DEPTH OF COHOMOLOGY SUPPORT LOCI FOR QUASI-PROJECTIVE VARIETIES VIA ORBIFOLD PENCILS

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ABSTRACT. The present paper describes a relation between the quotient of the fundamental group of a smooth quasi-projective variety by its second commutator and the existence of maps to orbifold curves. It extends previously studied cases when the target was a smooth curve. In the case when the quasi-projective variety is a complement to a plane algebraic curve this provides new relations between the fundamental group, the equation of the curve, and the existence of polynomial solutions to certain equations generalizing Pell's equation. These relations are formulated in terms of the depth which is an invariant of the characters of the fundamental group discussed in detail here.

1. INTRODUCTION

Let $\mathcal{X}$ be a smooth quasi-projective variety and let $\chi \in \mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*)$ be a character of its fundamental group. Viewing χ as a rank one local system, one associates to it the twisted cohomology groups. The purpose of this note is to extend known relations between holomorphic maps of $\mathcal{X}$ onto curves, i.e. holomorphic pencils, and dimensions of the twisted cohomology $H^1(\mathcal{X}, \chi)$.

The problem of the existence of holomorphic pencils can be traced back to almost one hundred years and in its projective version[1] goes back to Castelnuovo, deFrancis, Catanese, Green-Lazarsfeld, and Simpson (cf. [21] for a list of references). The quasi-projective case was considered in [3], where the structure of the jumping subsets of the variety of characters

$$(1.1) \qquad \mathring{V}_k(\mathcal{X}) := \{\chi \in \mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*) \mid \dim H^1(\mathcal{X}, \chi) = k\}$$

was studied together with its relation to pencils. In this context, if $\chi \in \mathring{V}_k(\mathcal{X})$ we say χ has *depth* k. The characteristic varieties $V_k(\mathcal{X})$ are defined analogously to $\mathring{V}_k(\mathcal{X})$, but replacing $=$ by $\geq$ in (1.1). This term was introduced in [20] for complements to plane curves and explicitly related to the structure of the fundamental group in [17, 20]. To be more precise, the characteristic varieties referred to above can be described as the zero sets of the Fitting ideals of the abelianization π_1'/π_1'' of the commutator of π_1, which coincide with the jumping loci (1.1) outside of the trivial character (see Theorem 2.3). In particular, the characteristic varieties (unlike the jumping sets for the higher cohomology spaces) depend only on the fundamental group. Fox calculus provides an effective method for calculating the characteristic varieties in the cases when a presentation of the fundamental group by generators and relators is known.

The results of [3] are as follows: each $V_k(\mathcal{X}) \subset \mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*)$ is a finite union of translated subgroups (i.e. cosets) of $\mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*)$. Moreover, for each component of positive dimension there exists a curve C with negative Euler characteristic

[1]and where local systems are replaced by holomorphic bundles.

such that this component has the form $\rho \cdot f^* \text{Hom}(\pi_1(C), \mathbb{C}^*)$ for some holomorphic map $f : \mathcal{X} \to C$. This was supplemented in [21] by showing that the zero-dimensional components have finite order.

A more precise version of this result can be found in [7] in terms of orbifolds. It includes some missing points regarding resonant conditions and extends the result from $V_1(\mathcal{X})$ to all characteristic varieties $V_k(\mathcal{X})$, $k \geq 1$.

If $\mathcal{X}$ is a complement to a plane projective curve, the target C of a holomorphic pencil mentioned above must be necessarily $C = \mathbb{P}^1 \setminus \{\text{points}\}$ and thus f extends to a rational pencil on $\mathbb{P}^2$. In this case, positive dimensional translated components $\rho \cdot f^* \text{Hom}(\pi_1(C), \mathbb{C}^*)$ of $V_1(C)$ have been shown (see Dimca [13]) to be related to the multiple fibers of such a pencil, see also [7].

For a generic non-isolated character $\chi \in V_k(C)$ in a component of $V_k(C)$ of dimension greater than one, the following formula for its depth holds:

$$(1.2) \qquad \dim H^1(C, \chi) = \begin{cases} \dim V_k(C) - 2 = -e(C) & \text{if } C \text{ is compact} \\ \dim V_k(C) - 1 = -e(C) & \text{otherwise,} \end{cases}$$

where $e(C)$ is the (topological) Euler characteristic of C.[2]

This provides a simple way to determine or at least to estimate the depth of characters on components having a *positive* dimension.

Isolated points in components $V_k(\mathcal{X})$ are common occurrence and below we describe the geometric significance of the depth of zero dimensional irreducible components of $V_k(\mathcal{X})$. We do so using *orbifold pencils* associated with such characters (as was mentioned, such characters must have a finite order).

It is worth mentioning that the nature of the cohomology of local systems is essentially different depending on whether $H_1(\bar{\mathcal{X}}, \mathbb{C})$ is trivial or not. In the latter case, due to the surjection $\pi_1(\mathcal{X}) \to \pi_1(\bar{\mathcal{X}}) \to 1$, some of the characters of $\pi_1(\mathcal{X})$ are the characters of the projective fundamental group (cf. [28] for a discussion on the difference between the projective and the quasi-projective case). In this paper (as in [3]) we shall focus on the case when $H_1(\bar{\mathcal{X}}, \mathbb{C}) = 0$. This includes the case of the complements to plane curves which provides many concrete and interesting examples.

For the basics on the theory of orbifolds we refer to [2] or, since we shall consider mainly orbifold curves, to [26] or [14]. An orbifold pencil is a (birational) dominant map $\mathcal{X} \to \mathcal{C}$, where $\mathcal{C}$ is the orbicurve such that the preimage of each point in $\mathcal{C}$ with stabilizer of order m is a multiple fiber of order a multiple of m. A proof of this lemma can be found in [7].

Lemma 1.1. *An orbifold pencil* $f : \mathcal{X} \to \mathcal{C}$ *defines a morphism of orbifold fundamental groups* $f_* : \pi_1(\mathcal{X}) \to \pi_1^{\text{orb}}(\mathcal{C})$.

We call (cf. Definition 4.7) the map described in the previous lemma a *marked orbifold pencil* $f : \mathcal{X} \to \mathcal{C}$. The *markings* are given by the pairs $(\mathcal{X}, \chi), \chi \in V_k(\mathcal{X})$, and $(\mathcal{C}, \rho), \rho \in V_k^{\text{orb}}(\mathcal{C})$ such that $f^*(\rho) = \chi$, where f^* is the map of groups of characters corresponding to f_*. Note that $V_k^{\text{orb}}(\mathcal{C})$ is the orbifold characteristic variety of $\mathcal{C}$ defined as V_k in (1.1) for $\pi_1^{\text{orb}}(\mathcal{C})$, which only depends on the group, as mentioned above.

A pair $(\mathcal{X}, \chi)$ can be marked by several orbifold pencils and we show that the number of such markings is related in an appropriate sense to the depth (cf. Theorem 1.2 below and section 4.3).

The relation between orbifold pencils and local systems with non-vanishing cohomology was studied in [10]. In this paper, the problem of finding a bound on

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[2]For example, if $C = \mathbb{P}^1 \setminus \{n \text{ points}\}$, then $\dim(H^1(C, \mathbb{C}^*)) = n - 1$ and $\dim(H^1(C, \chi)) = -e(C) = n - 2$, if χ is non-trivial.

the degree of the Alexander polynomial is discussed for plane curves with cusps and nodes as the only singularities (or curves with singularities in a more general class of δ-essential singularities). The connection with the cohomology of local systems is coming from the following: For an irreducible curve D in $\mathbb{P}^2$ one has $H_1(\mathbb{P}^2 \setminus D, \mathbb{Z}) = \mathbb{Z}/\deg D\mathbb{Z}$ i.e. $\mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*) = \mu_{\deg D}$ and if χ_ξ corresponds to $\xi \in \mu_{\deg D}$ then:

$$(1.3) \qquad \Delta(\xi) = 0 \Rightarrow \dim H^1(\mathbb{P}^2 \setminus D, \chi_\xi) \neq 0.$$

The key step in [10] for obtaining the bound on the degree of the Alexander polynomial (or equivalently the multiplicity of the root $\exp(\frac{2\pi i}{6})$) was to show the following:

Theorem 1.2. *The degree of the Alexander polynomial of a curve D having cusps and nodes as the only singularities coincides with the number of independent orbifold pencils $\mathbb{P}^2 \to \mathbb{P}^1_{2,3,6}$ such that D is the preimage of the orbifold point having the cyclic group of order six as the stabilizer. This number of independent pencils equals the rank of the group of quasitoric relations*

$$(1.4) \qquad u^2 + v^3 = w^6 F,$$

where $F = 0$ is a defining equation for D.

Theorem 1.2 can also be extended to general Alexander polynomials and non-reduced curves.

One of the main results of this paper is the following theorem (proven in section 4.3) providing the relations between orbifold pencils and depth. It shows that the number of independent pencils (with a given target) provides a lower bound for the depth of a character. Moreover, for an interesting class of characters this bound is exact.

Theorem 1.3. *Let $\mathcal{X}$ be a quasi-projective manifold together with a character χ.*

(1) *Assume that there are n strongly independent marked orbifold pencils with a fixed target $(\mathcal{C}, \rho)$ and let $d(\rho)$ denote the depth of the character ρ of $\pi_1^{orb}(\mathcal{C})$. Then $d(\chi) \geq nd(\rho)$.*

(2) *If in addition χ is a 2-torsion character and two is its only weight (cf. 2.8 for a definition of weights of a character), then there are exactly $d(\chi)$ strongly independent orbifold pencils on $\mathcal{X}$ whose target is the global $\mathbb{Z}_2$-orbifold $\mathcal{C} = \mathbb{C}_{2,2}$. These pencils are marked with the non-trivial character ρ of $\pi_1^{orb}(\mathbb{C}_{2,2})$ characterized by the condition that it extends to $\mathbb{P}^1_{2,2}$.*

Moreover, if χ is a d-torsion character, then (1) implies that $\phi_d(t)^{nd(\rho)} | \Delta_{X,\chi}(t)$ and (2) implies that $d(\chi)$ is the multiplicity of $\phi_2(t) = (t+1)$ as a factor of $\Delta_{X,\chi}(t)$, where $\phi_k(t)$ denotes the cyclotomic polynomial of order k.

See sections 4.1, 4.2, and 5.1 for the required definitions. The Hodge theoretical condition on χ of having two as its only weight, can be characterized as the requirement of the equality of the first Betti numbers of both the double cover of $\mathcal{X}$ defined by χ and its smooth compactification; see Theorem 5.1 for another characterization. We specialize these results to the case of complements to plane curves in section 5. The group $\pi_1(\mathbb{P}^2 \setminus D)$ is closely related to its central extension $\pi_1(\mathbb{C}^2 \setminus D)$ where $\mathbb{C}^2$ is obtained from $\mathbb{P}^2$ by deleting a generic line at infinity. In this case the group of characters is $(\mathbb{C}^*)^r$ where r is the number of irreducible components. The properties of the characters lying on coordinate components are essentially different and in section 5.1 we give a Hodge theoretical characterization of coordinate essential characters.

In the case of plane curves the orbifold pencils correspond to solutions of certain equations over the function field $\mathbb{C}(x, y)$. For example, as mentioned in Theorem 1.2,

the depth of characters of order 6 is related to the number of independent polynomial solutions in u, v, w of the quasitoric equation $u^2 + v^3 = w^6 F$ of type $(2, 3, 6)$. This also can be used to relate the cohomology of the Milnor fiber of arrangements of lines with triple points and solutions to the Catalan equation (cf. [22]).

