Q})$ and defined as

$$\int_X Td_X \cup f^*(\alpha) \tag{11}$$

where $f : X \to B(\pi_1(X))$ is the classifying map from X to the classifying space of the fundamental group of X. It asks if the higher arithmetic genus $\chi_\alpha(X)$ is a birational invariant. This question is motivated by the Novikov's conjecture: the higher signatures (i.e. the invariant defined for topological manifold X by (11) in which the Todd class is replaced by the L-class) are homotopy invariant (cf. [15]). Higher χ_y-genus defined by (11) with the Todd class replaced by Hirzebruch's χ_y class ([27]) comes into the correction terms describing the non-multiplicativity of the χ_y in topologically locally trivial fibrations $\pi : E \to B$ of projective manifolds with non trivial action of $\pi_1(B)$ on the cohomology of the fibers of π (cf. [11] for details).

Recall that two manifolds X_1, X_2 are called K-equivalent if there is a smooth manifold $\tilde{X}$ and a diagram:

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \phi_1 \swarrow & & \searrow \phi_2 & \\ X_1 & & & & X_2 \end{array} \tag{12}$$

in which ϕ_1 and ϕ_2 are birational morphisms and $\phi_1^*(K_{X_1})$ and $\phi_2^*(K_{X_2})$ are linearly equivalent.

The push forward formula (1.5) leads to the following:

Theorem 1.7. *For any $\alpha \in H^*(B\pi, \mathbb{Q})$ the higher elliptic genus $(\mathcal{E}LL(X) \cup f^*(\alpha), [X])$ is an invariant of K-equivalence. Moreover, if (X, D, G) and $(\hat{X}, \hat{D}, G)$ are G-normal and Kawamata log-terminal and if $\phi : (\hat{X}, \hat{D}) \to (X, D)$ is G-equivariant such that*

$$\phi^*(K_X + D) = K_{\hat{X}} + \hat{D} \tag{13}$$

then

$$Ell_\alpha(\hat{X}, \hat{D}, G) = Ell_\alpha(X, D, G)$$

In particular the higher elliptic genera (and hence the higher signatures and $\hat{A}$-genus) are invariant for crepant morphisms. The specialization into the Todd class is birationally invariant (i.e. condition of invariance (12) is not needed in Todd case).

Another consequence is possibility to define higher elliptic genus for singular varieties with Kawamata log-terminal singularities and for G-normal pairs (X, D) (cf. [10]).

1.5. **DMVV formula.** Elliptic genus comes into a beautiful product formula for the generating series for orbifold elliptic genus associated with the action of the symmetric group S_n on products $X \times ... \times X$ and for which the first case appears in [17] (with string-theoretical explanation). A general product formula for orbifold elliptic genus of triples is given in [9].

Theorem 1.8. *Let (X, D) be a Kawamata log-terminal pair. For every $n \geq 0$ consider the quotient of $(X, D)^n$ by the symmetric group S_n, which we will denote by $(X^n/S_n, D^{(n)}/S_n)$. Here we denote by $D^{(n)}$ the sum of pullbacks of D under n canonical projections to X. Then we have*

$$\sum_{n \geq 0} p^n Ell(X^n/S_n, D^{(n)}/S_n; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}}, \tag{14}$$

where the elliptic genus of (X, D) is

$$\sum_{m \geq 0} \sum_{l} c(m, l) y^l q^m$$

and $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$.

It is amazing that such simple-minded construction as LHS of (14) leads to the Borcherds lift (cf. [4]) of Jacobi forms.

1.6. **Other applications of elliptic genus.** In this section we point out other instances in which elliptic genus plays significant role.

1.6.1. *Chiral deRham complex.* In work [40] for a complex manifold X the authors construct a (bi)-graded sheaf Ω_X^{ch} of vertex operator algebras (with degrees called fermionic charge and conformal weight) with the differential d_{DR}^{ch} having fermionic degree 1 and quasi-isomorphic to the deRham complex of X. An alternative construction in using the formal loop space was given in [31]. Each component of fixed conformal weight has filtration so that graded components are:

$$\otimes_{n\geq 1}(\Lambda_{-yq^{n-1}}T_X^* \otimes \Lambda_{-y^{-1}q^n}T_X \otimes S_{q^n}T_X^* \otimes S_{q^n}T_X) \tag{15}$$

In particular it follows that:

$$Ell(X,q,y) = y^{\frac{-\dim X}{2}}\chi(\Omega_X^{ch}) = y^{\frac{-\dim X}{2}} Supertrace_{H^*(\Omega_X^{ch})} y^{J[0]} q^{L[0]} \tag{16}$$

($J[m], L[n]$ are the operators which are part of the vertex algebra structure)). The chiral complex for orbifolds was constructed in [21] and the extension of relation (16) to orbifolds (with discrete torsion) is discussed in [38].

1.6.2. *Mirror symmetry.* Physics definition of mirror symmetry in terms of conformal field theory suggests that for the elliptic genus, defined as an invariant of a conformal field theory, (by an expression similar to the last term in (16), cf. [54]) one should have for X and its mirror partner $\hat{X}$ the relation:

$$Ell(X) = (-1)^{dim X} Ell(\hat{X}) \tag{17}$$

This is indeed the case (cf. [6], Remark 6.9) for the mirror symmetric hypersurfaces in toric varieties in the sense of Batyrev .

1.6.3. *Elliptic genus of Landau-Ginzburg models.* Physics literature (cf. e.g. [33]) also associates to a weighted homogeneous polynomial a conformal field theory (Landau-Ginzburg model) and in particular the elliptic genus. Moreover it is expected that orbifoldized Landau-Ginzburg model will coincide with the conformal filed theory of hypersurface corresponding to this weighted homogeneous polynomial. In particular one expects a certain identity expressing equality of orbifoldized elliptic genus corresponding to weighted homogeneous polynomial (or a more general Landau Ginzburg model) and the elliptic genus of the corresponding hypersurface. In [41] the authors construct a vertex operator algebra which is related by such type of a correspondence to the cohomology of the chiral deRham complex of the hypersurface in $\mathbb{P}^n$ and in particular obtain the expression for the elliptic genus of a hypersurface as an orbifoldization. Moreover, in [26] the authors obtain expression for the one variable Hirzebruch's genus as an orbifoldization.

