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On Mordell–Weil groups of isotrivial abelian varieties over function fields

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Abstract We show that the Mordell–Weil rank of an isotrivial abelian variety with cyclic holonomy depends only on the fundamental group of the complement to the discriminant, provided the discriminant has singularities in CM class introduced here. This class of singularities includes all unibranched plane curves singularities. As a corollary, we describe a family of simple Jacobians over the field of rational functions in two variables for which the Mordell–Weil rank is arbitrarily large.

1 Introduction

Let $\mathcal{A}$ be an abelian variety over a function field K of characteristic zero. The group of K-points of $\mathcal{A}$ is an interesting *algebro-geometric* invariant. If $dim_K \mathcal{A} = 1$, $deg.tr.K/\mathbb{C} = n$, then it is closely related to the Neron-Severi group of the corresponding elliptic $(n+1)$-fold (cf. [34,41]). In this note we consider a class of abelian varieties $\mathcal{A}$ over the function field $K = \mathbb{C}(x, y)$ for which the Mordell–Weil rank can be described in *topological* terms. This description extends the results of [7] where the case of elliptic curves over $K = \mathbb{C}(x, y)$ was studied in detail.

We shall work with a non-singular projective model of $\mathcal{A}$, i.e. assume that $\mathcal{A}$ is a smooth projective variety together with a flat morphism

$$\pi : \mathcal{A} \to \mathbb{P}^2 \tag{1}$$

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such that fibers over closed points in the Zariski open subset of $X = \mathbb{P}^2$ are polarized abelian varieties over $\mathbb{C}$. Our main results relate the Mordell–Weil rank of $\mathcal{A}$ to the fundamental group of the complement to the discriminant Δ of (1).

The restrictions which we impose on the abelian variety $\mathcal{A}$, allowing one to express the Mordell–Weil rank topologically, are the following:

A1. $\mathcal{A}$ is isotrivial.
A2. The holonomy group of the family (1), (cf. Sect. 2.1) is cyclic.
A3. The singularities of the discriminant have CM type (cf. Sect. 3.3).

In the case of elliptic *surfaces* (i.e. $dim_K \mathcal{A} = 1$, $deg.tr.K/\mathbb{C} = 1$) satisfying the condition A1, the condition A2 is automatically fulfilled. However, the Mordell–Weil rank is far from being topological as the examples in [28] show. Besides giving bounds on the Mordell–Weil rank (cf. Theorem 1.2), we also present several classes of families $\mathcal{A}$ for which the rank can be calculated explicitly. In addition to the conditions A1–A3 above, we limit our-selves to the case of abelian varieties for which the discriminant is irreducible. We imposed this condition to simplify the exposition.

The data which affects the Mordell–Weil rank of $\mathcal{A}$ in fact requires only a small portion of the fundamental group of the complement to Δ. It is the same as the data controling the Betti numbers of cyclic branched covers of $\mathbb{P}^2$ with the ramification locus coinciding with the discriminant of morphism (1). As was shown in [25], Betti numbers of cyclic branched covers can be expressed in terms of the quotient π_1/π_1'' of the fundamental group $\pi = \pi_1(\mathbb{P}^2 - \Delta)$ by its second commutator π_1''. It is convenient to express them in terms of the Alexander invariant of $\Delta \subset \mathbb{P}^2$ i.e. the vector space $\pi_1'/\pi_1'' \otimes \mathbb{C}$ considered as a module over the group ring $\mathbb{C}[\pi_1/\pi_1']$ of the abelianization of π_1 and ultimately this Alexander invariant represents the topological data which controls Mordell–Weil ranks of abelian varieties (1).

Results of this note show that for abelian varieties considered below, the Mordell–Weil rank depends, besides the type of the generic fiber, the degree and the local type of singularities of the discriminant, on the dimensions of certain linear systems of curves determined by the local type of singularities of the latter. This is a consequence of known results showing that the Alexander invariants of plane singular curves depend only on this data (cf. [27] and Sect. 2.5 below and references therein). Recently in [18], a relation was obtained between the rank of the elliptic curves and the dimensions of such linear systems in the case when $\mathcal{A}$ has the discriminant *with cusps and nodes as the only singularities*, using methods different than those used in this paper (i.e. studying the syzygies of the locus of cusps of the discriminant).

One of the key ingredients in the proof of above mentioned results, having independent interest, is the decomposition theorem of the Albanese varieties of cyclic branched covers of $\mathbb{P}^2$. In the context of abelian varieties (1) these covers come up since the abelian varieties satisfying above conditions A1 and A2 become trivial over cyclic extensions of $\mathbb{C}(x, y)$. We show that the Albanese variety of the cyclic cover of $\mathbb{P}^2$, over which the pull back of $\mathcal{A}$ is trivial splits up to isogeny into a product of abelian varieties of CM type, assuming that the singularities of the branching locus of the cyclic cover of $\mathbb{P}^2$ have CM type. More precisely, we have the following (similar result was obtained in [7] in the case when $\mathcal{A}$ is an elliptic curve but with slightly different assumptions on singularities):

Theorem 1.1 *Let Δ be an irreducible and reduced curve in $\mathbb{P}^2$ such that all its singularities have CM type. Then the Albanese variety of a cyclic cover of $\mathbb{P}^2$ ramified along Δ is isogenous to a product of abelian varieties of CM type each having as its endomorphism algebra an etale algebra which is a product of cyclotomic fields.*

The definition of singularities of a CM type in the case of plane curves is given in terms of the local Albanese variety which is equivalent to the data of weight one part and its Hodge filtration for the mixed Hodge structure on the cohomology of the Milnor fiber of the singular points (cf. Definition 3.4). The local Albanese variety is the special case of the abelian variety associated by Deligne with 1-motif in [9]. We say that a plane curve singularity has a CM type if its local Albanese variety has a CM type. We refer to [31,36] or [30] for information on abelian varieties of CM type but recall that those are abelian varieties A with $End(A) \otimes \mathbb{Q}$ containing an (etale) $\mathbb{Q}$-subalgebra of rank $2\dim A$ isomorphic to a product of fields (in the case of CM-singularities, we show that these fields are cyclotomic).

The class of plane curve singularities of CM type is rather large: it includes all unibranched singularities (cf. Theorem 3.12), simple singularities, δ-essential singularities in the sense of [7] etc. However, ordinary multiple points of multiplicity greater than 3 do not have CM type in general (cf. Sect. 3).

The precise relation between the topology of the complement to the discriminant and the Mordell–Weil rank is given as follows.

Theorem 1.2 *Let $\mathcal{A}$ be an isotrivial abelian variety over field $\mathbb{C}(x, y)$, π be morphism (1) and A be its generic fiber. Let $\Delta \subset \mathbb{P}^2$ be the discriminant of π and let $G \subset Aut\, A$ be the holonomy group of $\mathcal{A}$ (cf. 2.1). Assume that:*

(a) *G is a cyclic group of order d acting on generic fiber A of (1) without fixed subvarieties of a positive dimension.*
(b) *The singularities of Δ have CM type and Δ is irreducible.*

Then

1. *the rank of the Mordell–Weil group of $\mathcal{A}$ is zero, unless the generic fiber of π is an abelian variety of CM-type with endomorphism algebra containing a cyclotomic field.*
2. *Assume that the generic fiber A of π is a simple abelian variety of CM type corresponding to the field $\mathbb{Q}(\zeta_d)$. Let s be the multiplicity of the factor $\Phi_d(t)$ of the Alexander polynomial of $\pi_1(\mathbb{P}^2 - \Delta)$ where $\Phi_d(t)$ is the cyclotomic polynomial of degree d. Then:*
$$\mathrm{rk} MW(\mathcal{A}, \mathbb{C}(x, y)) \leq s \cdot \phi(d) \tag{2}$$
(here $\phi(d) = deg \Phi_d(t)$ is the Euler function).
3. *Let A be an abelain variety as in 2. If d is the order of the holonomy of $\mathcal{A}$ and the Albanese variety $Alb(X_d)$ of the d-fold cover X_d of X ramified over Δ has A is its direct summand with multiplicity s then one has equality in (2).*

Theorem 1.2 has the following as an immediate consequence:

Corollary 1.3 *If $\mathcal{A}$ is a family (1) with generic fiber A such that $End(A)\otimes\mathbb{Q}=\mathbb{Q}(\zeta_d)$ holds and such that for each singular point of the discriminant the monodormy operator has no primitive roots of unity of degree d as an eigenvalue then $rkMW(\mathcal{A})=0$.*

On the other hand, for the Jacobian of the curve over $\mathbb{C}(x,y)$ given in (u,v) plane by the equation

$$u^p=v^2+(x^p+y^p)^2+(y^2+1)^p \tag{3}$$

one has $rkMW=p-1$ (cf. 5.2). The Jacobian of generic fiber of the family (3) is a simple abelian variety.

