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Topical Review

Elliptic genus of singular algebraic varieties and quotients

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Abstract

This paper discusses the basic properties of various versions of the two-variable elliptic genus with special attention to the equivariant elliptic genus. The main applications are to the elliptic genera attached to non-compact GITs, including the theories regarding the elliptic genera of phases on $N = 2$ introduced in Witten (1993 *Nucl. Phys.* B **403** 159–222).

Keywords: phases of $N = 2$ theories, elliptic genus, GIT quotients, Jacobi forms

1. Preface

This paper provides an overview of the main results of complex elliptic genera, while the second part focuses on the equivariant elliptic genus and contains additional details regarding the treatment of the elliptic genera theories of the $N = 2$ phases given in [40]. The first two sections give a chronological review of the highlights of the development of the elliptic genus as well as its relation to other problems since its introduction in the 1980s (section 2), recalling the key definitions and properties relating to the (complex, two-variable) elliptic genus (see section 3). Then we describe the equivariant elliptic genera using the approach to equivariant cohomology given in [20]. This gives a fast way of deriving the basic properties of the equivariant elliptic genus obtained in [52] from the non-equivariant version given in [9]. The final sections review the properties of the elliptic genera of Witten's $N = 2$ phase theories (see [56]), following [40], but also making explicit specializations of the elliptic genus to the χ_y-genus and the Euler characteristics, in the Landau–Ginzburg instance providing new links between the elliptic genus and the invariant of the singularities. Example 5.1 gives a direct calculation of the χ_y-genus of the LG phase (and can be a starting point for a reader interested in singularity theory), while in the last section it is obtained as a specialization of the elliptic

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genus. Two appendices record the well-known information of the basics of theta-functions and quasi-Jacobi forms introduced in [39].

2. Introduction

The elliptic genus appeared in the middle of the 1980s in the works of topologists and physicists. In mathematics, it was viewed either as the index of an operator on a graded infinite dimensional vector bundle with finite dimensional graded components (see (2) below) or (via Riemann–Roch) as a combination of characteristic numbers (see (3)). The motivation was the problem of finding rigid genera of differentiable spin-manifolds endowed with a circle action (see [36, 37]) extending the Atiyah–Hirzebruch rigidity of the $\hat{A}$-genus. In physics, elliptic genera appeared as indices of certain Dirac-like operators on a free loop space associated with spin-manifolds and also in connection with anomaly cancellations (see volume [38, 54] for an overview of the first results and references therein, e.g. [46]).

The initial versions of elliptic genera were given in the context of differentiable manifolds, but complex versions of the elliptic genus were proposed by Hirzebruch (see [28], see also [31, 36]), and by Witten ([38]) soon after. A different type of elliptic genus, associated with C^∞-manifolds with a vanishing first Pontryagin class, was proposed by Witten (see [38]), leading to important connections with homotopy theory and elliptic cohomology (see [1, 32]).

This first period culminated with the proof of Witten's rigidity conjecture by Bott and Taubes (see [10]). In the complex case, rigidity has been proven by Hirzebruch and in [36] (for somewhat different but closely related versions of the *complex* elliptic genus). A further study of rigidity was done in [41], and much of the material from this first period is summarized in Hirzebruch's book [29] to which we refer the reader.

The elliptic genus, as an invariant of superconformal field theory (SCFT) (or of a representation of a superconformal algebra), was considered at about the same time (see [55] and references to earlier works there). In the case of sigma models associated with manifolds, an invariant of SCFT becomes an invariant of an underlying manifold. There are, however, other backgrounds with which one can associate SCFT and obtain the invariants of such a background. Such notable examples are minimal model SCFTs and Landau–Ginzburg models. The latter are associated with weighted homogeneous polynomials that have isolated singularities. The elliptic genus, as a character of the representation of superconformal algebra in a complex setting (i.e. $N = 2$ SCFT), was given in [34], following [55].

The mathematical version of the elliptic genus as an invariant of the SCFT was presented in [42] where the authors constructed the chiral deRham complex of a manifold and the associated vertex operator algebra (VOA). The characters of vertex operator algebras, which are close relatives of the elliptic genus of SCFTs, were considered from the very beginning of the study of VOAs ([5, 17]), though the mathematical study of analogs of $N = 2$ superconformal algebras is still not very well developed at the moment.