1.6.4. *Concluding remarks.* There are several other interesting issues which should be mentioned in a discussion of elliptic genus. It plays important role in work J.Li, K.F.Liu and J.Zhou (cf. [37]) in connection with Gopakumar-Vafa conjecture (cf. also [24]). Elliptic genus was defined for proper schemes with 1-perfect obstruction theory ([20]). In fact one has well defined cobordism class in Ω^U associated to such objects (cf. [14]). In the case of surfaces with normal singularities, one can extend the above definition of elliptic genus beyond log-terminal singularities (cf. [49]). Elliptic genus is central in the study of elliptic cohomology ([45]). Much of the above discussion of can be extended to equivariant context (cf. [48]) and a survey of this is given in [50] in this volume.

2. Quasi-Jacobi forms

The Eisenstein series $e_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k}$ $(\tau \in \mathbb{H})$ fails to be modular for $k = 2$ but the algebra generated by functions $e_k(\tau), k \geq 2$, called the algebra of quasi-modular forms on $SL_2(\mathbb{Z})$, has many interesting properties (cf. [57]). In particular, there is a correspondence between quasi-modular forms and real analytic functions on $\mathbb{H}$ which have the same $SL_2(\mathbb{Z})$ transformation properties as modular forms. Moreover, the algebra of quasi-modular forms has a structure of $\mathcal{D}$-module and supports an extension of Rankin-Cohen operations on modular forms.

In this section we show that there is algebra of functions on $\mathbb{C}\times\mathbb{H}$ closely related to the algebra of Jacobi forms of index zero with similar properties. This algebra is generated by Eisenstein series $\sum \frac{1}{(z+\omega)^n}$ (sum over elements ω of a lattice $W \subset \mathbb{C}$). It has description in terms of real analytic functions satisfying functional equation of Jacobi forms and having other properties of quasi-modular forms mentioned in the last paragraph. It turns out that the space of functions on $\mathbb{C}\times\mathbb{H}$ generated by elliptic genera of arbitrary (possibly not Calabi Yaus) complex manifolds belong to this algebra of quasi-Jacobi forms.

Recall the following:

Definition 2.1. A weak (resp. meromorphic) Jacobi form of index $t \in \frac{1}{2}\mathbb{Z}$ and weight k for a finite index subgroup of the Jacobi group $\Gamma_1^J = SL_2(\mathbb{Z}) \propto \mathbb{Z}^2$ is a holomorphic (resp. meromorphic) function χ on $\mathbb{H}\times\mathbb{C}$ having expansion $\sum c_{n,r}q^n\zeta^r$ in $q = exp(2\pi\sqrt{-1}\tau)$ ($Im\tau$ sufficiently large) and satisfying the following functional equations:

$$\chi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}) = (c\tau+d)^k e^{\frac{2\pi itcz^2}{c\tau+d}}\chi(\tau,z)$$

$$\chi(\tau,z+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)}e^{-2\pi it(\lambda^2\tau+2\lambda z)}\chi(\tau,z)$$

for all elements $[\begin{pmatrix} a & b \\ c & d \end{pmatrix},0]$ and $[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},(a,b)]$ in Γ. The algebra of Jacobi forms is the bi-graded algebra $J = \oplus J_{t,k}$. and the algebra of Jacobi forms of index zero is the sub-algebra $J_0 = \oplus_k J_{0,k} \subset J$.

For appropriate l a Jacobi form can be expanded in (Fourier) series in $q^{\frac{1}{l}}$ (with l depending on Γ). We shall need below the following real analytic functions:

$$\lambda(z,\tau) = \frac{z-\bar{z}}{\tau-\bar{\tau}}, \qquad \mu(\tau) = \frac{1}{\tau-\bar{\tau}} \tag{18}$$

They have the following transformation properties:

$$\lambda(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}) = (c\tau+d)\lambda(z,\tau) - 2icz \tag{19}$$

$$\lambda(z+m\tau+n,\tau) = \lambda(z,\tau) + m$$

$$\mu(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2\mu(\tau) - 2ic(c\tau+d) \tag{20}$$

Definition 2.2. *Almost meromorphic Jacobi form* of weight k, index zero and depth (s,t) is a (real) meromorphic function in $\mathbb{C}\{q^{\frac{1}{t}}, z\}[z^{-1}, \lambda, \mu]$, with λ, μ given by (18) which

a) satisfies the functional equations (2.1) of Jacobi forms of weight k and index zero and

b) which has degree at most s in λ and at most t in μ.

Definition 2.3. *A quasi-Jacobi form* is a constant term of an almost meromorphic Jacobi form of index zero considered as a polynomial in the functions λ, μ i.e. a meromorphic function f_0 on $\mathbb{H} \times \mathbb{C}$ such that exist meromorphic functions $f_{i,j}$ such that $f_0 + \sum f_{i,j}\lambda^i\mu^j$ is almost meromorphic Jacobi form.

From algebraic independence of λ, μ over the field of meromorphic functions in q, z one deduces:

Proposition 2.4. *F is a quasi-Jacobi of depth (s,t) if and only if:*

$$(c\tau+d)^{-k} f(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = \sum_{i \le s, j \le t} S_{i,j}(f)(\tau,z)(\frac{cz}{c\tau+d})^i(\frac{c}{c\tau+d})^j$$

$$f(\tau, z+a\tau+b) = \sum_{i \le s} T_i(f)(\tau,z)a^i$$

To see some basic examples of quasi-Jacobi forms let us consider the following sequence of functions on $\mathbb{H} \times \mathbb{C}$.

Definition 2.5. (cf.[51][1]) $E_n(z,\tau) = \sum_{(a,b)\in\mathbb{Z}^2} \frac{1}{(z+a\tau+b)^n}$

Such series $E_n(z,\tau)$ converges absolutely for $n \ge 3$ and for $n = 1,2$ defined via "Eisenstein summation" as

$$\sum\nolimits_e = lim_{A\to\infty} \sum_{a=-A}^{a=A} (lim_{B\to\infty} \sum_{b=-B}^{b=B})$$

though we shall omit the subscript e. The series $E_2(z,\tau)$ is related to the Weierstrass function as follows:

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(a,b)\in\mathbb{Z},(a,b)\neq 0} \frac{1}{(z+a\tau+b)^2} - \frac{1}{(a\tau+b)^2} =$$

$$= E_2(z,\tau) - lim_{z\to 0}(E_2(z,\tau) - \frac{1}{z^2})$$

Moreover

$$e_n = lim_{z\to 0}(E_n(z,\tau) - \frac{1}{z^n}) = \sum_{(a,b)\in\mathbb{Z},(a,b)\neq 0} \frac{1}{(a\tau+b)^n}$$

is the Eisenstein series (notation of [51]). The algebra of functions of $\mathbb{H}$ generated by the Eisenstein series $e_n(\tau), n \ge 2$ is the algebra of quasi-modular forms for $SL_2(\mathbb{Z})$ (cf. [55], [57]).