It would be interesting to know if ranks of isotrivial abelian varieties over $\mathbb{C}(x,y)$ with fixed generic fiber are bounded (or have non-trivial bounds in terms of degree of discriminant). Note that inequality (2) only shows that with assumption of Theorem 1.2 such bounds are equivalent to the bounds on the possible multiplicities of Φ_d in the Alexander polynomial of the discriminant. Very little is known at the moment about bounds on such multiplicites (cf. [7] containing a discussion of the relation between the bounds on the rank and the degree of the latter).

The content of this paper is as follows. In the next section we recall the background material used below. The Sect. 3 discusses the local Albanese varieties of plane curve singularities and cases when they have CM type. The decomposability of the Albanese variety of cyclic branched covers (under certain conditions) is proved in Sect. 4. Section 5 contains the proof of the Theorem 1.2 and gives examples of specific situations in which the above theorem can be applied. The Theorem 1.2 in fact can be used in both directions: it gives many examples in which one obtains explicitly the rank of the Mordell–Weil group. On the other hand, it also provides a mean to give a bound on the complexity of the Alexander module of certain curves (cf. [7]). The Example 5.2, discussing the discriminant with largest known multiplcity of $\Phi_d(t)$ is given at the end of the last section.

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2 Abelian varieties over transcendental extensions of $\mathbb{C}$

2.1 Isotrivial abelian varieties, discriminant and holonomy

As in the Introduction, we fix a flat proper morphism of smooth complex projective varieties $\pi:\mathcal{A}\to X$ with generic fiber being an abelian variety over $\mathbb{C}$, i.e. an abelian variety over $\mathbb{C}(X)$.

A *rational* section (resp. a section) of π is a rational (resp. regular) map $s:X\to\mathcal{A}$ such that $\pi\circ s$ is the identity on the domain of s.

An abelian variety $\pi:\mathcal{A}\to X$ is called isotrivial if for an open set $U\subseteq X$ the fibers of π over any pair of points $x,y\in U$ are isomorphic as polarized abelian varieties with the polarization induced from $\mathcal{A}$. The generic fiber of π will be denoted A.

The discriminant locus Δ of π is the subvariety of X consisting of points x for which the fiber $\pi^{-1}(x)$ is not smooth. The map $\pi^{-1}(X-\Delta)\to X-\Delta$ is a locally trivial fibration. It follows from [24] (cf. also [20] and [33]) that there is an unramified Galois covering $s : X' - \Delta' \to X - \Delta$ such that

(a) the Galois group G is a subgroup of automorphisms of the fiber A preserving its polarization induced from the polarization of $\mathcal{A}$ and
(b) such that

$$\mathcal{A} = \{(X' - \Delta') \times A\}/G \tag{4}$$

with the action given by $g(x, a) = (gx, ga)(x \in X' - \Delta', a \in A)$. The equality (4) is a birational isomorphism which is biregular if one replaces the left hand side by the open subset $s^{-1}(X - \Delta)$ in $\mathcal{A}$.

We shall assume that $X' - \Delta'$ is an open set in its G-equivariant smooth compactification X', i.e $X' - \Delta'$ is the complement to a divisor $\Delta' \subset X'$ where X' is a G-equivariant resolution of singularities of a G-equivariant compactification of $X' - \Delta'$.

Definition 2.1 The holonomy group of an isotrivial abelian variety $\mathcal{A}$ is a group G which satisfies the conditions (a) and (b) above and such that no quotient of G satisfies them. The holonomy map is the composition

$$\pi_1(X - \Delta) \to G \to Aut A \tag{5}$$

Note that the first map in (5) can be described as the homomorphism corresponding to the covering map $X' - \Delta' \to X - \Delta$, i.e. having the kernel isomorphic to $\pi_1(X' - \Delta')$.

In this paper we are concerned only with the case $X = \mathbb{P}^2$. Since we assume in Theorem 1.2(b) that the image of the holonomy is non-trivial it follows that Δ has codimension one in X.

2.2 Chow trace of isotrivial families

Next recall Lang-Neron's finite generation result for abelian varieties over function fields starting with the definition of Chow trace (cf. [8,21]). Given an extension K/k of fields and an abelian variety A over K, there exist an abelian variety B over k (called the *Chow trace*) and homomorphism $\tau : B \otimes_k K \to A$[1] defined over k such that for any extension E/k disjoint from K, abelian variety C over E and morphism $\alpha : C \to A$ over KE there exists $\alpha' : C \to B$ such that $\alpha = \tau \circ \alpha'$ (after appropriate field extensions of A and B). A description of the Chow trace in the case of relative Picard schemes is given in [15] (cf. Proposition 2.2, also [35]). In the case of isotrivial abelian varieties, we have the following (which in the case of relative Picard schemes is a consequence from [15]):

Proposition 2.2 *Let $\mathcal{A} \to X$ be an isotrivial abelian variety over $\mathbb{C}(X)$ with holonomy G. Then $\mathbb{C}$-trace of $\mathcal{A}$ is isomorphic to the abelian subvariety A^G of A which is the maximal subvariety of A fixed by the holonomy group G.*

[1] $B \otimes_k K$ is the result of field extension of B.

Proof For any path $\gamma : [0,1] \to X - \Delta$ the identification (4) provides the map: $h_\pi : \pi^{-1}(0) \to \pi^{-1}(1)$ as the composition of a fixed identification of $\pi^{-1}(0)$ with the fiber over a point in $s^{-1}(\gamma(0))$ and a restriction of projection $(X' - \Delta') \times A \to (X' - \Delta')/G \subset \mathcal{A}$ in (4) on the end point of the s-lift of path γ. This is well-defined since the lift of s is unique, but a change of the path $\gamma(t)$ results in a composition of h_π with an automorphism from G. In particular, one has an identification of subvarieties A^G of any two fibers of π and the map $A^G \times (X - \Delta) \to \mathcal{A}$ can be defined using continuation along paths. This yields the trace map $\tau : A^G \otimes k(X) \to \mathcal{A}$.

Next, given a map $T : B \times (X - A) \to \mathcal{A}$ commuting with projections on $X - A$, restricting it on a loop γ in $X - \Delta$ yielding a holonomy transformation $g \in G$, one sees that $T|_{(B \times \gamma(0))} : B \times \gamma(0) \to \mathcal{A}_{\gamma(0)}$ (the fiber of $\mathcal{A}$ over $\gamma(0)$) has the image belonging to A^G i.e. we have factorization of T through τ. This implies the universality property in the definition of trace. □

An automorphism group G of a polarized abelian variety is finite (cf. [5] Chapter 5 Corollary 1.9) and (4) can be used to construct an isotrivial family of polarized abelian varieties for any etale covering of $X - \Delta$ with a Galois group $G \subset Aut A$.

With the notion of trace in place, one can state a function field version of the Mordell–Weil theorem as follows:

Theorem 2.3 (cf. [21]) *Let K be a function field of a variety over a field k. Let A be an abelian variety defined over K and $\tau : B \to \mathcal{A}$ is its trace. Then the Mordell–Weil group $A(K)/\tau B(k)$ is finitely generated.*

2.3 Examples of isotrivial families of abelian varieties with cyclic group of automorphisms

If the automorphism group of an abelian variety is cyclic, then *any* family of abelian varieties with such fiber has a cyclic holonomy group. Here is a way to obtain such examples. Jacobians of curves with cyclic automorphism groups have cyclic automorphism groups as well since by the Torelli theorem $Aut(J(C)) = Aut(C)/\pm I$ (resp. $Aut(J(C)) = Aut(C)$) for non-hyperelliptic curves (resp. for hyperelliptic curves). (cf. [23,42]). As an example of curves with a cyclic automorphism group one can consider the curves $C_{p-2,p}$ with the following equation of the affine part (cf. [22,3,17] and Sect. 3.4 below):

$$u^p = v^{p-2}(1 - v) \tag{6}$$

Example 2.4 Let $\Phi(x, y)$ be a curve in $\mathbb{C}^2$ which is the affine portion of a smooth projective curve having degree p. [2] Consider the curve over $\mathbb{C}(x, y)$ given by

$$u^p \Phi(x, y) = v^{p-2}(1 - v) \tag{7}$$

[2] Assumption of smoothness will be used below to show that the Mordell–Weil rank in this case is zero. The construction in this example yields an isotrivial family of Jacobians for any Φ.

Over the complement to $\Phi(x, y) = 0$ we have the family of curves isomorphic to the curve (6). This family is trivialized over X' given by $z^p = \Phi(x, y)$. The Jacobian of (7) over $\mathbb{C}(x, y)$ provides an example of an isotrivial abelian variety over this field.