In the late 90s, the focus of mathematical study shifted to the elliptic genera of singular varieties ([50]). On the one hand this was motivated by the Goresky–MacPherson problem (see [25, 50]) of determining the Chern numbers (rather than the Chern classes, which being homological, are not a part of the multiplicative structure and do not determine the Chern numbers) of singular varieties admitting a small resolution of singularities, which are independent of a resolution. The intersection homology groups, introduced in the early 80s, do possess this property, see [25].

On the other hand, mirror symmetry is inherently interwoven with singular varieties: in its very first example of a smooth quintic in $\mathbb{P}^4$ the mirror partner is *the orbifold*, which is the global quotient of such a quintic by the action of an abelian group of order 125 and exponent 5. The relation between the SCFTs, which is a physics definition of mirror symmetry, implies the following relation between the elliptic genera of the mirror partners (see [57])

$$Ell(X) = (-1)^{\dim X} Ell(X'). \tag{1}$$

Mathematically, the relations similar to (1) but involving Hodge numbers, Gromov–Witten invariants, derived categories, etc, serve either as a definition or a test of mirror symmetry.

The orbifold elliptic genus was proposed in the context of Landau–Ginzburg models by Witten (see [55]). In a geometric context (i.e. sigma-models), the elliptic genus of orbifolds was apparently already understood in physics terms right after the introduction of the orbifold Euler characteristic (see [15, 30]). The mathematical definition of the orbifold elliptic genus was given in [8]. This paper also contains a definition of the elliptic genus for a certain class of singular varieties, including the orbifolds, in terms of the resolutions of singularities. The relation between both notions of the elliptic genus of orbifolds, which is the so-called McKay correspondence for an elliptic genus, had been proven in [9]. It includes, as a very special case, the numerical relations, which in dimension 2 are consequences of the relation between the representations of finite subgroups of $SL_2(\mathbb{C})$ and the resolutions of the quotients of $\mathbb{C}^2$ due to McKay (see [43]).

The identity (1) for the hypersurfaces in toric varieties corresponding to the dual polyhedra (Batyrev's mirror symmetry, see [3]) was shown in [7]. One of the major applications of an orbifold elliptic genus to the elliptic genera of the Hilbert schemes of K3-surfaces was given in [14]. A vast generalization in the context of the orbifold elliptic genera of symmetric products was given in [8].

It is interesting to compare the elliptic genus with the other invariants of smooth and singular varieties appearing in the context of mirror symmetry: Hodge numbers, Gromov–Witten invariants, and derived or Fukaya categories (see [35]). Issues similar to those mentioned above in the context of elliptic genus, e.g. the search for the extension of the original definitions from smooth to singular varieties, behavior in mirror correspondence, MacKay correspondence, etc, appeared in the study of all these invariants. However, the results for one type rarely imply the results for others. For example, it is convenient to organize the Hodge numbers of smooth projective varieties in the E-function: $E(u,v) = \sum h^{p,q}u^p v^q$. This has Hirzebruch's genus (see [27]) $\chi_y = \sum \chi(\Omega^p)y^p = \sum_{p,q}(-1)^q h^{p,q}y^p$ as a specialization $y = u, v = -1$. The elliptic genus is related to the χ_y-genus via: $y^{\frac{-\dim X}{2}}\chi_{-y}(X) = \lim_{q\to 0} Ell(X)$. However, neither the E-function or $Ell(X)$ determine each other (see [7]). Both invariants factor through different universal rings of classes of manifolds: the K-group of varieties, in the case of the E-function, and the group of unitary cobordisms, in the case of the elliptic genus. In fact, $E(u,v)$ is a homomorphism $K_{\mathbb{C}}(Var) \to \mathbb{Z}[u,v]$, while the elliptic genus is a homomorphism from the cobordism ring $\Omega^U \to \mathbb{C}[a,b,c,d]$ to a certain polynomial algebra of functions on the product of $\mathbb{H}\times\mathbb{C}$ of the upper half and the whole complex plane respectively. Neither appears to have an extension with good properties to a bigger ring (see, however, [33]).