A similar result for characters of order two is shown in section 5.2. Let $D \subset \mathbb{P}^2$ be a projective plane curve, $X_D := \mathbb{P}^2 \setminus D$ its complement, and χ a 2-torsion character on $\pi_1(X_D)$. Denote by $Q_{(D,\chi)}$ the set of $(2, 2, 0)$-quasitoric relations associated to χ, that is,

$$(1.5) \quad Q_{(D,\chi)} := \{(f, g, h, U, V) \in \mathbb{C}[x, y, z]^5 \mid fU^2 - gV^2 = h, f \cdot g = F, h_{\mathrm{red}}|H\}/ \sim,$$

where $D := \{FH = 0\}$, χ is ramified exactly along $F = 0$ (see Definition 5.8) and $\sim$ is the appropriate equivalence relation. Then

Theorem 1.4. *The set of $(2, 2, 0)$-quasitoric relations $Q_{(D,\chi)}$ has a structure of a finitely generated abelian group and*

$$\operatorname{rank} Q_{(D,\chi)} \le d(\chi).$$

Moreover, if X_D and χ satisfy the conditions of Theorem 1.3(2), then

$$\operatorname{rank} Q_{(D,\chi)} = d(\chi).$$

We refer to section 5.2 for the exact definition of $Q_{(D,\chi)}$ and the equivalence $\sim$ among quasitoric relations. This result is illustrated both in section 6 of this paper and in [6] with several non-trivial examples aiming to describe a calculation method for the group structure of $Q_{(D,\chi)}$.

Finally, we note that there is a surprising connection between the polynomial equations considered in (1.5) and the Pell equations over the field of rational functions $\mathbb{C}(x, y)$. Investigations of the Pell equations

$$(1.6) \qquad\qquad\qquad u^2 - f(x)v^2 = 1$$

over the function field $\mathbb{C}(x)$ apparently go back to Abel [1] (cf. [24]). More recently, the equation (1.6) over $k[x]$ was considered by F.Hazama in [15], [16] where a group structure closely resembling the one described in Theorem 1.4 also appeared. A more detailed study of this connection is out of the scope of this note, but will appear elsewhere.

2. PRELIMINARIES

2.1. **Characteristic varieties.**

Recall the basic definitions and results related to characteristic varieties and homology of covering spaces. We will follow the original exposition given in [20], but rephrase it in a more general setting.

Throughout this section X will be considered a topological space of finite type (that is, X has the homotopy type of a finite CW-complex), $\pi_1'(X) \subset \pi_1(X)$ be the commutator of the fundamental group. We shall assume that $\pi_1(X)/\pi_1(X)' = H_1(X, \mathbb{Z})$ is a free abelian group of rank r. Basic examples are the complement to plane algebraic curves in $\mathbb{C}^2$ with r components and links in a 3-sphere with r components.

Consider the torus of characters of $\pi_1(X)$ i.e.

$$(2.1) \qquad\qquad\qquad \mathrm{Char}(X) := \mathrm{Hom}(\pi_1(X), \mathbb{C}^*).$$

Alternatively, since $\mathrm{Char}(X)$ depends only on $\pi_1(X)$ we refer to it as $\mathrm{Char}(\pi_1(X))$. This torus is canonically isomorphic to the spectrum $\mathrm{Spec}\,\mathbb{C}[H_1(X, \mathbb{Z})] = (\mathbb{C}^*)^r$ of the group ring of abelianized $\pi_1(X)$. Let $X_{\mathrm{ab}} \to X$ be the universal abelian cover i.e. the covering with the group $H_1(X, \mathbb{Z})$. The group $H_1(X, \mathbb{Z})$ acts on X_{ab} as a group of automorphisms and this provides $H_*(X_{\mathrm{ab}}, \mathbb{C})$ with a structure of a

$\mathbb{C}[H_1(X,\mathbb{Z})]$-module. Recall that with each R-module M over a commutative ring R one associates the support which is the subvariety of $\operatorname{Spec} R$ consisting of the prime ideals $\mathfrak{p}$ such that the localization $M_{\mathfrak{p}}$ does not vanish.

Definition 2.1. The *characteristic variety* $V_k(X)$ is the subvariety of the torus $\operatorname{Char}(X) = \operatorname{Spec} \mathbb{C}[H_1(X,\mathbb{Z})]$ given as the support of the module $\bigwedge^k(H_1(X_{\mathrm{ab}},\mathbb{C}))$ (the exterior power of the homology module). Alternatively, $V_k(X)$ can be given as the zero set of the k-th Fitting ideal of $H_1(X_{\mathrm{ab}},\mathbb{C})$, that is, the ideal generated by the $(n-k)\times(n-k)$ minors of the matrix Φ with coefficients in $\mathbb{C}[\pi_1(X)]$ of the map Φ:

$$(2.2) \qquad \mathbb{C}[\pi_1(X)/\pi_1'(X)]^m \xrightarrow{\ \Phi\ } \mathbb{C}[\pi_1(X)/\pi_1'(X)]^n \longrightarrow H_1(X_{\mathrm{ab}},\mathbb{C}) \longrightarrow 0.$$

We denote by $\mathring{V}_k(X)$ the set of the characters in $V_k(X)$ which do not belong to $V_j(X)$ for $j > k$. If a character χ belongs to $\mathring{V}_k(X)$, then k is called the *depth* of χ and denoted by $d(\chi)$. [3]

The following expresses the homology of finite abelian covers in terms of the depth of characters of π_1. The argument follows closely the one given in [19] and [20], but we will present some details here since the statement of Theorem 2.2 is in a more general context than in the references above. See also [8, 17, 25].

Theorem 2.2. *Let X be a finite CW-complex, let H be subgroup of $\pi_1(X)$ of finite index containing the commutator $\pi_1(X)'$ and let $K := \pi_1(X)/H$.*

Let $i_H : \operatorname{Char}(K) \to \operatorname{Char}(X)$ be the embedding of the character varieties induced by the surjection $\pi_1(X) \to K$. Let X_H be the covering of X corresponding to the subgroup H.

Then

$$(2.3) \qquad b_1(X_H) = b_1(X) + \sum_{\xi \in \operatorname{Char}(K)\setminus\{1\}} d(i_H(\xi)).$$

Proof. Consider the five term exact sequence corresponding to the spectral sequence:

$$(2.4) \qquad E_2^{p,q} = H_p(L, H_q(X_{\mathrm{ab}},\mathbb{C})) \Longrightarrow H_{p+q}(X_H,\mathbb{C})$$

which is the spectral sequence for the free action of the group $L := H/\pi_1'(X)$ on the universal abelian cover X_{ab}. It yields:

$$(2.5) \qquad H_2(L,\mathbb{C}) \longrightarrow H_1(X_{\mathrm{ab}})_L \longrightarrow H_1(X_H) \longrightarrow H_1(L,\mathbb{C}) \longrightarrow 0$$

(subscript in second left term denotes covariants). Next, after taking the tensor product of sequence (2.2) with the group ring $\mathbb{C}[H/\pi_1'(X)] = \mathbb{C}[L]$, using for a $\mathbb{C}[L]$-module M the identification of the covariants $M_{\mathbb{C}[L]}$ with $M \otimes_{\mathbb{C}[L]} \mathbb{C}$ applied to the second term in 2.5 and finally using the isomorphism:

$$(2.6) \qquad (\mathbb{C}[\pi_1(X)/\pi_1'(X)])^s \otimes_{\mathbb{C}[L]} \mathbb{C} = (\mathbb{C}[\pi_1(X)/\pi_1'(X)]/I_{\mathbb{C}[L]})^s = (\mathbb{C}[K])^s$$

(here $I_{\mathbb{C}[L]}$ is the augmentation ideal) one obtains:

$$(2.7) \qquad \mathbb{C}[K]^m \longrightarrow \mathbb{C}[K]^n \longrightarrow H_1(X_{\mathrm{ab}},\mathbb{C})_L \longrightarrow 0$$

Since $\operatorname{Spec}\mathbb{C}[K]$ has a canonical identification with $\operatorname{Char}(K)$ and the dimension of the cokernel of the left homomorphism in (2.7) is the sum of cokernels of localizations of (2.7) at the maximal ideal of $\xi \in \operatorname{Char}(K) \subset \operatorname{Char}(X)$, the dimension of cokernel in (2.7) is equal to $\sum_{\xi \in \operatorname{Char}(K)} d(i_H(\xi))$. To conclude the proof

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[3]cf. Theorem 2.3 for comparison of this definition and comments after (1.1)

we will show that contribution of the character $\xi = 1$ in the last sum is equal to the dimension of the image of the left homomorphism in (2.5) and that the right term in (2.5) is equal to $b_1(X)$. Indeed, since $b_1(X) = \operatorname{rank} \pi_1(X)/\pi_1'(X)$, $\dim H_1(L, \mathbb{C}) = \operatorname{rank} L$ and the group K is finite the second claim follows. The first one follows from consideration of the commutative square obtained by taking morphism of the sequence (2.5) into similar five term sequence replacing H by $\pi_1(X)$:

(2.8)
$$
\begin{array}{ccc}
H_2(L, \mathbb{C}) & \longrightarrow & H_1(X_{\mathrm{ab}})_L \\
\downarrow & & \downarrow \\
H_2(\pi_1(X)/\pi_1'(X), \mathbb{C}) & \longrightarrow & H_1(X_{\mathrm{ab}})_{\pi_1(X)/\pi_1'(X)}.
\end{array}
$$

The left vertical arrow is isomorphism (again since K is finite) and the right vertical arrow is surjection which is the isomorphism over the contribution of the trivial character in $H_2(L, \mathbb{C})$. Hence the identity (2.3) is verified. $\square$

Definition 2.1 allows algorithmic calculation of the characteristic varieties (using Fox calculus) provided a presentation of the fundamental group is known. See for example [6, 29] for explicit cases of such calculations.

On the other hand one has the following interpretation using local systems ([17, 20]). Recall that a (rank n) local system is a (n-dimensional) linear representation of the fundamental group $\pi_1(X)$. For treatment of the local systems and their cohomology we shall refer to [11].

A topological definition of the cohomology of rank one local systems can be given as follows. If X is a finite CW-complex, χ is a character of $\pi_1(X)$ and X_{ab} is the universal abelian cover, then one can define the twisted cohomology $H^k(X, \chi)$ as the cohomology of the complex:

(2.9)
$$
\ldots \longrightarrow C^k(X_{\mathrm{ab}}) \otimes_{\mathbb{C}[H_1(X, \mathbb{Z})]} \mathbb{C}_\chi \xrightarrow{\partial^k \otimes 1} \ldots
$$

where $C^k(X_{\mathrm{ab}})$ is the $\mathbb{C}$-vector space of i-cochains of X_{ab} considered as a module over the group ring of $H_1(X, \mathbb{Z})$ and the $\mathbb{C}_\chi$ is the one dimensional $\mathbb{C}$-vector space with the $\mathbb{C}[H_1(X, \mathbb{Z})]$-module structure given by the character χ. If X is a smooth manifold, $H^*(X, \chi)$ has a de Rham description (cf. [11]). The homology of a local system can be described using the dual chain complex. We have the following:

Theorem 2.3. *If $\chi \neq 1$, then*

(2.10)
$$
d(\chi) = \dim H^1(X, \chi).
$$

The connection with the cohomology of local systems allows one to apply general techniques on cohomology with twisted coefficients, which yield the following results on the structure of characteristic varieties:

Theorem 2.4 ([3])**.** *Each $V_k(X)$ is a finite union of cosets of subgroups of $\operatorname{Char}(X)$. Moreover, for each component V of $V_k(X)$ having a positive dimension there is a map $f : X \to C$ where C is a quasi-projective curve such that V is a coset of the subgroup $f^* H^1(C, \mathbb{C}^*) \subset \operatorname{Char}(X)$.*

Theorem 2.5 ([7])**.** *Let V be an irreducible component of $V_k(X)$. Then one of the two following statements holds:*

(1) *There exists an orbicurve $\mathcal{C}$, a surjective orbifold morphism $f : X \to \mathcal{C}$ and an irreducible component W of $V_k^{\mathrm{orb}}(\mathcal{C})$ such that $V = f^*(W)$.*
(2) *V is an isolated torsion point not of type (1).*

Recall the definition of V_k^{orb} after Lemma 1.1 and see section 4.1 for more details on $V_k^{\mathrm{orb}}(\mathcal{C})$ for orbicurves.