2.4 Abelian varieties of CM type

A large class of examples of abelian varieties *admitting* cyclic group of automorphisms, which will appear in several contexts below, is given by abelian varieties of CM type. Recall that a CM field is an imaginary quadratic extension of a totally real number field (cf. [31,36,30]). A CM-algebra is a finite product of CM-fields. Such an algebra E is endowed with an automorphism ι_E such that for any $\rho : E \to \mathbb{C}$ one has $\rho \circ \iota_E = \bar{\rho}$ (the conjugation of ρ). CM-type of a CM-algebra E is:

$$\{\Phi \subset Hom(E, \mathbb{C}) | Hom(E, \mathbb{C}) = \Phi \cup \iota_E(\Phi), \Phi \cap \iota_E(\Phi) = \emptyset\} \tag{8}$$

For a CM field K of degree g over $\mathbb{Q}$, a CM type Φ is a collection of pairwise, not conjugate, embeddings $\sigma_1, \ldots, \sigma_g$ of $K \to \mathbb{C}$.

For a CM algebra E with a chosen CM-type Φ and a lattice in E i.e. a subgroup Λ such that $E = \Lambda \otimes_{\mathbf{Z}} \mathbb{Q}$ (e.g. product of the rings of integers of each of CM-fields composing E) corresponds the torus $E \otimes \mathbb{R}/\Lambda$ with the complex structure induced from the identification $E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^{\frac{dim_{\mathbb{Q}} E}{2}}$ given by the direct sum of the homomorphisms $\phi \in \Phi$ where Φ is the CM type. This complex torus is an abelian variety (cf. [30,31,36]). The following example of a CM type appears below in the context of singularities:

Example 2.5 Let p be an odd prime. Consider the set Φ of roots of unity of degree p with positive imaginary part (i.e. $exp(\frac{2\pi ik}{p})$ for $1 \leq k \leq \frac{p-1}{2}$). The set of embeddings of $\mathbb{Q}(\zeta_p)$ induced by the maps $exp(\frac{2\pi i}{p}) \to \omega, \omega \in \Phi$ provides a CM type of $\mathbb{Q}(\zeta_p)$.

Note that this CM type is primitive in the sense that the corresponding abelian variety is simple (cf. [36], Section 8.4, p. 64).

More generally, for a pair of primes p, q, the set of primitive roots of unity of degree $p \cdot q$ with a positive imaginary part provides a CM type of the field $\mathbb{Q}(\zeta_{pq})$.

2.5 Alexander polynomials

The Alexander polynomial is an invariant of the fundamental group which allows one to state conditions under which the first Betti number of a cyclic cover is positive (cf. [27]). Recall that for a group G and a surjection $\sigma : G \to W$ onto a cyclic group W, one defines the Alexander polynomial as follows. Let $K = Ker\sigma$ and K/K' be the abelianization of K. It follows from the exact sequence

$$0 \to K'/K'' \to K/K'' \to W \to 0$$

that W acts on K'/K'' via conjugation on K'.

Definition 2.6 The Alexander polynomial $\Delta_G(t)$ of G relative to the surjection σ is the characteristic polynomial of a generator of W acting on the vector space $K/K' \otimes \mathbb{C}$ (this space has a finite dimension, cf. [27]; in cases when W is finite, one chooses the polynomial of the minimal degree among polynomials corresponding to different actions of generator). Moreover, one has a cyclic decomposition $K'/K'' \otimes \mathbb{C} = \oplus_i \mathbb{C}[W]/\lambda_i$[3] in terms of which $\Delta_D(t) = \Pi_i \lambda_i(t)$ where t acts as a generator of W.

The properties of the Alexander polynomials of the fundamental groups of the complements to the algebraic curves in $\mathbb{P}^2$ are summarized by the following:

Theorem 2.7 *Let $G = \pi_1(\mathbb{P}^2 - D)$ where D is a projective curve of degree d with arbitrary singularities and with r irreducible components. Let $\Delta_D(t)$ denote the Alexander polynomial of D relative to surjection $G = \mathbb{Z}_{\deg D} = W$ sending a loop to the class modulo $deg D$ of its total linking number with D (cf. [25])*

1. *For each singularity P of the curve D denote by $\Delta_P(t)$ the Alexander polynomial of the local fundamental group $\pi_1(B_P - B_P \cap D)$ where B_P is a small ball about P in $\mathbb{P}^2$ (as above, the Alexander polynomial is defined relative to surjection $\pi_1(B_P - B_P \cap D) \to \mathbb{Z}$ given by the total linking number with $B_P \cap D$).*
   Then the Alexander polynomial $\Delta_D(t)$ polynomial of $\pi_1(\mathbb{P}^2 - D)$ divides the product:

$$\Pi_{P \in Sing(H \cap D)} \Delta_P(t) \tag{9}$$

   In particular the Alexander polynomial of $\pi_1(\mathbb{P}^2 - D)$ is cyclotomic.
2. *Let X_N be an N-fold cyclic branched covering space of $\mathbb{P}^2$ ramified over D, and corresponding to a surjection of W onto a cyclic group of order N. Then the characteristic polynomial of the generator of W acting on $H_1(X_N, \mathbb{C})$ is equal to*

$$\sum_i gcd(t^N - 1, \lambda_i(t)) \tag{10}$$

3. *With each singularity P and a rational number $\kappa \in (0, 1)$ one associates the ideal $I(P, \kappa)$ in the local ring of P (the ideal of quasi-adjunction)* [4] *with the following properties. Let $\mathcal{I}_\kappa \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf for which the support of $\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_\kappa$ is the set of singularities of D different than nodes and stalk of $\mathcal{I}_\kappa$ at P is $I(P, \kappa)$ then*

$$\Delta_D(t) = (t-1)^{r-1} \Pi_\kappa \left[(t - exp(-2\pi i \kappa))\,(t - exp(2\pi i \kappa))\right]^{dim H^1\left(\mathbb{P}^2, \mathcal{I}_\kappa(d-3-\kappa d)\right)} \tag{11}$$

   where the product is over all $\kappa = \frac{i}{d}, 1 \le i \le d - 1$

[3] If W is finite, then $\mathbb{C}[W]$ is isomorphic to $\mathbb{C}[t, t^{-1}]/(t^{|W|} - 1)$ and $\lambda_i \in \mathbb{C}[W]$ are viewed as polynomials in $\mathbb{C}[t]$ having the minimal degree in its coset. This definition is slightly different from the one used in [25] where only infinite W was used. The reduction to the case when W is infinite was done by replacing the projective curve D by its affine portion such that the line at infinity L is transversal to D. If D is irreducible then $H_1(\mathbb{P}^2 - D, \mathbb{Z}) = \mathbb{Z}_{deg D}$ but $H_1(\mathbb{P}^2 - D \cup L, \mathbb{Z}) = \mathbb{Z}$. Moreover, for reduced D, the surjection $\pi_1(\mathbb{P}^2 - D \cup L) \to \mathbb{Z}$ given by the linking number with D yields the same polynomial as surjection $\pi_1(\mathbb{P}^2 - D) \to \mathbb{Z}_{deg D}$ given by the linking number with D.

[4] $I(P, \kappa)$ is defined in terms of the germ of the curve and $\kappa \in \mathbb{Q}$ (cf. [26]); there is identification of the ideals of quasi-adjunction and the multiplier ideals (ibid.).

In particular, for curves with singularities locally equivalent to $u^p = v^q$ only, one has:

$$\Delta_D(t) = \left[\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}\right]^s \tag{12}$$

where $s = dim H^1(\mathbb{P}^2, \mathcal{I}((\frac{1}{p}+\frac{1}{q})d-3))$ and $\mathcal{I}$ is the ideal sheaf defined by conditions:

(a) *$\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}$ is supported at singularities of D*

(b) *stalk $\mathcal{I}$ at each singular point is the maximal ideal of the local ring.*[5]

We refer to [25,27] for proofs of these results but note that much of the reasoning in the proof of the Theorem 4.1 is a Hodge theoretical refinement of the topological arguments in the proof of the first part of the Theorem 2.7. The local type of singularities of plane curves which come up in part 2 in Theorem 2.7 and associated mixed Hodge structures are discussed in the next section.

3 Local Albanese varieties and singularities of CM-type

The main result of this section is Theorem 3.12, describing the structure of the local Albanese varieties of unibranched singularities, showing that they have a CM type. Section 3.1 contains a description of several constructions of the mixed Hodge structures associated with plane curve singularities. The results mainly follow from previous discussions given in [7,26]. In Sect. 3.2 we recall the definitions of the local Albanese variety following [7]. Then we introduce plane curve singularities of a CM-type as those for which the local Albanese varieties will have a CM type. The assertion that unibranched singularities have a CM type is proven in Sect. 3.4.