An attempt to extend the mathematical treatment of the elliptic genus to a wider context, which includes the elliptic genera of singular varieties and Landau–Ginzburg models, was made in [40]. More specifically, for certain geometric invariant theory (GIT) quotients, one can define the elliptic genus such that for the quotients considered by Witten in [56] and corresponding to Calabi–Yau or the Landau–Ginzburg models, they reproduce respectively the elliptic genera of the Calabi–Yau manifolds considered in mathematics and physics literature and the elliptic genera of the Landau–Ginzburg models considered in physics. The approach of [40] is based on the use of the equivariant elliptic genus (in mathematics literature, the equivariant elliptic genus of compact varieties was considered by Waelder [52]). In particular, it implies LG/CY correspondence for the elliptic genus as a consequence of the equivariant McKay correspondence. In the following sections, we spell out some of the details about the elliptic genus of such GIT quotients.

3. Review of previous work

3.1. Complex manifolds

The two-variable elliptic genus, which is the subject of this paper, can be defined as the holomorphic Euler characteristic of a bi-graded bundle associated with the manifold. More precisely, given a vector bundle F on a complex manifold X, one associates it with the Poincare series $\Lambda_t F$ and $\mathrm{Sym}_t F$ (or just $S_t F$) defined respectively as $\Lambda_t = \sum \Lambda^i(F)t^i$ and $\mathrm{Sym}_t(F) = \sum \mathrm{Sym}^i(F)t^i$. Both series are elements of the ring of polynomials in the formal variable t with coefficients in the semi-ring generated by vector bundles. With these notations, the elliptic genus is given by the Fourier expansion with coefficients of the monomials $q^i y^j$ being the holomorphic Euler characteristics of the bi-graded components of the infinite tensor product of graded bundles with $q^i y^j$ providing the bi-grading (see [7, 31, 36, 50])[2]:

$$Ell(X) = y^{\frac{-\dim X}{2}} \chi(X, \otimes_{n \geqslant 1} \Big(\Lambda_{-yq^{n-1}} \Omega^1_X \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} \Omega^1_X \otimes S_{q^n} T_X \Big) \otimes K_X^{-k}). \tag{2}$$

(Here T_X, Ω^1_X, K_X are respectively the tangent, cotangent and canonical bundles of X, and k is a constant.)

The Riemann–Roch theorem implies that (2) is a linear combination of Chern numbers defined as follows. The evaluation of (2) for a compact complex manifold provides a homomorphism of the cobordism *ring* Ω^U of almost complex manifolds (see [48]). The target of this homomorphism is a ring of holomorphic functions on $\mathbb{H} \times \mathbb{C}$ where $\mathbb{H}$ is the upper half-plane if one interprets the formal variable in (2) as $q = \mathrm{e}^{2\pi\sqrt{-1}\tau}, y = \mathrm{e}^{2\pi\sqrt{-1}z}, \tau \in H, z \in \mathbb{C}$. Hirzebruch's formalism (see [27]) implies that any such a homomorphism $\phi : \Omega^U \to R$ with values in a commutative ring R (i.e. an R-valued genus) can be specified by a formal power series $Q(x) \in R[[x]]$ so that $\phi(X) = \prod Q(x_i)[X]$ is the evaluation of the product series at the Chern roots x_i on the fundamental class $[X] \in H_{\dim_{\mathbb{R}} X}(X)$ of X (the Chern roots x_i satisfy $\prod(1 + x_i) = c(X)$, where $c(X) \in H^*(X)$ is the total Chern class of X). In the case of (2), $Q(x)$ is the Taylor series in variables x of the function $\frac{x^{1-k}\theta(\frac{x}{2\pi\sqrt{-1}} - z, \tau)}{\theta(\frac{x}{2\pi\sqrt{-1}}, \tau)}$ and for $k = 0$ one has

$$Ell(X) = \prod_i \frac{x\theta(\frac{x}{2\pi\sqrt{-1}} - z, \tau)}{\theta(\frac{x}{2\pi\sqrt{-1}}, \tau)}[X]. \tag{3}$$