Now we shall describe the algebra of quasi-Jacobi forms for Jacobi group Γ_1^J. We have the following:

[1]These series were recently used in [23] under the name twisted Eisenstein series.

Proposition 2.6. *The functions E_n are weak meromorphic Jacobi forms of index zero and weight n for $n \geq 3$. E_1 is a quasi-Jacobi form of index 0 weight 1 and depth $(1,0)$. $E_2 - e_2$ is a weak Jacobi form of index zero and weight 2 and E_2 is a quasi-Jacobi form of weight 2, index zero and depth $(0,1)$.*

Proof. The first part is due to the absolute convergence of series (2.5) for $n \geq 3$. We have the following transformation formulas

$$E_1(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)E_1(\tau,z) + \frac{\pi ic}{2}z \tag{21}$$

$$E_1(\tau, z+m\tau+n) = E_1(\tau,z) - 2\pi im$$

and

$$E_2(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^2E_2(\tau,z) - \frac{1}{2}\pi ic(c\tau+d) \tag{22}$$

$$E_2(\tau, z+a\tau+b) = E_2(\tau,z)$$

First equality in (21) (resp. (22)) follows from $E_1(\tau,z) = \frac{1}{z} - \sum e_{2k}(\tau)z^{2k-1}$ (resp. $E_2(\tau,z) = \frac{1}{z^2} + \sum_k (2k-1)e_{2k}z^{2k-2}$ (cf. [51] Ch.3 (10))). The second in (22) is immediate form the definition of Eisenstein summation while the second equality in (21) follows from [51]. □

Remark 2.7. Eisenstein series $e_k(\tau), k \geq 4$ belong to the algebra of quasi-Jacobi forms. Indeed, one has the following identities (cf. [51], (7),(35) in ch IV):

$$E_4 = (E_2 - e_2)^2 - 5e_4; \quad E_3^2 = (E_2 - e_2)^2 - 15e_4(E_2 - e_2) - 35e_4$$

Proposition 2.8. *The algebra of Jacobi forms (for Γ_1^J) of index zero and weight t ($t \geq 2$) is generated by $E_2 - e_2, E_3, E_4$.*

A short way to show this is to notice that the ring of such Jacobi forms is isomorphic to the ring of cobordisms of SU-manifolds modulo flops (cf. section 1.1) via isomorphism sending a complex manifold X of dimension d to $Ell(X) \cdot (\frac{\theta'(0)}{\theta(z)})^d$. This ring of cobordisms in turn is isomorphic to $\mathbb{C}[x_1, x_2, x_3]$ where x_1 is cobordism class of a K3 surface and x_2, x_3 are cobordism classes of certain four- and six-manifolds (cf. [47]). The graded algebra $\mathbb{C}[E_2 - e_2, E_3, E_4]$ is isomorphic to the same ring of polynomials (cf. examples 2.14) and the claim follows.

Proposition 2.9. *The algebra of quasi-Jacobi forms is the algebra of functions on $\mathbb{H} \times \mathbb{C}$ generated by functions $E_n(z,\tau)$ and $e_2(\tau)$.*

Proof. First notice that the coefficient of λ^s for an almost meromorphic Jacobi form $F(\tau,z) = \sum_{i \leq s} f_i\lambda^i$ of depth $(s,0)$ is holomorphic Jacobi form of index zero and weight $k-s$ i.e. by above proposition is a polynomial in $E_2 - e_2, E_3, ...$ Moreover $f_0 - E_1^s f_s$ is a quasi-Jacobi form of index zero and weight at most $s-1$. Hence by induction the ring of quasi-Jacobi forms of index zero and depth $(*,0)$ can be identified with $\mathbb{C}[E_1, E_2 - e_2, E_3, ...]$. Similarly, the coefficient μ^t of an almost meromorphic Jacobi form $F = \sum_{j \leq t}(\sum f_{i,j}\lambda^i)\mu^j$ is an almost meromorphic Jacobi form of depth $(s,0)$ and moreover $F - (\sum_i f)i, s\lambda^i)E_2^t$ has depth (s',t') with $t' < t$. The claim follows. □

Here is an alternative description of the algebra of quasi-Jacobi forms:

Proposition 2.10. *The algebra of functions generated by the coefficients of the Taylor expansion in x of the function:*

$$\frac{\theta(x+z)\theta'(0)}{\theta(x)\theta(z)} - (\frac{1}{x}+\frac{1}{z}) = \sum_{i\ge 1} F_i x^i$$

is the algebra of quasi-Jacobi forms (for $SL_2(\mathbb{Z})$).

Proof. The transformation formulas for theta function (cf. [44]):

$$\theta(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = \zeta(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau+d}} \theta(\tau,z) \tag{23}$$

$$\theta'(\frac{a\tau+b}{c\tau+d}, 0) = \zeta(c\tau+d)^{\frac{3}{2}} \theta'(\tau,0)$$

$$\theta(\tau, z+m\tau+n) = (-1)^{m+n} e^{-2\pi i m z - \pi i m^2 \tau} \theta(\tau,z)$$

imply that

$$\Phi(x,z,\tau) = \frac{x\theta(x+z)\theta'(0)}{\theta(x)\theta(z)} \tag{24}$$

satisfies the following functional equations:

$$\Phi(\frac{a\tau+b}{c\tau+d}, \frac{x}{c\tau+d}, \frac{z}{c\tau+d}) = e^{\frac{2\pi i c z x}{c\tau+d}} \Phi(x,z,\tau) \tag{25}$$

$$\Phi(x, z+m\tau+n, \tau) = e^{2\pi i m x} \Phi(x,z,\tau)$$

In particular in expansion

$$\frac{d^2(log(\Phi))}{dx^2} = \sum H_i x^i \tag{26}$$

the left hand side is invariant under transformations in (25) and the coefficient H_i is a Jacobi form of weight i and index zero for any i. Moreover the coefficients F_i in $\Phi(x,z,\tau) = 1 + \sum F_i(z,\tau)x^i$ are polynomials in F_1 and H_i. What remains to show is that E_i's determine $F_1, H_i, i \ge 1$ and vice versa. Recall that E_i has index zero (invariant with respect to shifts) and weight i. We shall use the following expressions:

$$\Phi(x,z,\tau) = \frac{x+z}{z} exp(\sum_{k>0} \frac{2}{k!}[x^k + z^k - (x+z)^k] G_k(\tau)) \tag{27}$$

where

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{l=1}^{\infty} \sum_{d|l} (d^{k-1}) q^l \tag{28}$$