3.1 Mixed Hodge structures associated with a link

Let $f(x, y)$ be a germ of a plane curve singularity at the origin (0, 0). Recall (cf. [7]) the comparison of the limit of a mixed Hodge structure associated with degeneration $f(x, y) = t$ defined in the case of isolated singularities of arbitrary dimension in [37], and the mixed Hodge structure on the cohomology of a punctured neighborhood of the exceptional set of a resolution of the singularity of $z^n = f(x, y)$ (or equivalently the link of the latter surface singularity) constructed in [10].

Let V be a germ of an algebraic space having an isolated singularity at $P \in V \subset \mathbb{C}^N$. Let $H^*_P(V)$ be the local cohomology of V. It is shown in [38] (using a mapping cone construction) that $H^*_P(V)$ supports a mixed Hodge structure. The cohomology of the link L of singularity of V, i.e. the intersection of V with a small sphere in $\mathbb{C}^N$ centered at P, is related to the local cohomology as follows:

$$H^*(L) = H^{*+1}_P(V) \tag{13}$$

[5] Such a description of $\mathcal{I}$ is a consequence of a calculation yielding that the maximal ideal is the ideal of quasi-adjunction of $u^p = v^q$ and $\kappa = 1 - \frac{1}{p} - \frac{1}{q}$.

In particular the cohomology of L receives a canonical mixed Hodge structure (cf. [38]). On the other hand, L is a retract of a deleted neighborhood of the exceptional set of a resolution of singularity of V, which provides a description of this mixed Hodge structure using the presentation:

$$\tilde{V} - E = \tilde{V} \bigcap \bar{V} - E \tag{14}$$

where $\tilde{V}$ is a resolution of the germ V, E is the exceptional set of the resolution, $\bar{V}$ is a smooth projective variety containing $\tilde{V}$. Here one views $\tilde{V}$ as a small tubular neighborhood of the exceptional set E. In particular (cf. [10]), one has the Mayer–Vietoris sequence, which is a sequence of the mixed Hodge structures:

$$\longrightarrow H^k(\tilde{V}) \oplus H^k(\bar{V} - E) \to H^k(\tilde{V} - E) \to H^{k+1}(\bar{V}) \longrightarrow \tag{15}$$

The weights on $H^k(\tilde{V}) = H^k(E)$ (resp. $H^k(\bar{V} - E)$) are $0, \ldots, k$, since E is a normal crossing divisor, (resp. $k, \ldots, 2k$ since $\bar{V} - E$ is smooth). The weight of $H^{k+1}(\bar{V})$ is $k+1$ since $\bar{V}$ is smooth projective. However the Gabber purity theorem yields that for $0 \le k < n$ the weights on $H^k(L)$ are less than or equal to k and for $n \le k \le 2n-1$ are greater or equal than $k+1$ (cf. [12]).

Applying this to the case when V is the cyclic cover $V_{f,n}$ given by $z^n = f(x, y)$ one obtains for its link $L_{f,n}$ the mixed Hodge structure with weights on $H^1(L_{f,n})$ being 0, 1 and weights on $H^2(L_{f,n})$ being 3, 4.

On the other hand, the vanishing cohomology of the family of germs $f(x, y) = t$, or equivalently the cohomology of Milnor fiber M_f, supports the limit mixed Hodge structure $H^1_{lim}(M_f)$ (with weights (0, 1, 2)). The following comparison between the mixed Hodge structures on $H^1(M_f)$ and $H^2(L_f)$ is given for example in [7].

Proposition 3.1 *Let $f(x, y)$ be a germ of a plane curve (possibly reducible and non-reduced) with semi-simple monodromy of order N and the Milnor fiber M_f. Let $L_{f,N}$ be link of the corresponding surface singularity $z^N = f(x, y)$. Then there is the isomorphism of the mixed Hodge structures:*

$$Gr_3^W H^2(L_{f,N})(1) = Gr_1^W H^1(F_f) \tag{16}$$

where the mixed Hodge structure on the left is the Tate twist of the mixed Hodge structure constructed in [10] *and the one on the right is the mixed Hodge structure on vanishing cohomology constructed in* [37].

From this Proposition we obtain the following:

Corollary 3.2 *If the monodromy of $f(x, y) = t$ is semisimple then the mixed Hodge structure on the Milnor fiber of $f(x, y)$ has type* (1, 1), (1, 0), (0, 1). *The mixed Hodge structure on either side of* (16) *is pure of weight* 1 *and polarized.*

Proof Since the monodromy has a finite order, one has $Gr_0^W = 0$ (cf. [37] p. 547). Moreover, $Gr_1^W H^1(F) = \oplus H^1(D_i)$ where D_i are smooth curves appearing in the semistable reduction of the family $f(x,y)=t$. (ibid). Therefore we obtain the polarization of the term on the right hand side of (16).

Note that the action of the monodromy on $Gr_2^W H^1(F_f)$ (resp. $Gr_1^W H^1(F_f)$) is trivial (resp. does not have 1 as an eigenvalue). □

3.2 Local Albanese variety

Given a pure Hodge structure $(H_{\mathbb{Z}}, F)$ of weight -1, one associates to it a complex torus as follows (a more general case of mixed Hodge structures of type $(0,0), (0,-1), (-1,0), (-1,-1)$ is discussed in [9]):

$$A_H = H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^0 H_{\mathbb{C}} \tag{17}$$

In the case when the Hodge structure is polarized, A_H is an abelian variety.

Definition 3.3 Local Albanese variety Alb_f of a plane curve singularity $f(x,y)=0$ is the abelian variety (17) corresponding to the Hodge structure on homology $H_1(M_f,\mathbb{Z})$ of the Milnor fiber which is dual to the cohomological mixed Hodge structure considered in the Proposition 3.1.

3.3 CM-singularities

Recall that if the monodromy operator acting on the (co)homology of the Milnor fiber is semisimple then it preserves the Hodge filtration (cf. [37]).

Definition 3.4 A plane curve singularity is called a singularity of CM type if its local Albanese variety is isogenous to a product of simple abelian varieties of CM type.

The local Albanese variety has the monodromy operator of the singularity as its automorphism. The following provides a description of the eigenvalues of the induced action on its tangent space at identity.

Proposition 3.5 *Let Alb_f be the local Albanese variety of singularity $f(x,y)=0$. The eigenvalues of the automorphism induced on the tangent space to Alb_f at identity by the monodromy operator of f are the exponents $exp(2\pi i\alpha)$ of the elements $\alpha \in \mathbb{Q}$ of the spectrum of this singularity (cf.* [37]*) which satisfy* $0<\alpha<1$.

Proof Indeed the above tangent space can be identified with $Gr_F^0 H^1(M_f)$ and the claim follows from the definition of the spectrum of singularity. □

For unibranched singularities of plane curves the spectrum was calculated in [32].

Example 3.6 For unibranched curve singularities with one characteristic pair, i.e. singularities with links equivalent to the links of singularity $x^p=y^q$ where $gcd(p,q)=1$, the number of eigenvalues of the monodromy acting on $Gr_F^0 H^1(M)$ (M is the Milnor fiber) is equal to $\frac{(p-1)(q-1)}{2}$. More precisely, the action on $H_1(M)$ is semi-simple and has as the characteristic polynomial (cf. [39])

$$\Delta_{p,q} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \tag{18}$$

The characteristic polynomial of the action on $Gr_F^0 H^1(M)$ is

$$\Pi(t - exp(-2\pi\sqrt{-1}\alpha)), \tag{19}$$

where

$$\alpha = \frac{i}{p} + \frac{j}{q}, \quad 0 < \alpha < 1, \quad 0 < i < p, \quad 0 < j < q \tag{20}$$

(cf. [32] and references there). In particular for $f(x, y) = x^2 + y^3$ the only eigenvalue on F^0 is $exp(\frac{2\pi\sqrt{-1}}{6})$. More generally, for the singularity $x^2 + y^p$ where p is an odd prime, the field generated by the roots of (18) is $\mathbb{Q}(\zeta_p)$ and the CM type corresponds to subset set $exp(\frac{2\pi\sqrt{-1}j}{p})$ where $\frac{1}{2} + \frac{j}{p} < 1$ i.e. coincides with the CM type discussed in Example 2.5.