The holomorphic functions, which are the elliptic genera of manifolds, have important modularity properties. If $c_1(X) = 0$ then $Ell(X)$ is the Jacobi form for the semidirect product of $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$ (the Jacobi group), i.e. it obeys the following transformation laws:

$$\phi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k \mathrm{e}^{\frac{2\pi\sqrt{-1}tcz^2}{c\tau + d}} \phi(z, \tau) \tag{4}$$

$$\phi(\tau, z + \lambda\tau + \mu) = (-1)^{2t(\lambda+\mu)} \mathrm{e}^{2\pi\sqrt{-1}t(\lambda^2\tau + 2\lambda z)} \phi(\tau, z)$$

[2] Höhn [31] uses $y \to -y$.

$$a, b, c, d, \lambda, \mu \in \mathbb{Z}, ad - bc = 1.$$

Here $k, t \in \mathbb{Z}$ are the weight and index respectively of the (weak) Jacobi form $\phi(z, \tau)$. For the Calabi–Yau manifold of dimension d, $Ell(X)$, which is given by (2) or (3), is the Jacobi form of weight zero and index $\frac{d}{2}$[3].

Without the Calabi–Yau condition, (3) is a *quasi-Jacobi form* in the sense of [39] (see also appendix B below)[4]. It follows (see [39] theorem 2.12) that the elliptic genera of almost complex manifolds are polynomials in $\hat{E}_2(z,\tau) = (E_2(z,\tau) - e_2(\tau))(\frac{\theta(z,\tau)}{\theta'(0,\tau)})^2, \hat{E}_n(z,\tau) = E_n(z,\tau)(\frac{\theta(z,\tau)}{\theta'(0,\tau)})^n, n \geqslant 1$ where $E_n(z, \tau)$ are the two variable Eisenstein series (see example B.4)

$$E_n(z,\tau) = \sum_{a,b \in \mathbb{Z}^2} (\frac{1}{z + a\tau + b})^n \quad n \in \mathbb{Z}, n \geqslant 1 \tag{5}$$

(with an appropriate choice of summation order for $n = 1$, 2 see [39]) and $e_2(\tau)$ is the one variable Eisenstein series[5].

For example, the elliptic genus of a complex surface of degree D in $\mathbb{P}^3$ can be calculated as

$$(E_1^2(\frac{D^2}{2} - 4D + 8)D + (E_2 - e_2)(\frac{D^2}{2} - 2)D)(\frac{\theta(z,\tau)}{\theta'(0,\tau)})^2. \tag{6}$$

In particular for the K3-surface, i.e. for the case $D = 4$, one obtains

$$24(E_2 - e_2)(\frac{\theta(z,\tau)}{\theta'(0,\tau)})^2.$$

The elliptic genus considered in [7] is given by (3) or (2) with $k = 0$. Up to a factor depending only on the dimension (see [7], proposition 2.3), it coincides with the elliptic genus considered in [31, 36, 50]. The latter two works use the Weierstrass σ-function (see (A.7), in appendix A) writing the characteristic series as follows[6]. Let

$$\Upsilon(x,\tau) = \mathrm{e}^{-G_2(\tau)x^2 - \frac{x}{2}}\sigma(x,\tau) = \mathrm{e}^{-\frac{x}{2}} 2\sinh(\frac{x}{2}) \prod_{n=1}^{\infty} \frac{(1 - q^n \mathrm{e}^x)(1 - q^n \mathrm{e}^{-x})}{(1 - q^n)^2} \tag{7}$$

(here $G_2(\tau) = -\frac{1}{24} + \sum_{n \geqslant 1}(\sum d|n)q^n$ see (A.9), appendix A). Then

$$Q(x) = \mathrm{e}^{kx} x \frac{\Upsilon(x - z, \tau)}{\Upsilon(x)\Upsilon(-z, \tau)}$$

(which, for $k = 0$, is different from (3) by a factor which is $(\mathrm{e}^{-\pi\sqrt{-1}z}\Upsilon(-z,\tau))^{\dim X}$).