2.2. **Alexander polynomial associated with a character.**

A specialization of the characteristic variety of a topological space X of finite type to a special character of its fundamental group $G := \pi_1(X)$ can be defined and it is a natural generalization of the Alexander polynomial to this context. For the sake of simplicity, as at the beginning of section 2.1, we shall assume that an identification $G/G' \cong \mathbb{Z}^r$ was made.

Let X be a finite CW-complex and $\chi \in \mathrm{Char}(G) = (\mathbb{C}^*)^r$ [4] a torsion character, that is $\chi := (\xi_d^{\varepsilon_1}, \ldots, \xi_d^{\varepsilon_r})$, where ξ_d is a primitive d-th root of unity, $0 \le \varepsilon_i < d$, and d is the order of χ. Note that χ determines naturally an epimorphism $\varepsilon : G/G' = \mathbb{Z}^r \to \mathbb{Z}$ defined as $\varepsilon(e_i) := \varepsilon_i$. Let $K_\varepsilon = \ker \varepsilon$, and $K'_\varepsilon = [K_\varepsilon, K_\varepsilon]$ be the commutator of K_ε. By the Hurewicz Theorem $M_\chi := K_\varepsilon/K'_\varepsilon$ can be identified with the homology of the infinite cyclic cover of X corresponding to ε and hence it can be viewed as a module over the group ring $\Lambda := \mathbb{Q}[t^{\pm 1}]$, where t is a generator of the Galois group of covering transformations.

Definition 2.6. Let X and χ be as above, then the *Alexander polynomial of X associated with χ* is a generator of the order of the module M_χ and will be denoted by $\Delta_{X,\chi}(t)$.

The following is a direct consequence of the definition and Theorem 2.3.

Proposition 2.7. *Under the above conditions, if $\chi \ne 1$, then $d(\chi)$ is the multiplicity of the factor $\phi_d(t)$ in $\Delta_{X,\chi}(t)$, where $\phi_d(t)$ is the cyclotomic polynomial of order d.*

Proof. Using the same arguments as in [5, Theorem 2.26], the polynomial $\Delta_{X,\chi}$ is the order of the torsion of

$$(G'/G'' \otimes \mathbb{Q}) \otimes_\Lambda \Lambda/(t_1 - t^{\varepsilon(\gamma_1)}, \ldots, t_r - t^{\varepsilon(\gamma_r)}),$$

therefore $d(\chi)$ is the multiplicity of ξ_d as a root of $\Delta_{X,\chi}$. Since $\Delta_{X,\chi} \in \mathbb{Q}[t]$, the result follows. $\qquad\square$

2.3. **Weight of a character.**

Now let us assume that X is a smooth quasi-projective variety and let $\chi \in \mathrm{Char}(X)$ be a character of finite order. Let X_χ be the covering space corresponding to $\ker \chi \subset \pi_1(X)$. Then $H^1(X_\chi)$ supports a mixed Hodge structure with weights $1, 2$ (cf. [12]). The cyclic group $\mathrm{Im}(\chi)$ acts (freely) on X_χ preserving both the Hodge and weight filtration.

Definition 2.8. An integer w is called a weight of a character χ if the χ-eigenspace of a generator g of $\mathrm{Im}(\chi)$ acting on $\mathrm{Gr}_w^W H^1(X_\chi)$ has a positive dimension. Similarly, p is called a Hodge filtration of χ if the χ-eigenspace of automorphism of $\mathrm{Gr}_F^p H^1(X_\chi)$ induced by g has a positive dimension.

The following gives an expression for the weight of a character in terms of finite coverings with arbitrary Galois groups.

Proposition 2.9. *Let $X_G \to X$ be an abelian cover of X with a finite covering group G. Let $\chi \in \mathrm{Char}(G)$. Then*

$$(2.11) \qquad \dim(W_w H^1(X_G))_\chi = \dim W_w H^1(X_\chi).$$

In particular G has non-zero χ-eigenspace on $W_w H^1(X_G)$ iff χ has weight w.

Proof. The argument is similar to the one in the proof of Theorem 2.2. $\qquad\square$

Remark 2.10. Note that characters might have either no weights or more than one weight. In this paper we will be most interested in characters with only one weight, namely weight 2.

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[4] this identification depends in a choice of generators of $H_1(X, \mathbb{Z})$.

2.4. **Essential, non-essential and essential coordinate components.**

The constructions described in the previous section can be applied to the case when $X = \mathbb{P}^2 \setminus \bigcup_{i=0}^{r} D_i$, where D_i are irreducible curves. In this case $H_1(X, \mathbb{Z}) = \mathbb{Z}^{r+1}/(d_1, ..., d_r)$, where $d_i = \deg D_i$. If the degree of one of its components, say D_0, is equal to one, i.e. we have the complement to a plane curve in $\mathbb{C}^2$, then $H_1(X, \mathbb{Z})$ is a free abelian group of rank r. Let $D = \bigcup D_i$ denote the (reducible) curve in $\mathbb{C}^2$ formed by irreducible components D_i. One has a preferred surjection $\pi_1(\mathbb{C}^2 \setminus D) \to \mathbb{Z}^r$ given by the linking numbers of a loop representing the element of π_1 with the component D_i:

$$\gamma \mapsto (\dots, \mathrm{lk}(\gamma, D_i), \dots \dots).$$

This also yields the identification $\mathrm{Char}(\mathbb{C}^2 \setminus D) = (\mathbb{C}^*)^r$.

Let D' be a reducible subcurve of D in $\mathbb{C}^2$, that is $D' \subset D$. Then (cf. [20]) one has a surjection $\pi_1(\mathbb{C}^2 \setminus D) \to \pi_1(\mathbb{C}^2 \setminus D') \to 1$ and hence an embedding

(2.12) $i_{D'} : \mathrm{Char}(\mathbb{C}^2 \setminus D') \to \mathrm{Char}(\mathbb{C}^2 \setminus D).$

The image of $i_{D'}$ is formed by the factors of $(\mathbb{C}^*)^r$ corresponding to the components of D'. Moreover it was shown in [20] that if $\chi \in V_k(\mathbb{C}^2 \setminus D')$ then $i_{D'}(\chi) \in V_j$ with $j \geq k$.

Definition 2.11 ([20])**.** The components of the characteristic variety $V_k(D)$ obtained as the image of a component of $V_k(D')$ are called *non-essential*. A component of V_k is called *coordinate* if it belongs to $i_{D'}(\mathrm{Char}(\mathbb{C}^2 \setminus D'))$ for some $D' \subsetneq D$.

Given a surjection $\pi_1(\mathbb{C}^2 \setminus \bigcup C_k) \to G := \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_m}$ (for $m \leq r$), one can construct the unbranched covering space $X_{a_1,\dots a_m}$ corresponding to the kernel of the above surjection of the fundamental group. Moreover, there is a compactification of this unbranched cover and its morphism to $\mathbb{P}^2$ extending the covering map. Though this compactification (branched cover) is non-unique, its birational class is well defined. In particular the first Betti number of this branched cover is well defined. A compactification $\bar{X}_{a_1,\dots a_m}$ can be selected so that it supports a G-action extending the action of G on $X_{a_1,\dots a_m}$. A calculation of $H_1(\bar{X}_{a_1,\dots a_m}, \mathbb{C})$ as a G-module is given by the following:

Theorem 2.12 ([25])**.** *For each character χ of $G = \mathbb{Z}_{a_1} \oplus \dots \oplus \mathbb{Z}_{a_m}$ let*

(2.13) $W_\chi := \{v \in H_1(\bar{X}_{a_1,\dots a_m}, \mathbb{C}) \mid g \cdot v = \chi(g)v \text{ for any } g \in G\}$

be the eigenspace of G-action corresponding to the character χ. Let D_χ be the union of components of D over which the character χ is unramified [5]. Then $\dim W_\chi$ is equal to the depth of χ considered as the character of $\pi_1(\mathbb{C}^2 \setminus D_\chi)$.

This theorem was used in [20] to describe essential components of $\pi_1(\mathbb{C}^2 \setminus D)$ in terms of combinatorics of singularities of D and the superabundances of the linear systems of curves given by the local type of singularities, their position on $\mathbb{P}^2$ and the degree of D.

3. Albanese varieties of smooth quasi-projective varieties.

Recall (cf. [27]) that given a projective variety X there is a canonically associated abelian variety $\mathrm{Alb}(X)$ and the map $X \to \mathrm{Alb}(X)$ (unique up to a choice of the image of point in X) is universal with respect to the maps into abelian varieties, i.e. given an abelian variety A and a morphism $X \to A$ there is a unique (up to ambiguity as above) factorization $X \to \mathrm{Alb}(X) \to A$.

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[5]a character $\chi \in H^1(\mathbb{C}^2 \setminus D)$ is unramified along a component $D_i \subset D$ if for the boundary $\gamma_i \in H_1(\mathbb{C}^2 \setminus D)$ of a small disk transversal to D_i one has $\chi(\gamma_i) = 1$. The characters of $\pi_1(\mathbb{C}^2 \setminus D)$ unramified along D_i can be identified with the characters of $\pi_1((\mathbb{C}^2 \setminus D) \cup D_i)$.

This construction can be extended to the quasi-projective case so that the Albanese variety is a semiabelian variety which is universal with respect to the morphisms into algebraic groups. For example one can use Deligne's construction of 1-motif associated to the mixed Hodge structure on cohomology $H^1(X)$ of smooth quasi-projective variety (cf. [11]). More precisely one has the following:

Theorem 3.1. *Let X be a quasi-projective variety which is a complement to a divisor with normal crossings in a smooth projective variety $\bar{X}$.*

(1) *Then one has exact sequence:*

$$(3.1) \qquad 0 \to \mathcal{A} \to \mathrm{Alb}(X) \to \mathrm{Alb}(\bar{X}) \to 0$$

where $\mathcal{A}$ is an affine abelian algebraic group isomorphic to a product of $\mathbb{G}_m$. $\mathrm{Alb}(X)$ depends on X functorially i.e. a morphism $X_1 \to X_2$ induces the homomorphism $\mathrm{Alb}(X_1) \to \mathrm{Alb}(X_2)$.