Theorem 3.7 *Let $f(x, y)$ be a germ of a plane curve singularity such that the monodromy T_f on $H^1(M_f, \mathbb{C})_{\neq 1} = H^1(M_f)/Ker(T_f - Id)$ is semisimple. If the characteristic polynomial $\Delta_f(t)$ does not have multiple roots different from 1 and $\Delta_f(-1) \neq 0$, then the singularity $f(x, y)$ has CM-type.*

Proof Let T_f denote linear operator induced by the monodromy f of $Gr_1^W H^1(F_f)$. Since the monodromy T_f of the Milnor fiber is semisimple, as was mentioned earlier, it preserves the Hodge filtration, acts on Alb_f and hence the algebra $End^\circ(Alb_f) = End(Alb_f) \otimes \mathbb{Q}$ contains the algebra $\mathbb{Q}[T_f]$. The latter has the dimension equal to the degree of the minimal polynomial of T_f which is equal to $dim H^1(M_f, \mathbb{C})_{\neq 1}$ as follows from the assumption that the multiple eigenvalues of the monodromy different from 1 are absent.

Next consider, the isogeny decomposition of Alb_f into the product of abelian varieties X_i on which appropriate powers $T_f^{d_i}$ have as eigenvalues only primitive roots of unity (cf. Theorem 2.1 in [4]). Since $d_i \geq 3$ due to assumption $\Delta_f(-1) \neq 0$, each X_i has sufficiently many complex multiplications, i.e. $End^\circ X_i$ contains semi-simple commutative algebra of rank $2dim X_i$. This subalgebra coincides with the center of $End^\circ(X_i)$. This implies the second inequality in:

$$2dim Alb_f = rk H_1(M_f, \mathbb{C})_{\neq 1} \leq [End^\circ(Alb_f) : \mathbb{Q}]_{red} \tag{21}$$

(notations as in [30] p.10 [6]) while the purity of Hodge structure on $H^1(M_f)_{\neq 1}$ implies the first equality.

Now, the Proposition 3.1 in [30] yields that in fact one has an equality in (21) and the claim follows (cf. for example def. 3.2 [30]).

Alternatively, one can use the Theorem 3.2 in [4] (cf. also [13]) to see that isogeny components X_i have CM type (again using absence of multiple eignevalues of the monodromy acting on $H^1(M_f, \mathbb{C})_{\neq 1}$) and hence to obtain the conclusion of the Theorem 3.7. □

[6] i.e. $[\Pi B_i : k]_{red} = \sum [B_i : k_i]^{\frac{1}{2}} [k_i : k]$ for a product of simple algebras B_i over k, with respective centers k_i.

Example 3.8 Simple singularities have CM type (cf. [7]). Indeed, the characteristic polynomials of the monodromy of Milnor fiber of simple singularities are readily available. For singularity A_{2k}, it is equal to $\frac{(t^{2k}+1)(t-1)}{t^2-1}$

For singularity $x^2y + y^{n-1}$ of type D_n it is equal to

$$\Delta(t) = \left(t^{n-1} + (-1)^{n-1}\right)(t-1) \tag{22}$$

For singularities E_6, E_8 i.e. $y^3 + z^4$, $y^3 + z^5$, the characteristic polynomials of monodromy are given in Example 3.6 and for E_7 i.e. $yz^3 + y^3$ the characteristic polynomial is equal to $t^7 - 1$.

Example 3.9 Consider singularity $f(x, y) = \Pi_{i=1}^{i=4}(x-\alpha_i y) = 0$ where α_i are generic complex numbers. The characteristic polynomial of the monodromy is $(t-1)^3(t^2+1)^2(t+1)^2$ and so Theorem 3.7 cannot be applied. In fact the local Albanese coincides with the Jacobian of the only exceptional curve of the resolution of singularity $z^4 = f(x, y)$. This curve is a fourfold cyclic cover of $\mathbb{P}^1$ totally ramified at 4 points. It cannot generically have CM type since the Jacobian of such curve surjects onto a twofold cover of $\mathbb{P}^1$ branched at 4 points which hence represents a generic elliptic curve.

Example 3.10 Consider the singularity of plane curve with the Puiseux expansion:

$$x^{\frac{3}{2}} + x^{\frac{21}{10}} = x^{\frac{3}{2}} + x^{\frac{3}{2}+\frac{6}{2\cdot 5}} \tag{23}$$

The Puiseux pairs are $(k_1, n_1) = (3, 2)$, $(k_2, n_2) = (6, 5)$ which yields corresponding data $w_1 = 3$, $w_2 = w_1n_1n_2 + k_2 = 36$ (cf. [32]) and hence the characteristic polynomial of the monodromy of this singularity is ($\Delta_{p,q}$ is given by (18))

$$\Delta(t) = \Delta_{3,2}(t^5)\Delta_{36,5}(t) \tag{24}$$

(cf. [40] for formulas for the characteristic polynomial of the monodromy in terms of Puiseux expansion), i.e.

$$\Delta(t) = \left[t^{10} - t^5 + 1\right]\left[\frac{(t^{36\cdot 5} - 1)(t-1)}{(t^{36} - 1)(t^5 - 1)}\right] \tag{25}$$

Since the cyclotomic polynomial of degree 10 divides the polynomials in both brackets in (25), $\Delta(t)$ has multiple roots. Nevertheless the local Albanese for this singularity is abelian variety of CM type (cf. Theorem 3.12).

3.4 Structure of a local Albanese variety of singularities of CM type

Theorem 3.11 *Let $f(x, y) = 0$ be a singularity with a semi-simple monodromy and let N be the order of the monodromy operator. The Albanese variety of germ*

$f(x,y)=0$ is isogenous to a product of Jacobians of the exceptional curves of positive genus for a resolution of:

$$z^N = f(x,y) \tag{26}$$

Proof Denote by P the isolated singularity of a germ X of the surface (26) and consider a resolution $\tilde{X} \to X$ of X. The dual graph of such a resolution does not contain cycles (cf. [11]) since the monodromy is assumed to be semi-simple.[7] Let $E = \cup E_i$ be the decomposition of the exceptional set of the resolution of (26) into irreducible components. We shall use the identification $H^2(L) = H^3_P(X)$ and the exact sequence (cf. [38, Corollary (1.12)] of mixed Hodge structures on local cohomology:

$$0 \to H^3_P(X) \to H^3_E(\tilde{X}) \to H^3(E) \to 0 \tag{27}$$

The last term is trivial, i.e. one has the identification of the first two. Moreover, one has the duality isomorphism (cf. [38, (1.6)]):

$$H^3_E(\tilde{X}) = Hom(H^1(E), \mathbb{Q}(-2)) \tag{28}$$

Since $H^1(E) = \oplus_i H^1(E_i)$, we infer the isomorphism of Hodge structures:

$$H^3(L) = \oplus Hom(H^1(E_i), \mathbb{Q}(-2)) \tag{29}$$

The claim follows since for the curves E_i having positive genus the Jacobians and corresponding Albanese varieties are isomorphic. □

Theorem 3.12 *Unibranched plane curve singularities have CM type.*

The Proof of Theorem 3.12 will consist of two steps. First, we shall show that for $f(x,y)$ unibranched all exceptional curves in a resolution of singularity (26) are Belyi cyclic covers in the following sense:

Definition 3.13 A Belyi cyclic cover is a cyclic cover of $\mathbb{P}^1$ branched at at most three points.

Secondly we shall use the following (cf. [14, 19]):

Lemma 3.14 *The Jacobian of a Belyi cyclic cover is an abelian variety of CM type.*

Then Theorem 3.12 follows from the Theorem 3.11. A proof of Lemma 3.14 is given in the Appendix A for reader's convenience.

Proof (of Theorem 3.12) □

Lemma 3.15 *Exceptional curves of a resolution of the singularity* (26) *are Belyi cyclic covers.*

7 Note that without the assumption that the surface singularity has form (26), the finiteness of the order of monodromy is not sufficient to conclude the absense of cycles (cf. [1]).

A resolution of the singularity (26) can be obtained as follows. Let $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ be a sequence of blow ups of $\mathbb{C}^2$ containing the germ $f(x,y) = 0$ and yielding a resolution of the latter. Denote by $\tilde{f} : \tilde{\mathbb{C}}^2 \to \mathbb{C}$ the composition of π and $f : \mathbb{C}^2 \to \mathbb{C}$ and let $\lambda_N : \tilde{\mathbb{C}} \to \mathbb{C}$ be the N-fold cover of $\mathbb{C}$ branched at the origin. Then one has the map of the normalization $\tilde{\mathcal{X}}$ of the fiber product of the maps $\tilde{f}$ and λ_N:

$$\tilde{\mathcal{X}} \overset{\mathcal{N}}{\to} \tilde{\mathbb{C}}^2 \times_{\mathbb{C}} \tilde{\mathbb{C}} \to X \tag{30}$$

Here $\tilde{\mathcal{X}}$ has at most simple surface singularities and their standard resolution, composed with the maps in (30), provides a resolution of X. Moreover, already $\tilde{\mathcal{X}}$ contains all the curves of a positive genus appearing in a resolution of X.