The Hirzebruch–Witten elliptic genus of an almost complex manifold corresponds to the characteristic series:

$$\mathrm{e}^{\frac{\bar{k}}{N}x} \frac{\Upsilon(x - \alpha)}{\Upsilon(x)\Upsilon(-\alpha)} \quad \alpha = 2\pi\sqrt{-1}\frac{(a\tau + b)}{N}, \ \ 0 \leqslant a, b, \bar{k} < N. \tag{8}$$

The specialization $q = 0$ of (8) yields the holomorphic Euler characteristic $\chi_y(X, K^{\frac{k}{N}})$ and the specialization of (2) to $z = \frac{2\pi\sqrt{-1}(\alpha + \beta\tau)}{N}$, up to a factor depending on the dimension, gives

[3] Different normalizations in (3), used in some papers, may lead to a different weight and index.

[4] Zagier pointed out that, at least for some of these functions, the term quasi-elliptic would be more appropriate.

[5] I.e. $E_2(0, \tau)$ with the omitted summand corresponding to $a = b = 0$.

[6] Instead of Υ, these papers use the notation $\Phi(x, \tau)$, which is a little different from the one used in [29]; the notations here and below are the same as in [29].

the Hirzebruch–Witten elliptic genus (8) ([7], proposition 2.4). As (3), this is an invariant of almost complex manifolds but it has the following modular property: if $c_1(X) = 0 \bmod N$ then the Hirzebruch–Witten elliptic genus is a modular form for the subgroup $\Gamma_0(N)$ of the modular group. If $N = 2$ in (8) then the Hirzebruch–Witten genus depends on the Pontryagin (rather than the Chern) classes of X only and is an invariant of the C^∞-manifolds which is modular (for $\Gamma_0(2)$) if the manifold is spin. This is the first instance of an elliptic genus which appears in mathematics literature and is due to Ochanine–Landweber–Stong (see [38]).

3.2. Orbifold elliptic genus

The elliptic genus of orbifolds which are global quotients was defined in [8] as follows (this definition was extended to arbitrary orbifolds in [18]):

Let X be a smooth projective variety and let Γ be a finite group of its automorphisms. For an element $g \in \Gamma$ let X^g denote its fixed point set. For a connected component $\bar{X}^g$ of X^g we consider the decomposition into the eigenspaces of g for the restriction $TX|_{\bar{X}^g}$ of the tangent bundle of X on $\bar{X}^g$. We represent each eigenvalue of g acting on this restriction, in the form $\exp(2\pi\sqrt{-1}\lambda(g))$, where $0 \leqslant \lambda(g) < 1$, and denote the eigenbundle corresponding to this eigenvalue as $V_{\lambda(g)}$. In particular, V_0 is the tangent bundle to $\bar{X}^g$ and $TX|_{\bar{X}^g} = V_0 \oplus (\oplus_{\lambda(g)\neq 0} V_{\lambda(g)})$. We also denote by $F(g, \bar{X}^g) = \sum \lambda(g)$, 'the fermionic shift' corresponding to the component $\bar{X}^g$. Then we let

$$V_{h,\bar{X}^h\subset X} := \otimes_{k\geqslant 1}\big[\Lambda_{yq^{k-1}} V_0^* \otimes \Lambda_{y^{-1}q^k} V_0 \otimes \mathrm{Sym}_{q^k} V_0^* \otimes \mathrm{Sym}_{q^k} V_0 \otimes$$

$$[\otimes_{\lambda(h)\neq 0}(\Lambda_{yq^{k-1+\lambda(h)}} V^*_{\lambda(h)} \otimes \Lambda_{y^{-1}q^{k-\lambda(h)}} V_{\lambda(h)} \otimes \mathrm{Sym}_{q^{k-1+\lambda(h)}} V^*_{\lambda(h)} \otimes \mathrm{Sym}_{q^{k-\lambda(h)}} V_{\lambda(h)})]\big] \tag{9}$$