(2) *If Γ is a finite group of biholomorphic automorphisms then the sequence (3.1) is compatible with the action of Γ.*

An explicit construction can be given as follows (cf. also [18]). Let $\bar{X}$, as above, be a smooth compactification of X such that $\bar{X} \setminus X = D = \bigcup D_i$ is a union of smooth divisors having normal crossings. Then:

$$(3.2) \qquad \mathrm{Alb}(X) = H^0(\bar{X}, \Omega^1_{\bar{X}}(\log D))^* / H_1(X, \mathbb{Z}),$$

where the embedding of $H_1(X, \mathbb{Z})$ as a lattice is given by $\gamma(\omega) := \int_\gamma(\omega)$. The Albanese map is given by

$$(3.3) \qquad P \mapsto \int_{P_0}^{P} \omega$$

(here P_0 is a fixed point on X). The integral (3.3) depends (modulo periods of ω) only on the end points of the path since a holomorphic logarithmic form is closed (cf. [11]). One has the commutative diagram:

(3.4)

$$0 \longrightarrow \ker\left(\bigoplus_i H^0(\mathcal{O}_{D_i}) \to H^1(\Omega^1_{\bar{X}})\right)^* \longrightarrow H^0(\Omega^1_{\bar{X}}(\log D))^* \to H^0(\Omega^1_{\bar{X}})^* \to 0$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

$$0 \to \mathrm{coker}\left(H_2(\bar{X}, \mathbb{Z}) \to \bigoplus_i H_0(D_i, \mathbb{Z})\right) \longrightarrow H_1(X, \mathbb{Z}) \longrightarrow H_1(\bar{X}, \mathbb{Z}) \to 0$$

In this diagram the upper row is dual to the exact cohomology sequence corresponding to the sequence of sheaves given by the residue map (cf. [11, (3.1.5.2)]; below $j : D_i \to \bar{X}$):

$$(3.5) \qquad 0 \to \Omega^1_{\bar{X}} \to \Omega^1_{\bar{X}}(\log D) \overset{\oplus_i \mathrm{Res}_{D_i}}{\longrightarrow} \bigoplus_i j_* \mathcal{O}_{D_i} \to 0.$$

The lower row is the exact sequence of the pair $(\bar{X}, X)$ in which we used the identification:

$$(3.6) \qquad H_2(\bar{X}, X, \mathbb{Z}) = H^{2 \dim D}(D, \mathbb{Z}) = \bigoplus_i H^{2 \dim D}(D_i, \mathbb{Z}) = \bigoplus_i H_0(D_i, \mathbb{Z}).$$

One verifies that all vertical arrows are injective and the image of each provides the lattice in the corresponding complex vector space in the upper row. The rank of the lattice which is the image of the right (resp. left) vertical row is equal to the real (resp. complex) dimension of the target[6]. In the case of the left arrow, one

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[6]Note that the fact that one uses the *real* dimension of $H^0(\Omega^1_{\bar{X}})$ is that rank $H_1(X, \mathbb{Z}) = \dim_{\mathbb{C}} H^1(\mathcal{O}_{\bar{X}}) + \dim H^0(\Omega^1_{\bar{X}}(\log D))$ which follows from the degeneration of the Hodge-deRham spectral sequence in the E_1-term.

uses that the dual map to $H_2(\bar{X}, \mathbb{C}) \to \bigoplus_i H_0(D_i, \mathbb{C})$ factors as

$$\bigoplus_i H^0(D_i, \mathbb{C}) \to H^{1,1}(\bar{X}) \to H^2(\bar{X}, \mathbb{C}).$$

Hence the quotient of the right (resp. left) injection is a complex torus (resp. affine algebraic group isomorphic to a product of several copies of $\mathbb{C}^*$). This complex torus is isomorphic to the Albanese variety of $\bar{X}$ by the classical construction. The remaining assertions of the Theorem 3.1 follow from the description of the Albanese map given by (3.3).

Finally note that Theorem 3.1 implies the following:

Corollary 3.2. *Let ϕ be an involution of X, ϕ_* be the corresponding automorphism of $\mathrm{Alb}(X)$. Let $\mathrm{Alb}(X)^- = \{v \in \mathrm{Alb}(X) \mid i\phi_*(v) = -v\}$ and $\mathrm{Alb}(X)^+ = \{v \in \mathrm{Alb}(X) \mid i\phi_*(v) = v\}$. Then one has an isogeny:*

$$(3.7) \qquad\qquad \mathrm{Alb}(X) = \mathrm{Alb}(X)^- \oplus \mathrm{Alb}(X)^+$$

4. ORBIFOLD PENCILS AND CHARACTERS OF FUNDAMENTAL GROUPS OF QUASI-PROJECTIVE MANIFOLDS

4.1. **Orbicurves.**

Definition 4.1. An *orbicurve* is a complex orbifold of dimension equal to one [7]. An orbicurve $\mathcal{C}$ is called a *global quotient* if there exists a finite group G and a manifold C such that $\mathcal{C}$ is the quotient of C by G with standard orbifold structure.

A *marking* on an orbicurve $\mathcal{C}$ (resp. a quasi-projective variety $\mathcal{X}$) is a character of its orbifold fundamental group [8] (resp. its fundamental group) that is, an element of $\mathrm{Char}^{\mathrm{orb}}(\mathcal{C}) := \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathcal{C}), \mathbb{C}^*)$ (resp. $\mathrm{Char}(X)$) (warning: this terminology is different from the one used in [2]).

A *marked orbicurve* is a pair $(\mathcal{C}, \rho)$, where $\mathcal{C}$ is an orbicurve and ρ is a marking on $\mathcal{C}$. More generally, one defines a *marked quasi-projective manifold* as a pair (X, χ) consisting of a quasi-projective manifold X and a character of its fundamental group. From now on, all characters used as marking below will be assumed to have a finite order.

A marked *global quotient* is a marked orbicurve $(\mathcal{C}, \rho)$ such that if $C \setminus R \to \mathcal{C} \setminus \mathcal{R}$ is the unbranched cover corresponding to the global quotient $C \to \mathcal{C}$, with R being the set of fixed points of non-identity elements of the covering group and $\mathcal{R}$ being its image (or equivalently the set of orbifold points), then one has

$$(4.1) \qquad\qquad \pi_1(C \setminus R) = \ker(\pi_1(\mathcal{C} \setminus \mathcal{R}) \to \pi_1^{\mathrm{orb}}(\mathcal{C}) \xrightarrow{\rho} \mathbb{C}^*)$$

In other words, the above cover $C \setminus R \to \mathcal{C} \setminus \mathcal{R}$ is the cover of the minimal degree over which ρ becomes trivial.

Remark 4.2. The above existence condition for a marking on a global quotient implies that the quotient map over the regular part of the orbifold is a *cyclic* cover. More precisely, the covering group of the cover of the regular part of the orbifold via ρ can be identified with $\mathrm{Im}(\rho) \subset \mathbb{C}^*$ i.e. ρ can be viewed as a character of the covering group. If d is the order of this covering group then the number of possible markings is equal to the value of the Euler function $\phi(d)$.

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[7]i.e. a smooth complex curve with a collection R of points (called *the orbifold points*) with a multiplicity assigned to each point in R. The complement to R is called *the regular part* of the orbifold.

[8]Recall (cf. [2]) that the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{C})$ of an orbifold $\mathcal{C}$ is defined as the quotient $\pi_1(\mathcal{C} \setminus \mathcal{R})$ by the normal closure of the elements $\gamma_{p_i}^{m(p_i)}$ where $m(p_i)$ is the multiplicity of an orbifold point p_i and γ_i is a meridian of p_i.

Definition 4.3. Let $\mathcal{C}$ be a global orbifold quotient and ρ a marking. Let R be the set of orbifold points and $C \setminus R \to \mathcal{C} \setminus \mathcal{R}$ be the quotient map with the covering group G. The integer

$$(4.2) \qquad d(\rho) = \dim\{v \in H^1(C \setminus R, \mathbb{C}) \mid g \cdot v = \rho(g)v, \quad g \in G\}$$

is called the *depth* of a character ρ of the orbicurve $\mathcal{C}$.

Example 4.4. Let $\mathbb{C}_{n,n}$ be the orbifold supported on $\mathbb{C}$ with two orbifold points of multiplicity n. We shall identify $\mathbb{C}$ with $\mathbb{P}^1 \setminus \{[1:1]\}$ so that the orbifold points correspond to $[0:1], [1:0]$. This is the global quotient of a smooth curve C by the cyclic group $\mathbb{Z}/n$ where C is the complement in $\mathbb{P}^1$ to the set $S := \{[\xi_n^i : 1] \mid i = 0, 1 \ldots, n-1\}$ of n points (here ξ_n is a primitive root of unity of degree n) and the global quotient map is the restriction on the complement to S of the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^n$. We have $\pi_1^{\mathrm{orb}}(\mathbb{C}_{n,n}) = \mathbb{Z}/n * \mathbb{Z}/n$. Such free product decomposition implies that $V_1^{\mathrm{orb}}(\mathbb{C}_{n,n}) = \mu_n \times \mu_n$, where μ_n is the multiplicative cyclic group of order n. Consider a character $\rho \in V_1^{\mathrm{orb}}(\mathbb{C}_{n,n})$ taking values ζ, ζ^{-1} on respective generators of this direct sum where ζ is a primitive root of unity. It follows that if $\pi_1(\mathbb{P}^1 \setminus \{[1:0], [1:1], [0:1]\}) \to \pi_1^{\mathrm{orb}}(\mathbb{C}_{n,n})$ is the canonical surjection (in the above identification of $\mathbb{C}$ and $\mathbb{P}^1$ so that the point at infinity corresponds to $[1:1]$), then the pullback of ρ takes values $\zeta, 1, \zeta^{-1}$ on generators corresponding to $[1:0], [1:1], [0:1]$. In particular the covering space corresponding to such ρ is $\mathbb{P}^1 \setminus S$ and the dimension of the ρ-eigenspace is equal to one.

4.2. Orbifold pencils.

Definition 4.5. Let $\mathcal{X}$ be a quasi-projective manifold and $\mathcal{C}$ be an orbicurve. A holomorphic map ϕ between $\mathcal{X}$ and the underlying $\mathcal{C}$ complex curve is called an *orbifold pencil* if the index of each orbifold point p divides the multiplicity of each connected component of the fiber $\phi^*(p)$ over p.

Remark 4.6. Note that this definition implies that if Γ_i is the boundary of a small disk normal to $\phi^{-1}(p_i)$ at its smooth point then $\phi(\Gamma_i)$ belongs to the subgroup of $\pi_1(\mathcal{C} \setminus p_i)$ generated by $\gamma_i^{m(p_i)}$. In particular an orbifold pencil induces the map $\pi_1(\mathcal{X}) \to \pi_1^{\mathrm{orb}}(\mathcal{C})$.

Definition 4.7. Let $\mathcal{X}$ be a quasi-projective variety, C be a quasi-projective curve and $\mathcal{C}$ be an orbicurve which is a global quotient of C. A *global quotient orbifold pencil* is an orbifold pencil $\phi : \mathcal{X} \to \mathcal{C}$ such that there exists a morphism $\Phi : X_G \to C$, where X_G is a quasi-projective manifold endowed with an action of the group G which makes the diagram:

$$(4.3) \qquad \begin{array}{ccc} X_G & \xrightarrow{\Phi} & C \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{C} \end{array}$$

commutative, for which the vertical arrows are the quotients by the action of G.

If, in addition, $(\mathcal{X}, \chi)$ and $(\mathcal{C}, \rho)$ are marked, then the global quotient orbifold pencil $\phi : \mathcal{X} \to \mathcal{C}$ is *marked* if $\chi = \phi^*(\rho)$ where $\phi^* : \mathrm{Char}^{\mathrm{orb}}(\mathcal{C}) \to \mathrm{Char}(\mathcal{X})$ is the homomorphism dual to the surjection $\phi_* : \pi_1(\mathcal{X}) \to \pi_1^{\mathrm{orb}}(\mathcal{C})$ corresponding to the orbifold map ϕ. We will refer to the map of pairs $\phi : (\mathcal{X}, \chi) \to (\mathcal{C}, \rho)$ as a *marked global quotient orbifold pencil* in $(\mathcal{X}, \chi)$ of target $(\mathcal{C}, \rho)$.

Remark 4.8. Consider R the collection of non-manifold points in $\mathcal{C}$ and $F = \phi^{-1}(R)$ the collection of multiple fibers corresponding to ϕ. The orbifold relation $\chi = \phi^*(\rho)$

takes place if and only if the following holds:

$$(4.4) \qquad \hat{\chi} = i^*(\chi) \quad \hat{\chi} = \hat{\phi}^*(\hat{\rho}) \quad \hat{\rho} = p^*(\rho),$$

for the map of open manifolds $\hat{\phi} : \mathcal{X} \setminus F \to \mathcal{C} \setminus R$, induced by ϕ, where $i : \mathcal{X} \setminus \phi^{-1}(R) \to \mathcal{X}$ is the embedding and $p : \pi_1(\mathcal{C} \setminus R) \to \pi_1^{\mathrm{orb}}(\mathcal{C})$ is the canonical projection.