Note that $\mathcal{N}$ replaces each exceptional curve D of resolution $\tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ by its cyclic branched cover of degree $gcd(N, m)$ where m is the multiplicity of $\pi^*(f)$ along D. Moreover, the ramification occurs at the intersection points of D with the remaining exceptional curves. To finish the proof of Lemma 3.15, it is enough to show that each exceptional curve of π has at most three intersections with remaining exceptional curves. This is the case as one can see, for example, from an inductive argument observing that the collection of exceptional curves on say $(k+1)$th step in a resolution of $f = 0$, is obtained from the collection of exceptional curves on step k by blowing up up the intersection point of a proper preimage of f appearing on the kth step and the intersection point of exceptional curves of the kth and $(k-1)$ steps. Such triple intersection occurs iff the exceptional curve on $k-1$ step was tangent to the proper preimage of f on that step. This yields the above claim on the number of intersections each E can have with the remaining exceptional curves.

We shall conclude this section indicating how one can obtain the identification of the CM type of isogeny components of the local Albanese. The argument above implies that the components of the resolution of the surface singularity (26) having non-trivial Jacobians (i.e. the components with a positive genus) correspond to the rapture points of the resolution tree of $f(x,y) = 0$ (cf. [40]).[8] As follows from the discussion above, the valency of each rapture point is equal to 3. The degree d of the corresponding Belyi cover of a component D of the exceptional set corresponding to such rapture point is equal to $gcd(N, m(D))$, where $m(D)$ is the multiplicity of the pull back of the germ f on the resolution on D. The ramification points of the Belyi cover correspond to the intersections with other exceptional curves in the resolution. The ramification index at the intersection of D with another exceptional curve D' is equal to $\frac{m(D)}{gcd(m(D), m(D'))}$. This data consisiting of the degree and the ramification indices identifies the isomorphism type of the cyclic Belyi cover completely. Using the formulas in Lemma 6.1 one can derive the CM type of corresponding Jacobian and hence the isogeny components of local Albanese variety.

Example 3.16 Consider the singularity $x^2 + y^5$. The dual graph of its resolution has one rapture point. The multiplicity of the corresponding component is equal to 10 with multiplicities of other three intersecting curves equal to 5, 4, 1 respectively. The corresponding Belyi cover is

[8] i.e. the point of the dual graph of resolution where with valency greater than 2.

$$y^{10} = x^4(x - z)z^5 \tag{31}$$

(50) yields that the non-zero eigenvalues of the covering transformation are $e^{\frac{2\pi\sqrt{-1}}{10}}$ and $e^{\frac{-2\pi\sqrt{-1}}{10}}$. This determines the CM type of the Jacobian of the genus two curve (31) corresponding to $\mathbb{Q}(\zeta_{10})$.

Example 3.17 For $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$ the characteristic polynomial of the monodromy is

$$\Phi_{26}(t)\Phi_6(t^2) = \Phi_{26}(t)\Phi_{12}(t)$$

where $\Phi_n(t)$ denotes the cyclotomic polynomial of degree n. The corresponding local Albanese variety is the product of simple CM-abelian varieties corresponding to $\mathbb{Q}(\zeta_{26})$ and $\mathbb{Q}(\zeta_{12})$. The CM type of each field is given by (20).

4 Splitting of Albanese varieties

In this section we show that the Albanese variety of certain cyclic branched covers of $\mathbb{P}^2$ is isogenous to a product of abelian varieties of CM type. A similar result on the existence of an isogeny between the Albanese variety of a cyclic cover and a product of elliptic curves was obtained in [7], but with much stronger restrictions on the singularities of the discriminant.

Recall that a construction of a model of cyclic branched cover with a given ramification curve can be given as follows (cf. [25]). Let D be a reduced irreducible curve in $\mathbb{P}^2$ and let $\pi_1(\mathbb{P}^2 - D) \to \mathbb{Z}_N$ be a surjection onto a cyclic group.

The corresponding unramified cyclic covering of $\mathbb{P}^2 - D$ of degree N is uniquely defined just by D, since the surjection $\pi_1(\mathbb{P}^2 - D) \to \mathbb{Z}_N$ coincides (up to an automorphism of $\mathbb{Z}_N$) with the surjection given by the linking number with D. The affine portion of the N fold cyclic cover is given by

$$z^N = F(x, y) \tag{32}$$

where $F = 0$ is an equation of D. A compactification of the surface (32), combined with a resolution of singularities, yields a smooth model X_N of covering space of $\mathbb{P}^2$ branched over D. If $N = deg D$, then the projective closure of (32) yields a model with isolated singularities in $\mathbb{P}^3$. In the cases when $deg D > N$, a model with isolated singularities can be obtained by the normalization of the projective closure.

Theorem 4.1 *Let D be a curve in $X = \mathbb{P}^2$ with singularities of CM type only. Then for a smooth projective model X_N of N-fold cyclic cover of $\mathbb{P}^2$ branched over D (or equivalently the surface (32)), the Albanese variety $Alb(X_N)$ is isogeneous of a product of abelian varieties of CM type.*

Proof Let $\psi : X_N \to X = \mathbb{P}^2$ be the projection of a smooth model of the N-fold cyclic cover (32). Let $\mathcal{E} = \cup E_i$ be the exceptional set. We shall denote by $\bar{R}$ the proper preimage of the branching locus of ψ in X_N. This branching locus R

contains D and possibly the line at infinity in (x, y)-plane of the cover (32) (depending on the $gcd(deg D, N)$). The cohomology $H^1(X_N - \bar{R})$ supports a mixed Hodge structure of type $(1, 0)$, $(0, 1)$, $(1, 1)$ and hence one can consider the Albanese variety corresponding to the weight one part (cf. [2,16]).

Step 1 Albanese of branched and unbranched covers. We claim that one has the identification:

$$Alb(X_N - \bar{R} \cup E_i) = Alb(X_N) \tag{33}$$

We have the following exact sequence of the pair:

$$\begin{aligned} \to 0 = H^1(X_N, X_N - \bar{R} \cup E_i) \to H^1(X_N) \to H^1(X_N - \bar{R}) \to \\ H^2(X_N, X_N - \bar{R} \cup E_i) \to H^2(X_N) \end{aligned} \tag{34}$$

The identification $H^i(X_N, X_N - \bar{R} \cup E_i) = H_{4-i}(\bar{R} \cup E_i)$ shows that the left term is zero and that the right map is injective since the intersection form on $H^2(X_N)$ restricted on the subgroup generated by fundamental cycles of $\bar{R}$, E_i, is non-degenerate.

The sequence (34) is a sequence of mixed Hodge structures with the Hodge structure on $H^1(X_N - \bar{R} \cup E_i)$ having weights 1 and 2. Hence (34) induces the isomorphism (33).

Step 2 Homology of unbranched cover and homology of cover of punctured regular neighborhood of branching locus

Let U be a small regular neighborhood of R in $\mathbb{P}^2$. Since R is ample, there exists a divisor $R' \subset U$ such that $\pi_1(R' - R \cap R') \to \pi_1(X - R)$ is a surjection (by the Lefschetz hyperplane section theorem applied to quasi-projective manifold $X - R$). The latter surjection can be factored as

$$\pi_1(R' - R \cap R') \to \pi_1(U - R) \overset{i_{U-R}}{\to} \pi_1(X - R) \tag{35}$$

and hence the right map is surjective. If $K_{X-R} \subset \pi_1(X - R)$ (resp. $K_{U-R} \subset \pi_1(U - R)$) is the kernel of surjection $lk_N : \pi_1(X - R) \to \mathbb{Z}_N$ (resp. the kernel of composition $lk_N \circ i_{U-R}$), then $i_{U-R}|_{K_{U-R}} : K_{U-R} \to K_{X-R}$ is surjective as well. Hence denoting by $(U - R)_N$ the N-fold cover of $U - R$ corresponding to index N subgroup K_{U-R} on $\pi_1(U - D)$, we obtain the surjection:

$$H_1((U - R)_N, \mathbb{Z}) \to H_1(X_N - \bar{R} \cup E_i) \tag{36}$$

(one verifies that the points at infinity do not provide contributions since D is always assumed to be transversal to the line at infinity cf. [25]). Moreover, both groups support a mixed Hodge structure and hence the map (36) induced by embedding induces a surjection of mixed Hodge structures.

Step 3 Homology of punctured cover of regular neighborhood of branching locus and homology of links of singularities of cyclic cover.