(i.e. the tensor product over the positive integers k of the tensor products of the exterior and symmetric algebras $\Lambda_t, \mathrm{Sym}_t$, defined just before equation (2), of the eigenbundles $V_{\lambda(h)}$ of the automorphism h labeled by the logarithms $\lambda(h)$ of the eigenvalues of h acting on the normal bundle of a component $\bar{X}^h$ with the subscripts t of the exterior and symmetric algebras depending on k and $\lambda(h)$). With these notations one defines the orbifold elliptic genus as:

$$Ell_{\mathrm{orb}}(X, \Gamma; y, q) := y^{\dim X/2} \sum_{\{h\}\in \mathrm{Conj}(\Gamma), \bar{X}^h} y^{F(h,\bar{X}^h\subseteq X)} \frac{1}{C(h)} \sum_{g\in C(h)} L(g, V_{h,\bar{X}^h\subseteq X}) \tag{10}$$

where $\mathrm{Conj}(\Gamma)$ is the set of conjugacy classes in Γ, $C(h) \subseteq \Gamma$ is the centralizer of $h \in \Gamma$ and $L(g, V)$ is the holomorphic Lefschetz number of g with the coefficients in a holomorphic g-bundle V, i.e. $L(g, V) = \sum(-1)^i \mathrm{tr}(g, H^i(X^g, V))$. The equivalent form of (10) is

$$Ell_{\mathrm{orb}}(X, \Gamma; y, q) := y^{\dim X/2} \sum_{\{h\}\in \mathrm{Conj}(\Gamma), \bar{X}^h\subseteq X} y^{F(h,\bar{X}^h\subseteq X)} \chi(H^\bullet(V_{h,\bar{X}^h\subseteq X})^{C(h)}) \tag{11}$$

where $\chi(H^\bullet(V_{h,\bar{X}^h\subseteq X})^{C(h)})$ is the alternating sum of the dimensions of the $C(h)$-invariant subspaces of the cohomology of bundles $V_{h,\bar{X}^h\subseteq X}$.

The Atiyah–Bott holomorphic Lefschetz formula (see [2]), allows us to rewrite the combination of holomorphic Lefschetz numbers (10) in terms of characteristic classes as follows: for a pair of commuting elements $g, h \in \Gamma$, let $X^{g,h} = X^g \cap X^h$ denote the set of points in X fixed by both g and h (by abuse of notation, the summations below take place, however, over the *connected components* of $X^{g,h}$). Then (10) in terms of the characteristic classes looks as follows:

$$\frac{1}{|\Gamma|}\sum_{g,h,gh=hg}(\prod\nolimits_{\lambda(g)=\lambda(h)=0}x_\lambda)\prod\nolimits_\lambda\frac{\theta(\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h))}\mathrm{e}^{2\pi\mathrm{i}z\lambda(h)}[X^{g,h}] \tag{12}$$

where the products are taken over all the Chern roots x_λ (counted with their multiplicities) of the eigenbundles V_λ corresponding to the logarithms λ of the characters of the abelian subgroup of Γ generated by g, h. The term in this sum corresponding to the pair in which both g, h are the identities coincides with the elliptic genus of X. We shall call this term the *'trivial'* sector of the orbifold elliptic genus. It is a summand in the *'untwisted'* sector representing the sum of terms corresponding to the pairs $(1, h)$.

A notable application of the mathematical definition (10) is the following formula for the generating function for the orbifold elliptic genera of the symmetric products:

Theorem 3.1 (See [8]). *Let X be a smooth projective variety and let $Ell(X) = \sum_{m,l} c(m,l)y^l q^m$. Then*

$$\sum p^n Ell_{\mathrm{orb}}(X^n,\Sigma_n;y,q)=\prod\nolimits_{i=1}^{\infty}\prod\nolimits_{m,l}\frac{1}{(1-p^iq^my^l)^{c(mi,l)}}. \tag{13}$$

The formula (13) and physics proof of this identity (13) was discovered in [14].