Lemma 4.9. *Let $\mathcal{X}$ be a quasi-projective manifold, $\mathcal{C}$ a marked global quotient with marking $\rho \in \mathrm{Char}^{\mathrm{orb}}(\mathcal{C})$. If $\phi : \mathcal{X} \to \mathcal{C}$ is an orbifold map such that $\chi = \phi^*(\rho)$ then ϕ is a global quotient orbifold pencil i.e. ϕ can be extended to a commutative diagram (4.3).*

Proof. Let, as above, $\pi_G : X_G \to \mathcal{X}$ be the covering space corresponding to $\ker \chi \subset \pi_1(\mathcal{X})$. Since, as follows from Remark 4.8, $\hat{\chi} = \hat{\phi}^*(\hat{\rho})$ one has the commutative diagram:

$$(4.5) \qquad
\begin{array}{ccc}
X_G \setminus \pi_G^{-1}(F) & \xrightarrow{\ \ \Phi\ \ } & C \setminus R \\
{\scriptstyle i^*(\pi_G)} \downarrow & & \downarrow \\
\mathcal{X} \setminus F & \xrightarrow{\ \ \phi\ \ } & \mathcal{C} \setminus \mathcal{R}
\end{array}$$

where $i^*(\pi_G)$ is the restriction of π_G onto $X_G \setminus \pi_G^{-1}(F)$. We have to verify that Φ extends to the map $X_G \to C$ i.e. that is any pair of small loops $\tilde{\Gamma}_P, \tilde{\Gamma}'_P \in \pi_1(X_G \setminus \pi_G^{-1}(F))$ about points P, P' in a connected component F' of $\pi_G^{-1}(F)$ is mapped by Φ into a pair small loops $\tilde{\delta}, \tilde{\delta}'$, about the same point in $R \subset C$. The images of loops $\tilde{\delta}, \tilde{\delta}'$ in $\mathcal{C} \setminus \mathcal{R}$ are homotopic and hence they will be homotopic in $C \setminus R$ iff $\tilde{\delta}^{-1} \cdot \tilde{\delta}'$ belongs to the subgroup of $\pi_1(\mathcal{C} \setminus \mathcal{R})$ generated by δ^m (since preimages of points of $\mathcal{R}$ in C correspond to cosets in the covering group of $C \setminus R$ over $\mathcal{C} \setminus \mathcal{R}$ of the subgroup generated by the image of δ^m). So we claim that $\phi_* \circ (i^*(\pi_G))_*(\tilde{\Gamma}_P), \phi_* \circ (i^*(\pi_G))_*(\tilde{\Gamma}_{P'})$ belong to the same coset of (δ^m). Indeed, since P, P' belong to a connected curve F', $(i^*(\pi_G))_*(\tilde{\Gamma}_P^{-1}\tilde{\Gamma}'_P) \in \pi_1(\partial F)$ where ∂F is the boundary of a small neighborhood of a component of F. Since one has an exact sequence $(\Gamma) \to \pi_1(\partial F) \to \pi_1(F) \to 0$, where Γ is a small loop about the component of F, and the image of a lift into ∂F of a loop in a component of F is trivial in $\mathcal{C}$, the image of $\pi_1(\partial F)$ in $\pi_1(\mathcal{C})$ belongs to the subgroup generated by $\phi_*(\Gamma) = \delta^m$ and hence images of $\tilde{\Gamma}_P, \tilde{\Gamma}_{P'}$ are in the same coset of δ^m as was claimed. $\qquad\square$

Remark 4.10. The key point in the above argument is that images of $\Gamma_P, \Gamma_{P'}$ are the same in $\pi_1^{\mathrm{orb}}(\mathcal{C})$ and that the covering group of $C \setminus R \to \mathcal{C} \setminus \mathcal{R}$ is the quotient of the latter.

Definition 4.11. Global quotient orbifold pencils $\phi_i : (\mathcal{X}, \chi) \to (\mathcal{C}, \rho)$, $i = 1, ..., n$ are called *independent* if the induced maps $\Phi_i : X_G \to C$ constructed in Lemma 4.9 define $\mathbb{Z}[G]$-independent morphisms of modules

$$(4.6) \qquad \Phi_{i*} : H_1(X_G, \mathbb{Z}) \to H_1(C, \mathbb{Z}).$$

In addition, if $\bigoplus_i \Phi_{i*} : H_1(X_G, \mathbb{Z}) \to H_1(C, \mathbb{Z})^n$ is surjective we say that the pencils ϕ_i are *strongly independent.*

Remark 4.12. Note that if either $n = 1$ or $H_1(C, \mathbb{Z}) = \mathbb{Z}[G]$, then independence is equivalent to surjectivity of $\bigoplus_i \Phi_{i*} \otimes \mathbb{Q}$ since the matrix of the latter has as its columns the vectors corresponding to Φ_{i*}.

This definition is motivated by the following:

Proposition 4.13. *Let $\mathcal{C}$ be an orbicurve which is a global G-quotient of the algebraic group $\mathcal{A} = \mathbb{C}^*$. The global quotient orbifold pencils $\phi_i : \mathcal{X} \to \mathcal{C}$, $i = 1, ..., r$ on a global quotient orbifold $\mathcal{X} = X_G/G$ such that the first Betti number of a smooth compactification of X_G is zero, are independent in the sense of Definition 4.11 if and only if they define $\mathbb{Z}$-independent elements of the abelian group $\mathrm{Mor}_G(X_G, \mathcal{A})$ of equivariant morphisms.*

Proof. Note that $\mathrm{Mor}_G(X_G, \mathbb{C}^*) = \mathrm{Hom}_G(\mathrm{Alb}(X_G), \mathbb{C}^*)$ by the universal property of maps from Albanese into an algebraic group. The assumption on the first Betti number of compactification of X_G yields that $\mathrm{Alb}(X_G)$ is a torus of dimension $b_1(X_G)$. Also $\mathrm{Hom}_G(\mathrm{Alb}(X_G), \mathbb{C}^*) = \mathrm{Hom}(H_1(X_G; \mathbb{Z}), H_1(\mathbb{C}^*; \mathbb{Z}))$ holds equivariantly and the claim follows. $\qquad\square$

Remark 4.14. In [10] it was shown that the Proposition 4.13 is also true for special $\mathcal{X}$ in cases when $\mathcal{A}$ is a certain elliptic curve (depending on the Alexander polynomial of $\mathcal{X}$). This is so if $\mathcal{X}$ is the complement of a cuspidal curve C in $\mathbb{P}^2$. In this case $\mathrm{Mor}_G(X_G, \mathcal{A})$ can be identified with the Mordell-Weil group of K-points of the elliptic curve over $\mathbb{C}$ admitting an automorphism of order 6 where K is the field of rational functions on the 6-fold cover of $\mathbb{P}^2$ ramified along the curve i.e. the degree six extension of $\mathbb{C}(x, y)$. This is also the case for δ-curves discussed in [10].

On the other hand, in the case of orbifolds with a trivial orbifold structure there are very few marked orbifold pencils.

Lemma 4.15. *Let $\phi_i : (\mathcal{X}, \chi) \to (C, \rho)$ be a collection of strongly independent marked pencils. Assume that C has the trivial orbifold structure. Then $\rho = 1$ and hence $\chi = 1$.*

Proof. Consider the map $(..., \phi_i, ...) : \mathcal{X} \to C^n$ induced by pencils ϕ. Independence implies that the induced map $H_1(\mathcal{X}) \to H_1(C^n)$ is surjective and hence the dual map of cohomology is injective. If $p_i : C^n \to C$ is the projection on the i-th factor, then $p_i^*(\rho) = (..., \rho, ...)$ (the non-identity component is on the i-th coordinate). The compatibility condition together with the injectivity of $H^1(C, \mathbb{C}^*) \to H^1(\mathcal{X})$ implies that $p_i^*(\rho) = \pi_j^*(\rho)$ and hence $\rho = 1$. $\qquad\square$

Lemma 4.16. *Let ϕ be a marked orbifold pencil of $(\mathcal{X}, \chi)$ with target $(\mathcal{C}, \rho)$ having connected fibers. If $\mathcal{C}$ is a global quotient of a curve C then ϕ is a marked global quotient orbifold pencil.*

Proof. Let, as above, $\bar{\mathcal{X}}$ denotes a smooth compactification of $\mathcal{X}$. Let $R \subset \mathcal{C}$ be the set of non-manifold points. The map $\pi_1(\mathcal{X} \setminus \phi^{-1}(R)) \to \pi_1(\mathcal{C} \setminus R)$ is surjective since the fibers of ϕ are connected. Let K be the kernel of the homomorphism $\pi_1(\mathcal{X} \setminus \phi^{-1}(R)) \to \pi_1(\mathcal{X}) \to \mu_n$, where the right homomorphism is the character χ. Consider $\tilde{X}$ the unbranched cover of $\mathcal{X} \setminus \phi^{-1}(R)$ corresponding to K. Extend it to a branched cover $\pi : \bar{X} \to \bar{\mathcal{X}}$ and let $X := \pi^{-1}\mathcal{X}$. Since χ is the image of $\rho \in \mathrm{Char}^{\mathrm{orb}}(\mathcal{C})$, the map $\mathcal{X} \setminus \phi^{-1}(R) \to \mathcal{C} \setminus R$ extends to the map of unbranched covering spaces and thus to the map of branched covering spaces $X \to C$ presenting the pencil ϕ as the global quotient. $\qquad\square$

4.3. **Orbifold pencils, depth, and roots of Alexander polynomials.**

In this section we shall give a proof of Theorem 1.3.

Proof of Theorem 1.3. In order to prove part (1), let us consider $\phi_{1,*}, \ldots, \phi_{n,*}$, n strongly independent orbifold pencils. Since $H_1(X_G; \mathbb{Z}) \xrightarrow{\oplus_i \Phi_{i,*}} H_1(C; \mathbb{Z})^n$ is an equivariant epimorphism, the dual morphism $H^1(C; \mathbb{Z})_\rho^n \to H_1(X_G; \mathbb{Z})_\chi$ is injective and hence $\mathrm{rank}\, H^1(C; \mathbb{Z})_\rho^n = nd(\rho) \leq \mathrm{rank}\, H_1(X_G; \mathbb{Z})_\chi = d(\chi)$.

As for part (2), let χ be a 2-torsion character. We shall apply the formula for the first Betti number in Theorem 2.2 to the degree-two covering X_χ of $\mathcal{X}$ corresponding to the subgroup $\ker\chi$ of $\pi_1(\mathcal{X})$. The sum in (2.3) contains only one term and yields that $d(\chi)$ is the dimension of the χ eigenspace of $H_1(X_\chi,\mathbb{C})$:

$$(4.7) \qquad d(\chi) = \{v \in H_1(X_\chi,\mathbb{C}) \mid g \cdot v = \chi(g)v\}.$$

Moreover, the action of $g \in \mu_2, g \neq 1$ is the multiplication by -1. The action of g on $\mathrm{Alb}(X_\chi)$ induces the isogeny $\mathrm{Alb}(X_\chi) = \mathrm{Alb}(X) \oplus \mathrm{Alb}(X_\chi)^-$ where the second summand is the subvariety of $\mathrm{Alb}(X_\chi)$ of points on which the covering group acts as multiplication by -1 (cf. Corollary 3.2; note that $\mathrm{Alb}(X_\chi)^+ = \mathrm{Alb}(X)$). Since we assume that χ has only weight two, $\mathrm{Alb}(X_\chi)^-$ has no compact part. In particular, by Theorem 3.1,

$$(4.8) \qquad \mathrm{Alb}(X_\chi)^- = \ker(\mathrm{Alb}(X_\chi) \to \mathrm{Alb}(X)) = (\mathbb{C}^*)^{d(\chi)}$$

and the order two action is given by $z \mapsto z^{-1}$. Therefore the projections give $d(\chi)$ independent equivariant (due to Albanese functoriality) maps $X_\chi \to \mathbb{C}^*$ Since $\mathrm{Mor}_{\mathbb{Z}_2}(X_\chi,\mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}_2}(\mathrm{Alb}(X_\chi),\mathbb{C}^*)$ and $\mathrm{Hom}_{\mathbb{Z}_2}((\mathbb{C}^*)^{d(\chi)},\mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^{d(\chi)},\mathbb{Z})$, then $\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Mor}_{\mathbb{Z}_2}(X_\chi,\mathbb{C}^*) = d(\chi)$. Each map hence descends to an orbifold pencil $\mathcal{X} \to \mathbb{C}_{2,2}$. Since one has the commutative diagram:

$$(4.9) \qquad
\begin{array}{ccc}
X_\chi \setminus \pi_\chi^{-1}(F) & \xrightarrow{\ \Phi\ } & \mathbb{C}^* \setminus \{\pm 1\} \\
{\scriptstyle i^*(\pi_G)}\downarrow & & \downarrow{\scriptstyle \pi_\rho} \\
\mathcal{X} \setminus F & \xrightarrow{\ \phi\ } & \mathbb{C}_{2,2} \setminus \pi_\rho(\pm 1)
\end{array}$$

where π_χ (resp. π_ρ) is the projection $X_\chi \to X$ (resp. $\mathbb{C}^* \to \mathbb{C}_{2,2}$). This diagram induces the isomorphism of order two quotients:

$$(4.10) \qquad \pi_1(\mathcal{X} \setminus F)/\pi_1(X_\chi \setminus \pi_\chi^{-1}(F)) = \pi_1(\mathbb{C}_{2,2} \setminus \pi_\rho(\pm 1))/\pi_1(\mathbb{C}^* \setminus \{\pm 1\})$$

which shows that the pencils $\mathcal{X} \to \mathbb{C}_{2,2}$ preserve markings.