The covering space $(U-R)_N$ can be viewed as a regular neighborhood of the union of exceptional set $\mathcal{E}$ of X_N for the map of X_N onto the surface (32) and the proper preimage of $R - Sing R$ in X_N (where $Sing R$ is set of singular points of R). As such, it is a union of the N-fold cyclic covering $(U^*_{R-Sing R})_N \subset X_N$ of a regular neighborhood $U_{R-Sing R} - (R - Sing R)$ of $R - Sing R \subset \mathbb{P}^2$ with deleted $R - Sing R$ and punctured regular neighborhoods of the susbets $\mathcal{E}_P \subset \mathcal{E}$ of exceptional subset in resolution X_N of (32) each being the preimage of the corresponding point $P \in Sing R$.

Let is consider the following part of the Mayer–Vietoris sequence, corresponding to just mentioned decomposition:

$$(U-R)_N = \bigcup_P (U(\mathcal{E}_P) - \mathcal{E}_P) \cup (U^*_{R-Sing R})_N$$

(recall that $U(\mathcal{E}_P) - \mathcal{E}_P$ is a retract of the link $L_{N,P}$ of singularity of (32) above P):

$$\begin{aligned} &\oplus_{P\in Sing R} H_1(L_{N,P}) \oplus H_1((U^*_{R-Sing R})_N) \to H_1((U-R)_N) \to \\ &\to H_0\left(\left(\bigcup_P L_{N,P}\right) \cap \left(U^*_{R-Sing R}\right)_N\right) \end{aligned} \quad (37)$$

Observe that the first homomorphism in (37) is surjective. Indeed the homomorphism following in the Mayer–Vietoris sequence the right map in (37) is the map of a sum of zero dimensional homology groups equivalent to injective map $\mathbb{C}^{\mathrm{Card} Sing D} \to \mathbb{C}^{\mathrm{Card} Sing D + 1}$ and hence is injective.

Finally note that the image of the map $H_1((U^*_{R-Sing R})_N) \to H_1(X_N)$ is trivial. Indeed, the action of the covering group on $H_1((U^*_{R-Sing R})_N)$ is trivial and hence the image of this group in $H_1(X_N)$ is trivial since the eigenspace on $H_1(X_N)$ corresponding to eigenvalue 1 has the same rank as $H_1(X, \mathbb{Z})$ and therefore is zero.

Step 4 End of the proof

Step 3 implies that composition map $H_1(L_{N,P}) \to H_1((U-R)_N) \to H_1(X_N)$ is surjective (cf. [25]). This yields the surjection of direct sum of the Albanese varieties corresponding to the remaining summands in the left term of (37) (i.e. the local Albanese varieties of all singular points of D) onto $Alb(X_N)$ and the claim of the theorem follows from Poincare complete reducibility theorem (cf. [5]). □

5 Proof of the main theorem and Examples

In this section we shall finish the proofs of Theorem 1.2, the Corollary 1.3, and will discuss several examples.

Proof (*of Theorem* 1.2) Let $\pi_1(\mathbb{P}^2 - \Delta) \to \mathbb{Z}_d$ be the holonomy representation of the isotrivial family (1). Let X_d denote (a smooth model of) the d-fold cyclic cover of D branched over Δ and $\Delta' \subset X_d$ be such that $X_d - \Delta' \to \mathbb{P}^2 - \Delta$ is an unramified cyclic cover. The holonomy group $\mathbb{Z}_d$ acts on $\mathcal{A} \times_{(\mathbb{P}^2-\Delta)} X_d$ containing the $\mathbb{Z}_d$-invariant subset $(X_d - \Delta') \times A$ (with the diagonal action (cf. 2.1)). Since by assumption the

Chow trace of $\mathcal{A}$ is trivial, $MW(\mathcal{A})$ is the group of section of morphism π (cf. Theorem 2.3). We have the following:

Proposition 5.1 *One has the canonical identification*

$$MW(\mathcal{A}) = MW(X_d \times_{\mathbb{P}^2} \mathcal{A})^{\mathbb{Z}_d} = Hom(Alb(X_d), A)^{\mathbb{Z}_d} \tag{38}$$

Indeed, assigning to $s : \mathbb{P}^2 - \Delta \to \mathcal{A}$ the regular section $(X_d - \Delta') \times_{(\mathbb{P}^2 - \Delta)} s(\mathbb{P}^2 - \Delta)$ of $(X_d - \Delta') \times_{\mathbb{P}^2 - \Delta} \mathcal{A}$ (which is invariant under the action of $\mathbb{Z}_d$) provides the first isomorphism. The second follows from the identification:

$$MW(\mathcal{A}_d) = Mor(X_d, \mathcal{A}) = Hom(Alb(X_d), A) \tag{39}$$

Since $Alb(X_d)$ is abelian variety of CM type it follows that the group $Hom(Alb(X_d), A)$ is trivial unless the abelian variety A is of CM type as well. Moreover, if A is simple and corresponds to a cyclotomic field of degree d then $rkMW$ is positive only if the decomposition of $Alb(X_d)$ into simple components contains A. This yields the inequality (2). If A is a component of $Alb(X_d)$ with multiplicity s then $MW(A_d) = Hom(A^s, A)$ which has rank $s \cdot dim End^\circ(A) = s\phi(d)$.

Proof (*of Corollary* 1.3) If none of characteristic polynomials of local monodromy of singularities of Δ has a primitive dth root of unity as a zero then the global Alexander polynomial does not contain the factor Φ_d and hence the Albanese of X_d cannot have as a factor a variety of CM type corresponding to the field $\mathbb{Q}(\zeta_d)$. The example in the Corollary (1.3) discussed below. □

Part 3 of the Theorem 1.2 provides an effective way to calculate the Mordell–Weil ranks of abelian varieties in the class described in its statement. Below are several examples illustrating this procedure.

Example 5.2 Consider the curve $C_{p,2}$ in (u, v)-plane over $\mathbb{C}(x, y)$ given by

$$u^p = v^2 + (x^p + y^p)^2 + (y^2 + 1)^p \tag{40}$$

This curve over $\mathbb{C}(x, y)$ is isotrivial since all curves

$$u^p = v^q + c, \quad c \in \mathbb{C}, \quad c \neq 0 \tag{41}$$

are biholomorphic. Moreover (40) has as its discriminant the curve

$$C_{p,2} : (x^p + y^p)^2 + (y^2 + 1)^p \tag{42}$$

The Alexander polynomial of the complement is the cyclotomic polynomial of degree $2p$ ([25]):

$$\Phi_{2p} = \frac{(t^{2p} - 1)(t - 1)}{(t^2 - 1)(t^p - 1)} \tag{43}$$

The curve (41) is the Belyi cyclic cover and its Jacobian was described earlier as $A(\mathbb{Q}(\zeta_{2p}), \Phi)$ with the CM type Φ as in example 3.6. Moreover, the Albanese variety of the covering of degree $2p$ of $\mathbb{P}^2$ ramified along $C_{p,2}$ is isomorphic to $A(\mathbb{Q}(\zeta_{2p}), \Phi)$ as well. Since $End^0(A(\mathbb{Q}(\zeta_{2p})) = \mathbb{Q}(\zeta_{2p})$ the claim follows. Note that it follows that the above Jacobian is simple as a consequence of the discussion of example 3.6 since the CM type is primitive (cf. Example 2.5 or [36], Section 8.4, p. 64).

Example 5.3 The Jacobian of the curve (7) considered in Sect. 2.3 has a Mordell–Weil rank equal to zero unless the Alexander polynomial of the curve $\Phi(x, y)$ has a root in $\mathbb{Q}(\zeta_p)$. If $\Phi(x, y)$ is the Eq. (42) then the curve (7) is birational over $\mathbb{C}(x, y)$ to the curve (40) and hence the Mordell–Weil rank for the Jacobian of (7) for such $\Phi(x, y)$ is $p - 1$.

Remark 5.4 The Jacobian of the curve in Example 5.2 is a simple isotrivial abelian variety over $\mathbb{C}(x, y)$ such that rank of its Mordell–Weil group is equal to $p - 1$. In particular the rank of abelian varieties over $\mathbb{C}(x, y)$ can be arbitrarily large.