(cf. [56]). On the other hand one has:

$$E_n(z,\tau) = \frac{1}{z^n} + (-1)^n \sum_{2m\ge n}^{\infty} \binom{2m-1}{n-1} e_{2m} z^{2m-n} \tag{29}$$

(cf. [51] Ch.III.sec.7 (10)) where

$$e_{2m} = \sum{}' (\frac{1}{m\tau+n})^{2m} = \frac{2(2\pi\sqrt{-1})^k}{(k-1)!} G_k \quad (\text{for } k=2m) \tag{30}$$

(cf. [51] Ch.III, sect 7 and [55] p.220). We have:

$$\frac{d^2log(\Phi(x,z,\tau))}{dx^2}=\sum_{i\geq 1}\frac{(-1)^i i x^{i-1}}{z^{i+1}}+\sum_{i\geq 2}\frac{2}{(i-2)!}[x^{i-2}-(x+z)^{i-2}]G_i(\tau) \tag{31}$$

The coefficient of x^{l-2} for $l\geq 2$ in Laurent expansion hence is:

$$\frac{(-1)^{l-1}(l-1)}{z^l}-\sum_{i\geq 2,i>l}\frac{2}{(i-2)!}\binom{i-2}{l-2}z^{i-l}G_i(\tau)= \tag{32}$$

$$\frac{(-1)^{l-1}(l-1)}{z^{l-1}}--\sum_{i\geq 2,i>l}\frac{1}{(2\pi\sqrt{-1})^i}(i-1)\binom{i-2}{l-2}z^{i-l}e_i=$$

$$\frac{(-1)^{l-1}(l-1)}{z^l}-(l-1)\frac{1}{(2\pi\sqrt{-1})^l}\sum_{i\geq 2,i>l}\binom{i-1}{l-1}e_i(\frac{z}{2\pi\sqrt{-1}})^{i-l}$$

(using (30) and identities with binomial coefficients). This yields

$$H_{l-2}(2\pi\sqrt{-1}z,\tau)=(-1)^{l-1}\frac{(l-1)}{(2\pi\sqrt{-1})^l}(E_l-e_l)$$

and the claim follows since (15) in [56] yields:

$$F_1(z,\tau)=\frac{1}{z}-2\sum_{r\geq 0}G_{r+1}\frac{z^r}{r!}=\frac{1}{z}-\frac{1}{(2\pi\sqrt{-1})}\sum_{r\geq 0}e_r(\frac{z}{2\pi\sqrt{-1}})^r \tag{33}$$

i.e.

$$F_1(2\pi i\sqrt{-1}z,\tau)=\frac{1}{2\pi\sqrt{-1}}E_1(z,\tau) \tag{34}$$

□

Remark 2.11. The algebra of quasi-Jacobi forms $\mathbb{C}[e_2,E_1,E_2,...]$ in closed under differentiations ∂_τ,∂_z. Indeed one has:

$$2\pi i\frac{\partial E_1}{\partial\tau}=E_3-E_1E_2. \qquad \frac{\partial E_1}{\partial z}=-E_2$$

$$2\pi i\frac{\partial E_2}{\partial\tau}=3E_4-2E_1E_3-E_2^2. \qquad \frac{\partial E_2}{\partial z}=-2E_3$$

and hence $\mathbb{C}[..E_i..]$ is a $\mathcal{D}$-module where $\mathcal{D}$ be the ring of differential operators generated by over the ring of holomorphic Jacobi group invariant functions on $H\times\mathbb{C}$ by $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial z}$. As is clear from the above discussion, the ring of Eisenstein series $\mathbb{C}[...E_i...]$ has natural identification with the ring of real valued almost meromorphic Jacobi forms $\mathbb{C}[E_1^*,E_2^*,E_3,....]$ on $\mathbb{H}\times\mathbb{C}$ having index zero where

$$E_1^*=E_1+2\pi i\frac{Imx}{Im\tau},E_2^*=E_2+\frac{1}{Im\tau} \tag{35}$$

Theorem 2.12. *The algebra of quasi-Jacobi forms of depth* $(k,0),k\geq 0$ *is isomorphic to the algebra of complex unitary cobordisms modulo flops.*

In another direction the depth of quasi-Jacobi forms allowing to "measure" the deviation of elliptic genus of a non-Calabi Yau manifold from being Jacobi form.

Theorem 2.13. *Elliptic genera of manifolds of dimension at most d span the subspace of forms of depth $(d,0)$ in the algebra of quasi-Jacobi forms. If a complex manifold satisfies $c_1^k=0, c_1^{k-1}\ne 0$* [2] *then its elliptic genus is a quasi-Jacobi form of depth $(s,0)$ where $s\le k-1$.*

Proof. It follows from the proof of proposition (2.10) that

$$\frac{d^2 log\Phi}{dx^2}=\sum_{i\ge 2}(-1)^{i-1}\frac{i-1}{(2\pi\sqrt{-1})^i}(E_i-e_i)x^{i-2}$$

which yields:

$$\Phi=e^{E_1x}\prod_i e^{\frac{1}{i}(-1)^{i-1}\frac{i-1}{(2\pi\sqrt{-1})^i}(E_i-e_i)x^i} \tag{36}$$

The Hirzebruch's characteristic series is (cf. (1)):

$$\Phi(\frac{x}{2\pi i})(\frac{\theta(z)}{\theta'(0)})$$

Hence if $c(TX)=\Pi(1+x_k)$ then

$$Ell(X)=(\frac{\theta(z)}{\theta'(0)})^{dimX}\prod_{i,k}e^{E_1x_k}e^{\frac{(-1)^{i-1}(i-1)}{i}(E_i-e_i)x_k^i}[X]= \tag{37}$$

$$(\frac{\theta(z)}{\theta'(0)})^{dimX}e^{c_1(X)E_1}\prod_{i,k}e^{\frac{(-1)^{i-1}(i-1)}{i}(E_i-e_i)x_k^i}[X]$$