Finally, we will check that any such pencil $\Phi : X_\chi \to \mathbb{C}^*$ can be assumed to have connected fibers and hence the induced morphism Φ_* on cohomology is surjective, which will imply that the n pencils can be found to be strongly independent. Consider the induced orbifold pencil $\phi : \bar{\mathcal{X}} \to \mathbb{P}^1_{2,2}$ and its Stein factorization $\bar{\mathcal{X}} \xrightarrow{\phi} S \xrightarrow{\tilde{\sigma}} \mathbb{P}^1_{2,2}$. Since $\bar{\mathcal{X}}$ is a rational surface, one has $S = \mathbb{P}^1$ with an orbifold structure containing at least two orbifold points each locally being quotient by an order 2 automorphisms. The double cover of S ramified along these two orbifold points after removing the preimage of the point at infinity by $\tilde{\sigma}$ induces maps $X_\chi \xrightarrow{\Phi'} \mathbb{C}^* \xrightarrow{\sigma} \mathbb{C}^*$ where $\Phi = \sigma \circ \Phi'$ and Φ' has connected fibers.

The *moreover* part is a direct consequence of Proposition 2.7. $\square$

5. Pencils on the complements to plane curves and zero-dimensional components of characteristic varieties

5.1. Essential coordinate components and weight.

Essential coordinate components were discovered in [4]. The dimension of the essential coordinate component is zero since the pencil $f : \mathbb{C}^2 \setminus D \to C$ corresponding to a positive dimensional coordinate component of V_i, can be extended to a map $\bar{f} : \mathbb{C}^2 \setminus D' \to C$ with $D' \subsetneq C$ (cf. [20]). It follows from [21] that essential coordinate characters have a finite order. Now we can give a Hodge-theoretical characterization of such in terms of weights (cf. Definition 2.8).

Theorem 5.1. *Let $X = \mathbb{C}^2 \setminus D$ be the complement to a plane curve D. A character in the characteristic variety of $\pi_1(X)$ is essential and coordinate iff it has weight two.*

Proof. Let χ be an essential coordinate character and n be its order. Denote by X_n the covering space of the complement to the curve corresponding to the surjection $\pi_1(X) \to H_1(X, \mathbb{Z}/n\mathbb{Z})$. Let $\bar{X}_n$ be a smooth model of the compactification of X_n. Then $W_1 H^1(X_n, \mathbb{C}) = \mathrm{Im}\, H^1(\bar{X}_n, \mathbb{C})$ (cf. [11]). If $v \in W_1 H^1(X_n, \mathbb{C})$ is a χ-eigenvector of $H_1(X, \mathbb{Z}/n)$ then χ is the eigencharacter of the action on the cohomology of the branched cover. Since χ is a coordinate character, it follows from Sakuma's formula (cf. Theorem 2.12) that χ belongs to the characteristic variety of the curve D' with components corresponding to the trivial coordinate of χ. Using Proposition 2.9, we obtain that essential coordinate characters have weight two.

Conversely, if a character has weight two, then it must be coordinate since by Theorem 2.12 non-coordinate characters belong to the image of $H^1(\bar{X}_n, \mathbb{C}) \to H^1(X_n, \mathbb{C})$ and hence have weight one. This is essential since otherwise Theorem 2.12 would imply that it appears as the eigencharacter of a weight one subspace. $\square$

5.2. **2-torsion characters and quasitoric relations.**

As a consequence of Theorem 1.3.(2) one has the following interpretation of depth for coordinate 2-torsion characters in terms of quasitoric relations of type $(2, 2, 0)$ (cf. [10] for a detailed treatment of quasitoric relations of elliptic type).

Let $D \subset \mathbb{P}^2$ be a plane curve and let χ be a 2-torsion character of $\pi_1(\mathbb{P}^2 - D)$ having two as the only weight (i.e. as in Theorem 5.1). Let S be the collection of the irreducible divisors of D and let $G(S)$ the subgroup of the group of divisors of $\mathbb{P}^2$ generated by S.

Let us fix a generic line at infinity and identify the coordinate ring of the affine plane with $\mathbb{C}[x, y]$. An element in $\mathbb{C}(x, y)$ is an S-unit (cf.[23]) if it is the quotient of two polynomials such that the irreducible components of their zero locus belong to S. This is a multiplicative group denoted by $E(S)$. Note that one has the identification $E(S)/\mathbb{C}^* \cong G(S)$.

An S-unit is called *primitive* if it is a square-free polynomial. The set S splits into two subsets $S = S_0 \cup S_1$ depending on whether or not χ ramifies along each divisor, namely, $\chi(\gamma_0) = 1$ (resp. $\chi(\gamma_1) = -1$) for the boundary γ_0 (resp. γ_1) of a small disk transversal to any divisor in S_0 (resp. S_1). Let $D_1 = \Sigma_{D_{1,i} \in S_1} D_{1,i}$ and $D_0 = \Sigma_{D_{0,i} \in S_0} D_{0,i}$. Note that D_1 has necessarily even degree. In other words, D admits an equation $FH = 0$, where $(F) = D_1$ is a polynomial of even degree and $(H) = D_0$.

We shall use the following notations:

- $\mathbb{C}[x, y]_{E(S)}$ is the localization of $\mathbb{C}[x, y]$ at $E(S)$, the group of S-units. Hence $E(S)$ is the group of units of $\mathbb{C}[x, y]_{E(S)}$.
- $E(S)^+ = E(S) \cap \mathbb{C}[x, y]$ i.e. $E(S)^+$ is the multiplicative monoid generated by $\mathbb{C}^*$ and the polynomials in $E(S)$.
- $\mathbb{K}_F := \mathbb{C}(x, y)[\sqrt{F}]$ is the quadratic extension of $\mathbb{C}(x, y)$.
- $\mathbb{K}_S := \mathbb{C}(x, y)[\sqrt{S}]$, the abelian extension of $\mathbb{C}(x, y)$ generated by those rational multivalued functions whose square has an associated divisor which is a formal sum in S.
- $E(\sqrt{S})$ is the multiplicative group generated by $\mathbb{C}^*$ and $\sqrt{S}$ and $E(\sqrt{S})^+$ is its associated monoid.
- $E_P(\sqrt{S})$ is the set of primitive elements of $E(\sqrt{S})$, that is, $E_P(\sqrt{S})^+ := \{\alpha \in E(\sqrt{S}) \mid \alpha^2 \in E(S)^+ \text{ is square-free}\}$.

We have the following inclusions:

$$\begin{array}{ccccccc} \mathbb{C}[x,y] & \subset & \mathbb{C}[x,y]_{E(S)} & \subset \mathbb{C}(x,y) \subset \mathbb{K}_F \subset & \mathbb{K}_S \\ \cup & & \cup & & \cup \\ E(S)^+ & \subset & E(S) & & E(\sqrt{S}) \supset E(\sqrt{S})^+ \supset E_P(\sqrt{S})^+. \end{array}$$

We consider the following set:

$$G_S := \{(\bar{u}, \bar{v}) \in (\mathbb{K}_S)^2 \mid \bar{u}^2 - \bar{v}^2 = 1\},$$

with a group structure given by:

$$(5.1) \qquad (\bar{u}_1, \bar{v}_1) \cdot (\bar{u}_2, \bar{v}_2) := (\bar{u}_1 \bar{u}_2 + \bar{v}_1 \bar{v}_2, \bar{u}_1 \bar{v}_2 + \bar{v}_1 \bar{u}_2).$$

Note that G_S is isomorphic to $\mathbb{K}_S^*$ via the following map

$$(5.2) \qquad \begin{array}{cccc} \pi: & \mathbb{K}_S^* & \to & G_S \\ & t & \mapsto & \left(\frac{t+t^{-1}}{2}, \frac{t-t^{-1}}{2}\right). \end{array}$$

Also, the map π is equivariant with respect to the automorphisms $t \mapsto t^{-1}$ of $\mathbb{K}_S^*$ and $(u, v) \mapsto (u, -v)$ of G_S.

Definition 5.2. Let $(\bar{u}, \bar{v}) \in G_S$. We say that $(\bar{u}, \bar{v})$ is an *F-pair* if

(1) $\bar{u}, \bar{v} \in E(\sqrt{S}) \cdot \mathbb{C}[x,y]_{E(S)}$ and
(2) there exists a decomposition $\bar{u} = \alpha \cdot u$, $\bar{v} = \beta \cdot v$, for some $\alpha, \beta \in E(\sqrt{S})$ and $u, v \in \mathbb{C}[x,y]_{E(S)}$, such that $\alpha \cdot \beta \in \sqrt{F} \cdot E(S)$.

A decomposition satisfying (2) is called an *F-decomposition* of the *F*-pair.

Remark 5.3. Note that any decomposition $\bar{u} = \alpha \cdot u$, $\bar{v} = \beta \cdot v$, for some $\alpha, \beta \in E(\sqrt{S})$ and $u, v \in \mathbb{C}[x,y]_{E(S)}$ of an *F*-pair is an *F*-decomposition.

Lemma 5.4. *The set G of F-pairs is a subgroup of G_S.*

Proof. Since $(\bar{u}, \bar{v})^{-1} = (\bar{u}, -\bar{v})$ it is enough to prove that the condition of *F*-pair is preserved by the product. Consider $(\bar{u}_1, \bar{v}_1), (\bar{u}_2, \bar{v}_2) \in G$ and let $\bar{u}_i = \alpha_i \cdot u_i$, $\bar{v}_i = \beta_i \cdot v_i$, $i = 1, 2$, be *F*-decompositions of these pairs. Hence:

$$(\bar{u}_1, \bar{v}_1) \cdot (\bar{u}_2, \bar{v}_2) = (\alpha_1 \alpha_2 u_1 u_2 + \beta_1 \beta_2 v_1 v_2, \alpha_1 \beta_2 u_1 v_2 + \beta_1 \alpha_2 v_1 u_2).$$

For the first coordinate we have

$$\alpha_1 \alpha_2 u_1 u_2 + \beta_1 \beta_2 v_1 v_2 = \alpha_1 \alpha_2 \left(u_1 u_2 + (\alpha_1 \beta_1)(\alpha_2 \beta_2) \frac{v_1 v_2}{\alpha_1^2 \alpha_2^2} \right).$$

Note that $\alpha_i^2, \alpha_i \cdot \beta_i \in E(S)$, $i = 1, 2$, and hence we have a decomposition of this first coordinate. In a similar way we obtain a decomposition of the second coordinate where the first factor is $\alpha_1 \beta_2$. Since

$$(\alpha_1 \alpha_2)(\alpha_1 \beta_2) = \alpha_1^2 (\alpha_2 \beta_2) \in \sqrt{F} \cdot E(S)$$

the result follows. □

Definition 5.5. Let $(\bar{u}, \bar{v}) \in G$ be an *F*-pair and let $\bar{u} = \alpha \cdot u$, $\bar{v} = \beta \cdot v$, be an *F*-decomposition of $(\bar{u}, \bar{v})$. This decomposition is said to be *normal* if:

(1) $\alpha = \frac{\tilde{\alpha}}{\gamma}$, $\beta = \frac{\tilde{\beta}}{\gamma}$, where $\tilde{\alpha}, \tilde{\beta}, \gamma \in E_P(\sqrt{S})^+$, $\gcd(\tilde{\alpha}, \tilde{\beta}, \gamma) = 1$.
(2) $u = \frac{\tilde{u}}{w}$, $v = \frac{\tilde{v}}{w}$, $\tilde{u}, \tilde{v} \in \mathbb{C}[x,y]$, $w \in E(S)^+$, $\gcd(\tilde{u}, \tilde{v}, w) = 1$.
(3) $(\tilde{\alpha}\tilde{\beta})^2 = F$.
(4) γ^2 is a divisor of H.