In [7] it was shown that finding a bound on the rank of Mordell–Weil group for isotrivial elliptic curves over $\mathbb{C}(x, y)$ with discriminant having only nodes and cusps as its singularities is equivalent to finding a bound[9] on the multiplicity of the factor $t^2 - t + 1$ in the Alexander polynomial of the discriminant (in [7] more general cases including ADE singularities were also considered). For curves with nodes and cusps the largest known at the moment multiplicity is 4 (cf. [7]). Similarly, for abelian varieties $\mathcal{A}$ with generic fiber being a simple abelian variety of CM type corresponding to $\mathbb{Q}(\zeta_d)$ and a CM type as in Example 3.6 the rank of $MW(\mathcal{A})$ is related to the multiplicity of the factor $\phi_d(t)$ (the cyclotomic polynomial of degree d) in the Alexander polynomial of the discriminant. Note that there are very few known examples of plane curves with non-trivial Alexander polynomials and singularities beyond those of ADE type (cf. [7,27]). In particular, the largest multiplicity of ϕ_{pq} for $p, q > 3$ is achieved for curves studied in [6]. They correspond to threefolds given in the example below:

Example 5.5 Let

$$u^2 = v^{2k+1} + \left(x^{2(2k+1)} + y^{2(2k+1)} + 1 - 2x^{2k+1} + 2(xy)^{2k+1} + y^{2k+1}\right) \quad (44)$$

be the curve over $\mathbb{C}(x, y)$. The discriminant is given by the second summand in the right hand side of (44). This is the curve studied in [6] where it was shown that the Alexander polynomial is

$$\left(\frac{t^{2k+1} + 1}{t + 1}\right)^3 \quad (45)$$

Generic fiber of fibration (44) is hyperelliptic curve. For $2k + 1 = p$ its Jacobian is a simple abelian variety of a CM type and the rank of Mordell–Weil of the corresponding to (44) family of Jacobians is $3(p - 1)$.

9 Either constant or depending on the degree of the discriminant.

Appendix: Jacobians of Belyi covers

In this appendix we shall prove the Lemma 3.14 i.e. that the Jacobians of Belyi cyclic covers are abelian varieties of the CM type. Though the Lemma 3.14 is apparently not new (cf. [14, 19]) the proof given below for convenience contains explicit formulas for the eigenvalues of the automorphisms of Belyi covers acting on the space of holomorphic 1-forms.

Proof (*of Lemma* 3.14) We claim that a generator of the group of deck transformations of a cyclic Belyi cover acting on H_1 does not have multiple eigenvalues. Once this is established, an argument as in the proof of the Theorem 3.7, shows that for the Jacobian of such cover one has $2 dim J = dim End^{\circ}(J)$ i.e. J has a CM type.

Let $C \to \mathbb{P}^1$ be a Belyi cyclic cover and let d be its degree, i.e. the group of roots of unity of degree μ_d acts on C with three points having non-trivial stabilizers. Let a, b, c be the indices these stabilizers in μ_d. As a model of such Belyi cover (suitable for our calculations below, cf. Proof of Lemma 6.1), one can choose the normalization of plane curve:

$$y^d = x^a (x - z)^b z^c, \quad a + b + c = d. \tag{46}$$

The action of μ_d is given by $T : (x, y, z) \to (x, e^{\frac{2\pi i}{d}} y, z)$. Let T_* be the induced map on $H_1(X, \mathbb{C})$. Now, the proper preimage for the map

$$(x, y) \to (x^d, y^b x^a) \tag{47}$$

of the affine model of (46), i.e.

$$y^d = x^a (x - 1)^b \tag{48}$$

yields a curve which has a component the Fermat curve

$$y^d = x^d - 1. \tag{49}$$

Since the Jacobian of Fermat curve is a product of abelian varieties of CM type (cf. [19]) this implies that the same is the case for cyclic Belyi covers. □

The following allows one effectively to calculate the CM type of local Albanese varieties in many cases. These formulas extend the special case presented in [43]. We have the following:

Lemma 6.1 1. *The multiplicity of the eigenvalue* $\omega_d^j = e^{\frac{2\pi\sqrt{-1} j}{d}}$ *of* T_* *acting on the space of holomorphic 1-forms of Belyi cyclic cover as above is equal to:*

$$-\left(\left[-\frac{aj}{d}\right] + \left[-\frac{bj}{d}\right] + \left[\frac{(a+b)j}{d}\right] + 1\right) \tag{50}$$

where $[\cdot]$ *denotes the integer part. In particular this multiplicity is equal either to zero or one.*

2. *Let* $gcd(a, b, c, d) = 1$ *(i.e. the Belyi cover is irreducible). Then the characteristic polynomial of the deck transformation acting on* H_1 *is given by*

$$\Delta(t) = \frac{(t^d - 1)(t - 1)^2}{(t^{gcd(a,d)} - 1)(t^{gcd(b,d)} - 1)(t^{gcd(c,d)} - 1)} \tag{51}$$

Proof (*of Lemma* 6.1) First note that the indices of stabilizers for the branching points of the cover (46) as the subgroups of the covering group are $gcd(a, d)$, $gcd(b, d)$, $gcd(c, d)$ respectively. Hence Riemann–Hurwitz formula yields that the genus of C is given by (cf. [17])

$$g = \frac{d - gcd(a, d) - gcd(b, d) - gcd(c, d) + 2}{2}. \tag{52}$$

We shall represent explicitly the cohomology classes of $H^0(\Omega^1_C)$ and calculate the action of covering group on holomorphic 1-forms. Recall that the space of holomorphic 1-forms on a plane curve of degree d can be identified with the space of adjoint curves of degree $d - 3$, i.e. the curves of degree $d - 3$ which equations at each singular point satisfy the adjunction conditions or equivalently belong to the adjoint ideal of this singularity. This can be made explicit since any holomorphic 1-form can be written as the residue of 2-form on its complement, i.e. as

$$\frac{P(x, y)dx}{y^{d-1}} \quad \deg P \le d - 3 \tag{53}$$

The curve (46) may have singular points only at $(0, 1)$, $(1, 0)$, $(1, 1)$ and near each the local equation is equivalent to $x^l + y^d = 0$ (by abuse of language we shall refer to these points as "singular" even if the curve is smooth there). To calculate the number of adjunction conditions we shall use the following (cf. [29]):

Proposition 6.2 *The conditions of adjunction for the singularity* $y^d + x^l$ *are the vanishing of the coefficients of monomials* $x^i j^j$ *such that* $(i + 1, j + 1)$ *is below or on the diagonal of the rectangle with vertices* $(0, 0)$, $(0, d)$, $(l, 0)$, (l, d). *The number of adjunction conditions for singularity* $y^d + x^l$ *is equal to*

$$\frac{(d - 1)(l - 1) + gcd(d, l) - 1}{2} \tag{54}$$

This implies that the dimension of the space of curves of degree $d - 3$ satisfying the conditions of adjunction at all three singularities is greater or equal:

$$\begin{aligned}
&\frac{(d-1)(d-2)}{2} - \frac{(d-1)(a-1) + gcd(d, a) - 1}{2} \\
&\quad - \frac{(d-1)(b-1) + gcd(d, b) - 1}{2} - \frac{(d-1)(c-1) + gcd(d, c) - 1}{2} \\
&= \frac{d + 2 - gcd(a, d) - gcd(b, d) - gcd(c, d)}{2}.
\end{aligned}$$

Comparison of this with the genus formula (52) shows that the conditions of adjunction imposed by three singular points are independent, i.e. one has the exact sequence

$$0 \to H^0(\Omega^1_{\tilde{C}}) \to H^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}(d)) \to \oplus_{s \in Sing C} M_s \to 0$$

where $\tilde{C}$ is the normalization of C and M_s is the quotient of the local ring of singular point by the adjoint ideal.

To calculate the action of T_* on $H^1(\tilde{C}, \mathbb{C})$, we shall use the identification (53) of adjoints with the forms and that the action of T^* on the monomials is given by $g(x^i y^j) = \omega_d^j x^i y^j$. Also note that the cardinality of the set of solutions to linear inequality (for a fixed j) is given as follows:

$$Card\{i | 0 < i,\ di + aj \leq da\} = a + \left[-\frac{aj}{d}\right] \tag{55}$$

The multiplicity of the eigenvalue corresponding to the monomial $x^i y^{j-1}$ (i.e. ω_d^{j-1}) in representation of μ_d in $H^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}(d)) = H^0(\Omega^1_{\bar{C}})$ where $\bar{C}$ is a smoothing of C is $\mathrm{Card}\{i | 0 < i, i + j - 1 \leq d - 3\} = d - 1 - j$. Hence the multiplicity of the eigenvalue ω^j is equal to

$$\begin{aligned} d - 1 - j - a - \left[-\frac{aj}{d}\right] - b - \left[-\frac{bj}{d}\right] - c - \left[-\frac{cj}{d}\right] = \\ = -\left(\left[-\frac{aj}{d}\right] + \left[-\frac{bj}{d}\right] + \left[\frac{(a+b)j}{d}\right] + 1\right). \end{aligned} \tag{56}$$

The last assertion of Lemma 6.1, i.e. that the multiplicity does not exceed 1, follows from the property $[x + y] \leq [x] + [y] + 1$.

The formula for the characteristic polynomial can be derived using the additivity of zeta function similarly to the expression for the euler characteristic obtained earlier. □

To finish a proof of Lemma 3.14, just note that absence of multiple eigenvalues implies that the Jacobian must have a CM type (as in the Proof of Theorem 3.7).

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