(here $[X]$ is the fundamental class of X) i.e. if $c_1=0$ elliptic class is polynomial in E_i-e_i with $i\ge 2$ and hence elliptic genus is a Jacobi form (cf. [35]). Moreover of $c_1^k=0$ then the degree of this polynomial is at most k in E_1 and the claim follows. □

Example 2.14. Expression (37) can be used to get formulas for the elliptic genus of specific examples in terms of Eisenstein series E_n. For example for a surface in $\mathbb{P}^3$ having degree d one has

$$(E_1^2(\frac{1}{2}d^2-4d+8)d+(E_2-e_2)(\frac{d^2}{2}-2)d)(\frac{\theta(z)}{\theta'(0)}^2)$$

In particular for $d=1$ one obtains:

$$(\frac{9}{2}E_1^2-\frac{3}{2}(E_2-e_2))(\frac{\theta(z)}{\theta'(0)})^2$$

One can compare this with the double series which is a special case of the general formula for elliptic genus of toric varieties in [6]. This lead to a two variable version of the identity discussed in Remark 5.9 in [6]. In fact following [5] one can define sub-algebra “toric quasi-Jacobi forms” of the algebra of quasi-Jacobi forms extending toric quasi-modular forms considered in [5]. This issue will be addressed elsewhere.

Next let us consider one more similarity between meromorphic Jacobi forms and modular forms: there is a natural non commutative deformation of the ordinary product of Jacobi forms similar to the deformation of the product modular forms constructed using Rankin-Cohen brackets (cf. [57]). In fact we have the following Jacobi counterpart of the Rankin-Cohen brackets:

[2] more generally, k is the smallest among i with $c_1^i\in Ann(c_2,..c_{dimM})$; an example of such manifold is an n-manifold having $n-k$-dimensional Calabi Yau factor.

Proposition 2.15. *Let f (resp. g) be a Jacobi form of index zero and weight k (resp. l). Then*

$$[f,g]=k(\partial_\tau f-\frac{1}{2\pi i}E_1\partial_z f)g-l(\partial_\tau g-\frac{1}{2\pi i}E_1(z,\tau)\partial_z g)f$$

is a Jacobi form of weight $k+l+2$. More generally, let

$$D=\partial_\tau-\frac{1}{2\pi i}E_1\partial_z$$

Then the Cohen-Kuznetsov series (cf. [57]):

$$\tilde{f}_D(z,\tau,X)=\sum_{n=0}^{\infty}\frac{D^nf(z,\tau)X^n}{n!(k)_n}$$

(here $(k)_n=k(k+1)...(k+n-1)$ is the Pochhammer symbol) satisfies:

$$\tilde{f}_D(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},\frac{z}{c\tau+d},\frac{X}{(c\tau+d)^2})=(c\tau+d)^k exp(\frac{c}{c\tau+d}\frac{X}{2\pi i})f_D(\tau,z,X)$$

$$\tilde{f}_D(\tau,z+a\tau+b,X)=\tilde{f}_D(\tau,z,X)$$

In particular the coefficient

$$\frac{[f,g]_n}{(k)_n(l)_n}$$

of X^n in $\tilde{f}_D(\tau,z,-X)\tilde{g}_D(\tau,z,X)$ is a Jacobi form of weight $k+l+2n$. It is given explicitly in terms of D^if,D^jg by the same formulas as the classical RC brackets.

Proof. The main point is that the operator $\partial_\tau-\frac{1}{2\pi i}E_1\partial_z$ has the same deviation from transforming Jacobi form to Jacobi as ∂_τ on modular forms. Indeed:

$$(\partial_\tau-\frac{1}{2\pi i}E_1\partial_z)f(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=$$

$$\{kc(c\tau+d)^{k+1}f(\tau,z)+zc(c\tau+d)^{k+1}\partial_z f(\tau,z)+(c\tau+d)^{k+2}\partial_\tau f(\tau,z)$$

$$-\frac{1}{2\pi i}[(c\tau+d)E_1(\tau,z)+2\pi icz](c\tau+d)^{k+1}\partial_z f)\}=$$

$$(c\tau+d)^{k+2}[f(\tau,z)-\frac{1}{2\pi i}E_1f_z]+kc(c\tau+d)^{k+1}f(\tau,z)$$

Moreover:

$$(\partial_\tau-\frac{1}{2\pi i}E_1\partial_z)f(\tau,z+a\tau+b)=f_\tau+af_z-\frac{1}{2\pi i}(E_1-2\pi ia)f_z=(\partial_\tau-\frac{1}{2\pi i}E_1\partial_z)f(\tau,z)$$

The rest of the proof is the same as in [57]. □

Remark 2.16. The brackets introduced in Prop. 2.15 are different from the Rankin-Cohen bracket introduced in [13].

3. Real singular varieties

Ochanine genus of an oriented differentiable manifold X can be defined using the following series with coefficients in $\mathbf{Q}[[q]]$ as the Hirzebruch characteristic power series (cf. volume [36] and references there):

$$Q(x) = \frac{x/2}{\sinh(x/2)} \prod_{n=1}^{\infty} [\frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}]^{(-1)^n} \tag{38}$$

As was mentioned in section 1.2, this genus is a specialization of the two variable elliptic genus (at $z = \frac{1}{2}$). Evaluation of Ochanine genus of a manifold using (38) and viewing the result as function of τ on the upper half plane (where $q = e^{2\pi i \tau}$) yields a modular form on $\Gamma_0(2) \subset SL_2(\mathbb{Z})$ (cf. [36]).

In this section we discuss elliptic genera for real algebraic varieties. It particular we address B.Totaro proposal that "it should be possible to define Ochanine genus for a large class of compact oriented real analytic spaces" (cf. [47]). In this direction [47] contains the following result:

Theorem 3.1. *Quotient of MSO by ideal generated by oriented real flops and complex flops* [3] *is*

$$\mathbb{Z}[\delta, 2\gamma, 2\gamma^2, 2\gamma^4..]$$

with $\mathbb{CP}^2$ (resp. $\mathbb{CP}^4$) corresponding to δ (resp. $2\gamma + \delta^2$). This quotient ring is the the image of MSO_* under the Ochanine genus.

In particular Ochanine genus of a small resolution is independent of its choice for singular spaces having singularities only along non-singular strata and having in normal directions only singularities which are cones in $\mathbb{R}^4$ or $\mathbb{C}^4$.

Our goal is to find a wider class of singular real algebraic varieties for which Ochanine genus of a resolution is independent of a choice of the latter.