Remark 5.6. The group $\mathbb{C}^*$ acts on the set of *F*-normal decompositions via:

$$\lambda \cdot (\alpha \cdot u, \beta \cdot v) \left((\lambda\alpha) \cdot \left(\lambda^{-1} u\right), \left(\lambda^{-1}\beta\right) \cdot (\lambda v) \right)$$

This action of $\mathbb{C}^*$ will be referred to as *proportionality*.

The following result shows that a normal F-decomposition of an F-pair is almost determined by the pair.

Lemma 5.7. *Any F-pair admits a normal F-decomposition. Moreover, such F-decomposition is unique (up to proportionality).*

Proof. The uniqueness part is straightforward. Let us start with an arbitrary F-decomposition $\bar{u} = \alpha \cdot u$, $\bar{v} = \beta \cdot v$. We can express for instance $\alpha = \frac{\tilde{\alpha}}{\gamma}$, where $\tilde{\alpha}, \gamma \in E(\sqrt{S})^+$ have no common factors. Note that an element in $E(\sqrt{S})^+$ is not primitive if an only if it contains a factor in $E(S)^+$. If $\tilde{\alpha}$ is not primitive, then $\tilde{\alpha} = \tilde{\alpha}_1 u_1$, where $u_1 \in E(S)^+$. In that case, one can rewrite $\bar{u} = (\frac{\tilde{\alpha}_1}{\tilde{\gamma}})(u_1 u)$. One can assume that both decompositions $\bar{u} = \alpha \cdot u$ and $\bar{v} = \beta \cdot v$ are such that α and β have primitive numerators and denominators. Then after taking a common denominator for α and β (resp. for u and v), we can assume that the decomposition

$$
(5.3) \qquad
\begin{aligned}
\bar{u} &= \alpha \cdot u = \left(\tfrac{\tilde{\alpha}}{\gamma}\right) \cdot \left(\tfrac{\tilde{u}}{w}\right), \\
\bar{v} &= \beta \cdot v = \left(\tfrac{\tilde{\beta}}{\gamma}\right) \cdot \left(\tfrac{\tilde{u}}{w}\right)
\end{aligned}
$$

satisfies (1) and (2).

By Remark 5.3, the decomposition $(\alpha \cdot u, \beta \cdot v)$ is an F-decomposition and hence $\alpha\beta = \frac{\tilde{\alpha}\tilde{\beta}}{\gamma^2} \in \sqrt{F} \cdot E(S)$. Let $\sigma \in E_P(\sqrt{S})$ denote an irreducible element. Let us denote by $m(\sigma, \alpha)$ the multiplicity of σ in the decomposition of α into irreducible factors. Since $\tilde{\alpha}, \tilde{\beta}, \gamma \in E_P(\sqrt{S})^+$ one concludes that $m(\sigma, \bullet) \in \{0, 1\}$ for $\bullet = \tilde{\alpha}, \tilde{\beta}, \gamma$. Moreover, by condition (2) in Definition 5.2, if $\sigma^2 | F$, then $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta}) - 2m(\sigma, \gamma)$ is odd and hence so is $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta})$. The previous two conditions imply that:

- $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta}) - 2m(\sigma, \gamma) = 1$,
- $m(\sigma, \gamma) = 0$, and thus
- $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta}) = 1$.

Hence the second property implies condition (4).

Similarly, if $\sigma^2 | H$, then condition (2) in Definition 5.2, implies that $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta}) - 2m(\sigma, \gamma)$ is even and hence so is $m(\sigma, \tilde{\alpha}) + m(\sigma, \tilde{\beta})$. As above, this and the fact that $m(\sigma, \bullet) \in \{0, 1\}$ for $\bullet = \tilde{\alpha}, \tilde{\beta}, \gamma$ imply that:

- $m(\sigma, \tilde{\alpha}) = m(\sigma, \tilde{\beta}) = 0$ and $m(\sigma, \gamma) \in \{0, 1\}$, or
- $m(\sigma, \tilde{\alpha}) = m(\sigma, \tilde{\beta}) = 1$ and $m(\sigma, \gamma) = 0$.

In order to show condition (3) it is enough to prove that the last case can be avoided. In order to do so we rewrite $\bar{u}$ and $\bar{v}$ as

$$
(5.4) \qquad
\begin{aligned}
\bar{u} &= \left(\tfrac{\tilde{\alpha}/\sigma}{\sigma\gamma}\right) \cdot \left(\tfrac{\sigma^2 \tilde{u}}{w}\right) = \alpha_1 \cdot u_1, \\
\bar{v} &= \left(\tfrac{\tilde{\beta}/\sigma}{\sigma\gamma}\right) \cdot \left(\tfrac{\sigma^2 \tilde{v}}{w}\right) = \beta_1 \cdot v_1,
\end{aligned}
$$

where $\alpha_1 := \frac{\tilde{\alpha}_1}{\gamma_1}$, $\beta_1 := \frac{\tilde{\beta}_1}{\gamma_1}$, $\tilde{\alpha}_1 := \frac{\tilde{\alpha}}{\sigma}$, $\tilde{\beta}_1 := \frac{\tilde{\beta}}{\sigma}$, $\gamma_1 := (\sigma\gamma)$, $u_1 := \frac{\tilde{u}_1}{w}$, $v_1 := \frac{\tilde{v}_1}{w}$, $\tilde{u}_1 := \sigma^2 \tilde{u}$, and $\tilde{v}_1 := \sigma^2 \tilde{v}$. Therefore, the second case can be avoided. After a finite number of steps one can assume that $\bar{u} = \alpha_1 \cdot u_1$, $\bar{v} = \beta_1 \cdot v_1$ satisfies:

(1) $m(\sigma, \tilde{\alpha}_1) + m(\sigma, \tilde{\beta}_1) = 1$ for all σ such that $\sigma^2 | F$,
(2) $m(\sigma, \tilde{\alpha}_1) = m(\sigma, \tilde{\beta}_1) = 0$ and $m(\sigma, \gamma_1) \in \{0, 1\}$, for all σ such that $\sigma^2 | H$.

Hence it satisfies (3). Finally, based on the construction, it is easy to check that this new decomposition also satisfies (1),(2), and (4), that is:

(1) $\tilde{\alpha}_1, \tilde{\beta}_1 \in E_P(\sqrt{S})^+$, $\gcd(\tilde{\alpha}_1, \tilde{\beta}_1, \gamma_1) = 1$,
(2) $\tilde{u}_1, \tilde{v}_1 \in \mathbb{C}[x, y]$, $w \in E(S)^+$, $\gcd(\tilde{u}_1, \tilde{v}_1, w) = 1$, and
(4) γ_1^2 is a divisor of H. $\qquad\square$

Consider a quintuple (f, g, h, U, V) of polynomials in $\mathbb{C}[x, y]$ satisfying the functional equation

$$(5.5) \qquad fU^2 - gV^2 = h,$$

where $f \cdot g = F$, h_{red} (a generator of the radical of (h)) divides H and F, H are defined as at the beginning of this section. Note that $(\mathbb{C}^*)^2$ acts on the set of such quintuples as follows: given $\lambda, \mu \in \mathbb{C}^*$, then

$$(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{U}, \tilde{V}) = \left(\left(\lambda^2 f \right), \left(\lambda^{-2} g \right), \left(\mu^2 h \right), \left(\mu \lambda^{-1} U \right), \left(\mu \lambda V \right) \right).$$

As in the case of F-pairs we will denote such action as *proportionality*.

Definition 5.8. A $(2, 2, 0)$-*quasitoric relation* associated to the character χ is a proportionality class (in the above sense) of quintuple (f, g, h, U, V) of polynomials in $\mathbb{C}[x, y]$ satisfying the functional equation $fU^2 - gV^2 = h$, where $f \cdot g = F$, h_{red} divides H and F, H are defined as at the beginning of this section.

Remark 5.9. As mentioned before Definition 5.2, the set of $(2, 2, 0)$-quasitoric relations has a natural order two action defined as $(f, g, h, U, V) \mapsto (f, g, h, U, -V)$. Note that $(f, g, h, -U, V)$ is proportional to $(f, g, h, U, -V)$ and $(f, g, h, -U, -V)$ is proportional to (f, g, h, U, V).

Our purpose now is to establish an isomorphism between $(2, 2, 0)$-quasitoric relations and normal F-decompositions.

Denote by $Q_{(D, \chi)}$ the set of $(2, 2, 0)$-quasitoric relations of D associated to the character χ, that is,

(5.6)
$$Q_{(D, \chi)} := \{ (f, g, h, U, V) \in \mathbb{C}[x, y]^5 \mid fU^2 - gV^2 = h, f \cdot g = F, \text{ and } h_{\mathrm{red}} | H \} / \sim .$$

Proposition 5.10. *The set of $(2, 2, 0)$-quasitoric relations $Q_{(D, \chi)}$ has a group structure isomorphic to the group G described above (Lemma 5.4), where the isomorphism is equivariant with respect to the order two actions on both groups.*

Proof. Let $(\bar{u}, \bar{v})$ be an F-pair. We are going to associate to $(\bar{u}, \bar{v})$ a $(2, 2, 0)$-quasitoric relation $QT(\bar{u}, \bar{v})$. To this end, consider a normal F-decomposition, $u = \alpha \frac{U}{W}$, $v = \beta \frac{V}{W}$, where $\alpha := \frac{\tilde{\alpha}}{\gamma}$ and $\beta := \frac{\tilde{\beta}}{\gamma}$. Let $f := \tilde{\alpha}^2$, $g := \tilde{\beta}^2$, $h_0 := \gamma^2$. From $\bar{u}^2 - \bar{v}^2 = 1$, we deduce:

$$fU^2 - gV^2 = h_0 W^2.$$

It is enough to show that W_{red} divides H. We already know that it divides FH, hence it is enough to show that no irreducible component of F divides W. Otherwise it should divide either f or g, say f for simplicity, then it divides V (and not g). Therefore its multiplicity is odd in fU^2 and even in gV^2 as well as in $h_0 W^2$, which is a contradiction. If $h := h_0 W^2$, then $QT(\bar{u}, \bar{v}) := (f, g, h, U, V)$ is a $(2, 2, 0)$-quasitoric relation of χ.

Notice that up to proportionality, one has $u = (\lambda \alpha) \frac{\lambda^{-1} \mu U}{\mu W}$, $v = (\lambda^{-1} \beta) \frac{\lambda \mu V}{\mu W}$, where $\lambda \alpha := \frac{\lambda \tilde{\alpha}}{\gamma}$ and $\lambda^{-1} \beta := \frac{\lambda^{-1} \tilde{\beta}}{\gamma}$. Define $\tilde{f} := \left(\lambda^2 f \right)$, $\tilde{g} := \left(\lambda^{-2} g \right)$, $\tilde{W} := \left(\mu^2 W^2 \right)$, $\tilde{U} := \left(\mu \lambda^{-1} U \right)$, and $\tilde{V} := \left(\mu \lambda V \right)$. From $\bar{u}^2 - \bar{v}^2 = 1$, we deduce:

$$\tilde{f} \tilde{U}^2 - \tilde{g} \tilde{V}^2 = h_0 \tilde{W}^2,$$

and hence $(\tilde{f}, \tilde{g}, h_0 \tilde{W}^2, \tilde{U}, \tilde{V}) \sim (f, g, h, U, V)$.