3.3. Elliptic genus of pairs

The same work [8], besides the definition of the elliptic genus of global quotients, contains an approach to the elliptic genus of singular varieties, based on the resolution of singularities, and in which the assumption that the singularities are the quotients is replaced by an assumption coming from the birational geometry:

Definition 3.2 ($\mathbb{Q}$-Gorenstein varieties with klt singularities). A normal variety X is called $\mathbb{Q}$-Gorenstein if a Weil $\mathbb{Q}$-divisor, which is a multiple of the divisor of the top degree differential form, is a Cartier. A $\mathbb{Q}$-Gorenstein variety is call klt (i.e. having a Kawamata log-terminal singularities) if there exists a resolution of the singularities $\hat{X} \to X$ such that the coefficients of decomposition $K_{\hat{X}} = f^*(K_X) + \sum \alpha_k E_k$ satisfy $\alpha_k > -1$.

To define the elliptic genus of singular varieties with singularities as in 3.2, one first defines the elliptic genus of the pairs X, E, where X is smooth and projective and E is a $\mathbb{Q}$-divisor on X, i.e. $E = \sum \alpha_k E_k$ is a formal sum such that the components E_k are smooth divisors on X intersecting transversally. Moreover, one assumes that $\alpha_k > -1$ for all k. In this situation one defines the cohomology class, called the *elliptic class* of the pair (X, E)[7]:

$$\mathcal{E}LL(X,E)=\prod\nolimits_i\frac{x_i\theta(\frac{x_i}{2\pi\sqrt{-1}}-z,\tau)\theta'(0,\tau)}{\theta(\frac{x_i}{2\pi\sqrt{-1}},\tau)\theta(-z,\tau)}\prod\nolimits_k\frac{\theta(\frac{e_k}{2\pi\sqrt{-1}}-(\alpha_k+1)z,\tau)\theta(-z,\tau)}{\theta(\frac{e_k}{2\pi\sqrt{-1}}-z,\tau)\theta(-(\alpha_k+1)z,\tau)}. \tag{14}$$

The elliptic genus of a pair is then the evaluation of the elliptic class on the fundamental class of X: $Ell(X, E) = \mathcal{E}LL(X, E)[X]$.

[7] In $H^*(X,\mathbb{Q})$ tensored with a ring of functions in z,τ appearing in the expansion (14) in x. A ring of quasi-Jacobi forms described in appendix B can be used. Often below we shall, by abuse of terminology, say that we consider the elliptic class in cohomology (or Chow groups), meaning in fact that this class is in the cohomology (or Chow theory) that has been extended in such a way.

The fundamental property of the elliptic class of pairs, which allows us to define the elliptic genus of a singular variety as the elliptic genus of a pair consisting of a resolution and a certain divisor on the latter, is the compatibility in the blowups:

Theorem 3.3 (See [9], where a more general statement concerning the orbifold elliptic class of pairs endowed with a Γ-action.). *Let (X, E) be a pair as described above after definition 3.2, Z is a submanifold of X transversal to the irreducible components E_k of $E = -\sum \alpha_k E_k$*[8]*, $f : \hat{X} \to X$ is the blow up of X with the center at Z, $Exc(f)$ is its exceptional divisor, $\hat{E}_k$ are the proper preimages of components E_k, α and $\hat{E}$ are such that $\hat{E} = -\sum \alpha_k \hat{E}_k - \alpha Exc(f)$ and $K_{\hat{X}} + \hat{E} = f^*(K_X + E)$. Then $\alpha > -1$ and*

$$f_*(\mathcal{E}LL(\hat{X}, \hat{E})) = \mathcal{E}LL(X, E). \tag{15}$$

In particular $Ell(\hat{X}, \hat{E}) = Ell(X, E)$.