3.1. **Real Singularities.** For the remainder of this paper "real algebraic variety" means an *oriented* quasi-projective variety $X_{\mathbb{R}}$ over $\mathbb{R}$, $X(\mathbb{R})$ is the set of its $\mathbb{R}$-points with Eucledian topology, $X_{\mathbb{C}} = X_{\mathbb{R}} \times_{Spec\mathbb{R}} Spec\mathbb{C}$ is the complexification and $X(\mathbb{C})$ the analytic space of its complex points. We also assume that $dim_{\mathbb{R}} X(\mathbb{R}) = dim_{\mathbb{C}} X(\mathbb{C})$.

Definition 3.2. A real algebraic variety $X_{\mathbb{R}}$ as above is called $\mathbb{Q}$-Gorenstein log-terminal if the analytic space $X(\mathbb{C})$ is $\mathbb{Q}$-Gorenstein log-terminal.

Example 3.3. Affine variety

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0 \tag{39}$$

in $\mathbb{R}^4$ is 3-dimensional Gorenstein log-terminal and admit a crepant resolution.

Indeed it is well known that complexification of Gorenstein singularity (39) admits a small (and hence crepant) resolution having $\mathbb{P}^1$ as its exceptional set.

Example 3.4. The 3-dimensional complex cone in $\mathbb{C}^4$ given by $z_1^2+z_2^2+z_3^2+z_4^2 = 0$ considered as codimension two sub-variety of $\mathbb{R}^8$ is a $\mathbb{Q}$-Gorenstein log-terminal variety over $\mathbb{R}$ and its complexification admits a crepant resolution.

---

[3] i.e. the ideal generated by $X' - X$ where X' and X are related by real or complex flop

Indeed, this codimension two sub-variety is a real analytic space which is the intersection of two quadrics in $\mathbb{R}^8$ given by

$$a_1^2+a_2^2+a_3^2+a_4^2-b_1^2-b_2^2-b_3^2-b_4^2=0 \tag{40}$$
$$a_1b_1+a_2b_2+a_3b_3+a_4b_4=0$$

(here $a_i = Rez_i, b_i = Imz_i$). The complexification, is the cone over complete intersection of two quadrics in $\mathbb{P}^7$. Moreover, the defining equations of this complete intersection after the change of coordinates: $x_i = a_i + \sqrt{-1}b_i, y_i = a_i - \sqrt{-1}b_i$ become:

$$x_1^2+x_2^2+x_3^2+x_4^2=0 \tag{41}$$
$$y_1^2+y_2^2+y_3^2+y_4^2=0$$

The singular locus is the union of two disjoint 2-dimensional quadrics and singularity along each is A_1 (i.e. the intersection of the transversal to it in $\mathbb{P}^7$ has A_1-singularity). To resolve (40), one can blow up $\mathbb{C}^8$ at the origin which results in $\mathbb{C}$-fibration over complete intersection (41). It can be resolved by small resolutions along two non-singular components of singular locus of (41). A direct calculation shows [4] that we have a log-terminal resolution of Gorenstein singularity which is the complexification of (40).

3.2. **Elliptic genus of resolutions of real varieties with $\mathbb{Q}$-Gorenstein log-terminal singularities.** Let X be a real algebraic manifold and let $D = \sum \alpha_k D_k$ ($\alpha_k \in \mathbb{Q}$) be a divisor on the complexification $X_{\mathbb{C}}$ of X (i.e. D_k are irreducible components of D). Let x_i denote the Chern roots of the tangent bundle of $X_{\mathbb{C}}$ and d_k are the classes corresponding to D_k (cf. section 1).

Definition 3.5. Let X be a real algebraic manifold and D a divisor on complexification $X_{\mathbb{C}}$ of X. The Ochanine class $\mathcal{E}LL_{\mathcal{O}}(X,D)$ of pair X, D is specialization:

$$\mathcal{E}LL(X_{\mathbb{C}}, D, q, z=\frac{1}{2})$$

of the two variable elliptic class of pair $\mathcal{E}LL(X_{\mathbb{C}}, D, q, z)$ given by:

$$\Big(\prod_l \frac{(\frac{x_l}{2\pi \mathrm{i}})\theta(\frac{x_l}{2\pi \mathrm{i}}-z)\theta'(0)}{\theta(-z)\theta(\frac{x_l}{2\pi \mathrm{i}})}\Big) \times \Big(\prod_k \frac{\theta(\frac{d_k}{2\pi \mathrm{i}}-(\alpha_k+1)z)\theta(-z)}{\theta(\frac{d_k}{2\pi \mathrm{i}}-z)\theta(-(\alpha_k+1)z)}\Big) \tag{42}$$

Ochanine elliptic genus of pair (X, D) as above is

$$Ell(X_{\mathbb{R}}, D) = \sqrt{\mathcal{E}ll(X_{\mathbb{C}}, D, q, \frac{1}{2})} \cup cl(X(\mathbb{R}))[X(\mathbb{C})] \tag{43}$$

Here $\sqrt{\mathcal{E}ll}$ denotes the class corresponding to the unique series with constant term equal to 1 and having $\mathcal{E}ll$ as its square.

The above class of pair is the class (3) considered in definition 1.4 in section 1 with group G being trivial. One can defined orbifold version of this class as well specializing (3) to $z = \frac{1}{2}$. See [8] for further discussion of the class $\mathcal{E}ll(X, D)$.

The relation with Ochanine's definition is as follows: if D is trivial divisor on $X_{\mathbb{C}}$ then the result coincides with the genus [42]. More precisely, we have the following:

[4]e.g. by considering the order of the pole of the form $\frac{dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4}{x_1 y_1}$ along the intersection of the exceptional locus of blow up $\tilde{\mathbb{C}^8}$ of $\mathbb{C}^8$ with the proper preimage of (41) in $\tilde{\mathbb{C}^8}$

Lemma 3.6. *Let $X_{\mathbb{R}}$ be a real algebraic manifold with non singular complexification $X_{\mathbb{C}}$. Then*

$$Ell(X_{\mathbb{R}}) = \sqrt{\mathcal{E}ll(T_{X(\mathbb{C})})} \cup cl(X(\mathbb{R}))[X(\mathbb{C})]$$