Conversely, let us fix a $(2, 2, 0)$-quasitoric relation $QT = (f, g, h, U, V)$. Let $\tilde{\alpha} := \sqrt{f}$ and $\tilde{\beta} := \sqrt{g}$ and write down $h := h_0 W^2$ where h_0 is square-free. Denote $\gamma := \sqrt{h_0}$. Then

$$\left(\frac{\tilde{\alpha}}{\gamma} \frac{U}{W}, \frac{\tilde{\beta}}{\gamma} \frac{V}{W} \right)$$

is the normal F-decomposition of the F-pair (u, v) such that $QT(u, v) = QT$.

Finally, note that if $(\bar{u}, \bar{v})$ is the normal F-decomposition associated with the $(2, 2, 0)$-quasitoric relation (f, g, h, U, V) as at the beginning of this proof, then $(\bar{u}, -\bar{v})$ will be associated with $(f, g, h, U, -V)$, and hence the result follows.   □

Consider X_2 the double cover of $X = \mathbb{C}^2 \setminus D$ associated to the order two character χ.

Proposition 5.11. *The group $Q_{(D,\chi)}$ is isomorphic to $\mathrm{Mor}_{\mathbb{Z}_2}(X_2, \mathbb{C}^*)$.*

Proof. We will give a constructive proof of this result. Suppose one has a quasitoric relation $(f, g, h, U, V) \in Q_{(D,\chi)}$. There is a rational map $\mathbb{C}^2 \to \mathbb{P}^1$ defined by $(x, y) \mapsto [fU^2 : gV^2]$ such that $fU^2 - gV^2 = h$. This map, restricted to $\tilde{X} = \{fgh = 0\}$ defines an orbifold morphism $\tilde{X} \to \mathbb{C}_{2,2} = \mathbb{P}^1_{(2,[0:1]),(2,[1:0])} \setminus \{[1 : 1]\}$. Finally, since $\{fg = 0\} = \{F = 0\}$ and $\{h = 0\} \subset \{H = 0\}$ by definition, one can restrict this to a well-defined orbifold morphism $(X, \chi) \to (\mathbb{C}_{2,2}, \rho)$, which induces an equivariant morphism in $\mathrm{Mor}_{\mathbb{Z}_2}(X_2, \mathbb{C}^*)$.

Conversely, any equivariant morphism in $\mathrm{Mor}_{\mathbb{Z}_2}(X_2, \mathbb{C}^*)$ induces a morphism of marked orbifolds on the quotient $(X, \chi) \to (\mathbb{C}_{2,2}, \rho)$. Extending this to $\mathbb{P}^2$ one obtains a rational morphism $\mathbb{P}^2 \dashrightarrow \mathbb{C}_{2,2} = \mathbb{P}^1_{(2,[0:1]),(2,[1:0])} \setminus \{[1 : 1]\}$, which on a generic affine chart can be defined by $(x, y) \mapsto [fU^2 : gV^2]$, where $\{fg = 0\}$ corresponds to the ramified part of D and $fU^2 - gV^2 = h$ with $\{\tilde{h} = 0\} \subset \{H = 0\}$ the unramified part of D. This quasitoric relation defines an element of $Q_{(D,\chi)}$ and the order-two action corresponds with the covering transformations.

Finally, let us check that this bijection is in fact a homomorphism. Note that an element $p := (f, g, h, U, V) \in Q_{(D,\chi)}$ produces the following commutative diagram:

$$[x : y : z : w] \longmapsto [\alpha u : \beta v]$$

(5.7)

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\Psi_p} & \mathbb{P}^1 \setminus \{[1 : 1], [1 : -1]\} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi_p} & \mathbb{P}^1_{(2,[1:0]),(2,[0:1])} \setminus \{[1 : 1]\}
\end{array}
$$

$$[x : y : z] \longrightarrow [fU^2 : gV^2],$$

where X_2 is contained in $\{[x : y : z : w] \in \mathbb{P}^3 \mid w^2 = fg = F\}$ and $(\alpha u, \beta v)$ is the normal F-decomposition associated with p according to Proposition 5.10. For convenience, let us change the coordinates of $\mathbb{P}^1$ so that $[1 : 1] \mapsto [1 : 0]$ and $[1 : -1] \mapsto [0 : 1]$. In that case, $\mathbb{P}^1 \setminus \{[1 : 1], [1 : -1]\}$ becomes $\mathbb{C}^* = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$ and the new equation of $\Psi_p : X_2 \to \mathbb{C}^*$ becomes $\Psi_p(x, y, z, w) = (\alpha u + \beta v)^2$. Moreover, diagram (5.7) becomes

$$[x : y : z : w] \mapsto (\alpha u + \beta v)^2$$

(5.8)

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\Psi_p} & \mathbb{C}^* \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi_p} & \mathbb{C}_{(2,\sqrt{-1}),(2,-\sqrt{-1})}
\end{array}
$$

$$[x : y : z] \longrightarrow 2\alpha\beta uv,$$

where the vertical right map is the double cover $t \mapsto -\frac{t-t^{-1}}{2}$ ramified at $t = \pm\sqrt{-1}$ with values $\mp\sqrt{-1}$ and where $(\alpha u + \beta v)^{-1} = (\alpha u - \beta v)$.

Given two quasitoric relations $p_i := (f_i, g_i, h_i, U_i, V_i) \in Q_{(D,\chi)}$, $i = 1, 2$ where $f_1 g_1 = f_2 g_2 = F$ and $h_i | H$. Consider $(\bar{u}_i, \bar{v}_i) = (\alpha_i u_i, \beta_i v_i)$, $i = 1, 2$ the normal F-decompositions associated with p_i according to Proposition 5.10. From (5.1) one can see that

$$\Psi_{p_1 p_2} = (\bar{u}_1 \bar{u}_2 + \bar{v}_1 \bar{v}_2 + \bar{u}_1 \bar{v}_2 + \bar{u}_2 \bar{v}_1)^2 = (\bar{u}_1 + \bar{v}_1)^2 (\bar{u}_2 + \bar{v}_2)^2 = \Psi_{p_1} \Psi_{p_2}. \qquad \square$$

As a consequence, under the conditions of Theorem 1.3 one obtains Theorem 1.4.

Proof of Theorem 1.4. The result follows from Proposition 5.11 and the fact that

$$\mathrm{Mor}_{\mathbb{Z}_2}(X_2, \mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}_2}(\mathrm{Alb}(X_2)^-, \mathbb{C}^*) = \mathrm{Mor}_{\mathbb{Z}_2}((\mathbb{C}^*)^d, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z}) = \mathbb{Z}^d,$$

see (4.8), where $d \leq d(\chi)$ and the equality holds if the character only has weight 2.
$\square$

6. EXAMPLES

We refer to [6] for explicit examples of cases of characters of depth greater than one and corresponding orbifold pencils in the case of complements to plane reducible curves.

In this section we will present two examples illustrating interesting phenomena about 2-torsion characters that have come up during the preparation of this paper and that might be of interest to the reader.

6.1. **Orbifold pencils of type** $(2, 2, 2, 2)$. Note that Theorem 1.3(2) refers only to weight two characters of order two. Characters of order two and weight one might be associated with elliptic orbifold pencils of type $(2, 2, 2, 2)$ as the following example seems to suggest.

Consider the Hesse arrangement $\mathcal{H}$ of lines, given by the twelve lines $\{\ell_1, \ldots, \ell_{12}\}$ joining the inflection points of a smooth plane cubic. It is easy to check that the system of cubics sharing the nine inflection points is a pencil with exactly four singular fibers. Each one of these fibers is a completely reducible curve given by three lines in general position. The four cubics $\mathcal{C}_k := \{\ell_{3k+1} \ell_{3k+2} \ell_{3k+3} = 0\}$, $k = 0, 1, 2, 3$ belong to a pencil and their union gives the Hesse arrangement. After blowing up the base points of this pencil one obtains an elliptic fibration onto $\mathbb{P}^1$ where $P_k \in \mathbb{P}^1$ is the image of the special fiber $\mathcal{C}_k$, $k = 0, 1, 2, 3$. If one further blows up one of the three double points in each special fiber, one obtains a rational surface $\tilde{\mathbb{P}}^2$, four exceptional vertical divisors E_0, E_1, E_2, E_3 (not sections) and twelve strict transforms $\tilde{\ell}_i$, $i = 1, \ldots, 12$. The surface $X := \tilde{\mathbb{P}}^2 \setminus \cup_{i=1}^{12} \tilde{\ell}_i$ together with the elliptic fibration induces a well-defined orbifold map onto $\mathbb{P}^1_{(2,P_0),(2,P_1),(2,P_2),(2,P_3)}$ since the preimage of P_k in X is given by $2E_k$ (in divisor notation).

6.2. **Ceva Arrangements.** Note that the polynomials f and g in the group of quasitoric relations 5.6 must satisfy $fg = F$ but such partition of F might be different for different quasitoric relations as the following example shows. Consider the following set of lines:

$$(6.1) \qquad \begin{aligned} \ell_1 &:= x, \quad \ell_2 := y, \quad \ell_3 := z, \quad \ell_4 := (y - z), \\ \ell_5 &:= (x - z), \quad \ell_6 := (x - y), \quad \ell_7 := (x - y - z). \end{aligned}$$

The curve $D := \left\{ \prod_{i=1}^7 \ell_i = 0 \right\}$ is a realization of the Special Ceva arrangement $\mathrm{CEVA}(2, 1)$ (cf. [9, Section 2.3.J, pg. 81]) otherwise known as the non-Fano plane. In [6], a computation of the 2-torsion characters of D is presented via orbifold pencils. In particular, consider $G := \pi_1(\mathbb{P}^2 \setminus D)$ (whose abelianization is $\frac{\mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_7}{\gamma_1 + \cdots + \gamma_7}$). A

character on G can be represented by a septuple of complex numbers whose product is 1, the i-th coordinate representing the image of any meridian γ_i around ℓ_i. The element $\chi = (1, -1, -1, 1, -1, -1, 1)$ represents hence a character on G. In fact, it is well known that its depth is two. Note that $F = \ell_2\ell_3\ell_5\ell_6$ whereas $H = \ell_1\ell_4\ell_7$ according to the notation introduced in section 5.2. Note that

$$(6.2) \qquad \begin{aligned} \ell_2\ell_5 - \ell_3\ell_6 &= \ell_1\ell_4, \\ \ell_2\ell_6 - \ell_3\ell_5 &= \ell_4\ell_7 \end{aligned}$$

are quasitoric relations of type $(2, 2, 0)$ corresponding to the quintuples $q_1 := (\ell_2\ell_5, \ell_3\ell_6, \ell_1\ell_4, 1, 1)$ and $q_2 := (\ell_2\ell_6, \ell_3\ell_5, \ell_4\ell_7, 1, 1)$ respectively. It is not hard to show that q_1 and q_2 are strongly independent and they generate $\mathcal{Q}_{(D,\chi)}$.

Finally, note that

$$\ell_2\ell_5\ell_8^2 - \ell_3\ell_6\ell_9^2 = \ell_1\ell_4\ell_7^2,$$

where $\ell_8 := (y - z - x)$ and $\ell_9 := (z - x - y)$ is another quasitoric decomposition corresponding to $q_3 = (\ell_2\ell_5, \ell_3\ell_6, \ell_1\ell_4\ell_7^2, \ell_8, \ell_9) \in \mathcal{Q}_{(D,\chi)}$. One can check that in fact $q_3 = -q_1 + 2q_2$ (in additive notation).

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