Proof. Indeed, we have

$$0 \to T_{X(\mathbb{R})} \to T_{X(\mathbb{C})}|_{X(\mathbb{R})} \to T_{X(\mathbb{R})} \to 0 \tag{44}$$

with the identification of the normal bundle to $X_{\mathbb{R}}$ with its tangent bundle given by the multiplication by $\sqrt{-1}$. Hence $\mathcal{E}ll(X_{\mathbb{R}})^2 = i^*(\mathcal{E}ll X_{\mathbb{C}})$ where $i : X_{\mathbb{R}} \to X_{\mathbb{C}}$ is the canonical embedding. Now the lemma follows from the identification of the characteristic series (38) and specialization $z = \frac{1}{2}$ of the series in (1) (cf. [6]) and the identify which is just a definition of the class $cl_Z \in H^{\dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} Z}$ of a submanifold Z of a manifold Y: $cl_Z \cup \alpha[Y] = i^*(\alpha) \cap [Z]$ for any $\alpha \in H^{\dim_{\mathbb{R}} Z}(Y)$. Indeed, we have the following:

$$Ell(X_{\mathbb{R}}) = \mathcal{E}ll(T_{X(\mathbb{R})})[X(\mathbb{R})] = \sqrt{\mathcal{E}ll(T_{X(\mathbb{C})})|_{X(\mathbb{R})})}[X(\mathbb{R})] = \tag{45}$$

$$\sqrt{\mathcal{E}ll(T_{X(\mathbb{C})})} \cup cl(X(\mathbb{R}))[X(\mathbb{C})]$$

□

Our main result in this section is the following:

Theorem 3.7. *Let $\pi : (\tilde{X}, \tilde{D}) \to (X, D)$ be a resolution of singularities of a real algebraic pair with $\mathbb{Q}$-Gorenstein log-terminal singularities i.e. $K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D)$. Then the elliptic genus of the pair $(\tilde{X}, \tilde{D})$ is independent of resolution. In particular if real algebraic variety X has a crepant resolution then its elliptic genus is independent of a choice of crepant resolution.*

Proof. Indeed for a blow up $f : (\tilde{X}, \tilde{D}) \to (X, D)$ we have:

$$f_*(\sqrt{\mathcal{E}LL(\tilde{X}, \tilde{D}, q, \frac{1}{2}}) = \sqrt{\mathcal{E}LL(X, D, q, \frac{1}{2}}) \tag{46}$$

This is a special case of the push-forward formula (4) in theorem 1.5 (with G being trivial group). Hence

$$\mathcal{E}ll_{\mathcal{O}}(X_{\mathbb{R}}, D) = \sqrt{\mathcal{E}ll(X_{\mathbb{C}}, D, q, \frac{1}{2})} \cup cl(X_{\mathbb{R}})[X_{\mathbb{C}}] = \tag{47}$$

$$\sqrt{\mathcal{E}LL(\tilde{X}_{\mathbb{C}}, \tilde{D}, q, \frac{1}{2})} \cup f^*([X_{\mathbb{R}}] \cap [X_{\mathbb{C}}]) = \mathcal{E}LL(\tilde{X}_{\mathbb{R}}, \tilde{D})$$

as follows from projection formula since $f^*(cl[X_{\mathbb{R}}]) = [cl\tilde{X}_{\mathbb{R}}]$ and since f_* is identity on H_0.

For crepant resolution one has $D = 0$ and hence by lemma 3.6 the elliptic genus of $X_{\mathbb{R}}$ is the Ochanine genus of the real manifold which is its crepant resolution.

□

Remark 3.8. Examples 3.3 and 3.4 show that singularities admitting crepant resolution include real 3-dimensional cones and real points of complex 3-dimensional cones.

References

[1] D. Abramovich, K. Karu, K. Matsuki, J. Włodarsczyk, *Torification and Factorization of Birational Maps*, Journal AMS, 15 (2002) no.3 p. 531-572.

[2] V.Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. J. Eur. Math. Soc. (JEMS) 1 (1999), no. 1, 5–33.

[3] J.Block, S.Weinberger, Higher Todd classes and holomorphic group actions. (English summary) Pure Appl. Math. Q. 2 (2006), no. 4, part 2, 1237–1253.

[4] R.Borcherds, Automorphic forms on $O_{s+2,2}(\mathbf{R})^+$ and generalized Kac-Moody algebras. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 744–752, Birkhuser, Basel, 1995.

[5] L.Borisov, P.Gunnells, Toric modular forms, Inventiones Math. 144 (2001) p.297-325.

[6] L.Borisov, A.Libgober, Elliptic genera of toric varieties and applications to mirror symmetry. Invent. Math. 140 (2000), no. 2, 453–485.

[7] L.Borisov, A.Libgober, Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex. Mirror symmetry, IV (Montreal, QC, 2000), 325–342, AMS/IP Stud. Adv. Math., 33, Amer. Math. Soc., Providence, RI, 2002.

[8] L.Borisov, A.Libgober, Elliptic genera of singular varieties. Duke Math. J. 116 (2003), no. 2, 319–351.

[9] L.Borisov, A.Libgober, McKay correspondence for elliptic genera. Ann. of Math. (2) 161 (2005), no. 3, 1521–1569.

[10] L.Borisov, A.Libgober, Higher elliptic genera. Math. Res. Lett. 15 (2008), no. 3, 511–520.

[11] S.Cappell, A.Libgober, L.Maxim, J.Shaneson, Hodge genera and characteristic classes of complex algebraic varieties. Electron. Res. Announc. Math. Sci. 15 (2008), 1–7.

[12] K.Chandrasekharan *Elliptic functions*, Fundamental Principles of Mathematical Sciences, **281**, Springer-Verlag, Berlin-New York, 1985.

[13] Y.J.Choie, W.Eholzer, Rankin-Cohen operators for Jacobi and Siegel forms. J. Number Theory 68 (1998), no. 2, 160–177.

[14] I.Ciocan-Fontanine, M.Kapranov, Virtual fundamental classes via dg-manifolds, arXiv:math/0703214.

[15] J.Davis, *Manifold aspects of the Novikov conjecture*, Surveys in surgery theory, vol.1, 195-224, Ann. of Math. Studies, Princeton Univ. Press. Princeton, N.J. 2000.

[16] P.Deligne, Theorie de Hodge. II, Publ. Math. IHES 40, 5-58 (1971). Theorie de Hodge III, Publ. Math. IHES 44, 5-77, (1974)

[17] R. Dijkgraaf, D. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Comm. Math. Phys. **185** (1997), no. 1, 197–209.

[18] J.Denef, F.Loeser, Geometry on arc spaces of algebraic varieties. European Congress of Mathematics, Vol. I (Barcelona, 2000), 327–348, Progr. Math., 201, Birkhuser, Basel, 2001

[19] C.Dong, K.F. Liu, X.Ma, On orbifold elliptic genus. Orbifolds in mathematics and physics (Madison, WI, 2001), 87–105, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[20] B.Fantechi, L.Göttsche Riemann-Roch theorems and elliptic genus for virtually smooth Schemes. arXiv:0706.0988.

[21] E.Frenkel, M.Szczesny, Chiral de Rham complex and orbifolds. J. Algebraic Geom. 16 (2007), no. 4, 599–624.

[22] W. Fulton, *Intersection Theory, second edition*, Springer-Verlag, Berlin, 1998.

[23] M.Gaberdiel, C.Keller, Differential Operators for elliptic genera. hep-th 0904.1831.

[24] E.Gasparim, Melissa Liu, The Nekrasov Conjecture for Toric Surfaces, arXiv:0808.0884

[25] M.Goresky, R.MacPherson. Problems and bibliography on intersection homology. ed. A.Borel et al.221-233. Birkhaüser, Boston (1984).

[26] V.Gorbounov,S.Ochanine, Mirror symmetry formulae for the elliptic genus of complete intersections. J. Topol. 1 (2008), no. 2, 429–445.

[27] F. Hirzebruch, *Topological methods in Algebraic Geometry*, translated from German and Appendix One by R. L. E. Schwarzenberger. With a preface to the third English edition by the author and Schwarzenberger. Appendix Two by A. Borel. Reprint of the 1978 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[28] F.Hirzebruch, *Elliptic genera of level N for complex manifolds*, Differential Geometric methods in Theoretical Physics (Como 1987). K. Bleuer, M. Werner Editors, NATO Adv. Sci.Inst.Ser. C: Math.Phys. Sci; 250. Dordrecht, Kluwer Acad.Publ.,1988.

[29] F.Hirzebruch,T.Berger,R.Jung, Manifolds and modular forms. With appendices by Nils-Peter Skoruppa and by Paul Baum. Aspects of Mathematics, E20. Friedr. Vieweg and Sohn, Braunschweig, 1992.

[30] M.Kaneko, D.Zagier, A generalized Jacobi theta function and quasimodular forms. The moduli space of curves (Texel Island, 1994), 165–172, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995.

[31] M.Kapranov, E.Vasserot, Vertex algebras and the formal loop space. Publ. Math. Inst. Hautes tudes Sci. No. 100 (2004), 209–269.

[32] M.Kapranov, E.Vasserot Formal loops IV: Chiral differential operators, arXiv:math/0612371.

[33] T. Kawai, Y. Yamada, and S.-K. Yang, Elliptic genera and $N = 2$ superconformal field theory, Nuclear Phys. B 414 (1994), no. 1–2, 191–212.

[34] Y.Kawamata, K.Matsuda, K.Matsuki, Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[35] I.Krichever, Generalized elliptic genera and Baker-Akhiezer functions, Math. Notes, 47 (1990), 132-142.

[36] Elliptic curves and modular forms in algebraic topology, P. S. Landweber, ed., Springer-Verlag (1988)

[37] J.Li, K.F.Liu and J.Zhou. Topological string partition functions and equivariant indices, Asian J. Math, 10 (2006) no.1 p.81-114.

[38] A.Libgober, M. Szczesny, Discrete torsion, orbifold elliptic genera and the chiral de Rham complex. Pure Appl. Math. Q. 2 (2006), no. 4, part 2, 1217–1236.

[39] E. Looijenga, *Motivic measures*, Séminaire Bourbaki, Vol. 1999/2000. Astérisque **276** (2002), 267–297.

[40] F.Malikov, V.Schechtman, A.Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys. **204** (1999), no. 2, 439–473.

[41] F.Malikov, V.Gorbounov. Vertex algebras and the Landau-Ginzburg/Calabi-Yau correspondence. Mosc. Math. J. 4 (2004), no. 3, 729–779, 784.

[42] S.Ochanine, Sur les genres multiplicatifs definis par des intgrales elliptiques. Topology 26 (1987), no. 2, 143–151.

[43] J.Rosenberg, An analogue of the Novikov conjecture in complex algebraic geometry. Trans. Amer. Math. Soc. 360 (2008), no. 1, 383–394

[44] J.Tannery, J.Molk. Elements de la theorie des fonctions elliptiques. Bronx, NY, Chelsea Publ. Company. 1972.

[45] C.B.Thomas, Elliptic cohomology. The University Series in Mathematics. Kluwer Academic/Plenum Publishers, New York, 1999.

[46] B.Totaro, Chern numbers for singular varieties and elliptic homology. Ann. of Math. (2) 151 (2000), no. 2, 757–791.

[47] B.Totaro. The elliptic genus of a singular variety. Elliptic cohomology, 360–364, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[48] R.Waelder, Equivariant elliptic genera, Pacific Math. J. 235(2):345-277, 2008.

[49] R.Waelder, Singular McKay correspondence for normal surfaces, math.AG/0810.3634

[50] R.Waelder, Rigidity of differential operators and Chern numbers of singular varieties. These Proceedings.

[51] A.Weil, Elliptic functions according to Eisenstein and Kronecker. Springer-Verlag, Berlin, 1999.

[52] C.L.Wang, K-equivalence in birational geometry and characterizations of complex elliptic genera. J. Algebraic Geom. 12 (2003), no. 2, 285–306.

[53] E.Witten, Elliptic curves and modular forms in algebraic topology, P. S. Landweber, ed., Springer-Verlag (1988

[54] E.Witten, Mirror manifolds and topological field theory Essays on mirror manifolds. Edited by Shing-Tung Yau. International Press, Hong Kong, 1992. (120–158)

[55] D. Zagier, Note on the Landweber-Stong elliptic genus, Elliptic curves and modular forms in algebraic topology, P. S. Landweber, ed., Springer-Verlag (1988), 216–224.

[56] D.Zagier, Periods of modular forms and Jacobi theta functions. Invent. Math. 104 (1991), no. 3, 449–465.

[57] D.Zagier, Elliptic modular forms and their applications. The 1-2-3 of modular forms, 1–103, Universitext, Springer, Berlin, 2008.

Department of Mathematics, University of Illinois, Chicago, IL 60607
E-mail address: libgober@math.uic.edu