Corollary 3.4. *Let X be a $\mathbb{Q}$-Gorenstein projective variety with at most klt singularities. Let $\hat{X} \to X$ be a resolution of its singularities and $\hat{E} = \sum \alpha_k \hat{E}_k$ be a normal crossing divisor on $\hat{X}$ such that $f^*(K_X) = K_{\hat{X}} + \hat{E}$. Then $Ell(\hat{X}, \hat{E})$ depends only on X, i.e. it is independent of a choice of $(\hat{X}, \hat{E})$ (and called a (singular) elliptic genus of X), which will be denoted $Ell_{\mathrm{sing}}(X)$.*

The fundamental relation between the singular and orbifold elliptic genera is given by the so-called MacKay correspondence for the elliptic genus:

Theorem 3.5. *Let X be a smooth projective variety on which a group Γ acts effectively via biholomorphic transformations. Let $\mu : X \to X/\Gamma$ be the quotient map. Assume that μ does not have ramification divisors, i.e. fixed points of elements of Γ have a codimension greater than one. Then*

$$\mathcal{E}LL_{\mathrm{orb}}(X, \Gamma; z, \tau) = \left(\frac{2\pi\sqrt{-1}\theta(-z, \tau)}{\theta(0,)}\right)^{\dim X} \mathcal{E}LL_{\mathrm{sing}}(X/\Gamma, z, \tau). \tag{16}$$

In particular, the orbifold elliptic genus coincides with the elliptic genus of any crepant[9] *resolution of the quotient (if such exist).*

We refer to [9] theorem 5.3 for a more general statement in the category of Kawamata log-terminal pairs and for the case of quotient maps admitting ramification divisors.

An immediate corollary is a reinterpretation of the series in theorem 3.1 in the case when $\dim X = 2$, as the generating series of the elliptic genera of the Hilbert schemes:

Corollary 3.6. *Let X be a smooth projective surface and $Ell(X) = \sum_{m,l} q^m y^l$. Then*

$$\sum p^n Ell(Hilb_n, q, y) = \prod_{i=1}^{\infty} \prod_{m,l} \frac{1}{(1 - p^i q^m y^l)^{c(mi,l)}}. \tag{17}$$

Indeed, in the case of surfaces, the morphism $Hilb_n(X) \to X^n/\Sigma_n$ is a smooth crepant resolution.

[8] Recall that the divisors E_k are assumed to be smooth.

[9] Recall that a resolution of singularities of a $\mathbb{Q}$-Gorenstein variety is called crepant if the α_k from definition 3.2 are all equal to zero.

4. Equivariant elliptic genus

In this section we discuss an equivariant version of the elliptic genus. In particular we shall describe the equivariant analog of the push forward formula (i.e. theorem 3.3) for an elliptic class, equivariant McKay correspondence, the equivariant localization and push forward properties of the contributions of compact components of fixed point sets into an elliptic class. Our approach is based on the equivariant intersection theory as developed in [20] (see also [50]). It allows us to derive equivariant results from their non-equivariant counterparts, already discussed in section 3.3, applied in an appropriately formulated context. As in [9, 20], instead of ordinary cohomology, we work in Chow theory, but a reader of the course can interpret all the statements as those in ordinary cohomology.

4.1. Equivariant intersection theory

We start by working in the category of quasi-projective normal varieties (over $\mathbb{C}$) with various assumptions on the singularities such as the $\mathbb{Q}$-Gorenstein and klt conditions (see section 3.3). We also assume that a reductive algebraic group G, $\dim G = g$ acts on such an X via a linearized action. The latter means that an ample line bundle L is presented on X together with a G-action on the total space of L such that the bundle projection on X is equivariant (see [44]). We shall refer to [20] section 6 for precise conditions of the action, which assure that the constructions needed for an equivariant intersection theory to run will work.

Let $V, \dim V = l$ be a representation of G, $U \subset V$ be an open set such that G acts on U freely and $\mathrm{codim} V \setminus U$ is sufficiently large. Then U/G is smooth, and for a given n, the Chow groups $A^{n'}(U/G) = A_{l-g-n'}(U/G)$ are well defined for $l \gg n$, and so are the products among them for all $n' < n$. The Chow ring $A^*(BG)$ is defined as the graded ring with $A^n(U/G)$ for $n \ll l$ as its graded components; again, they are independent of l as long as l is large enough.

Since G acts freely on U, the diagonal G-action on $X \times U$ is free as well, the quotient space $X_G = (X \times U)/G$ does exist and the equivariant Chow group $A_i^G(X)$ can be defined as the usual Chow group $A_{i+l-g}(X_G)$. Again, it is independent of V,U as long as $\mathrm{codim} V \setminus U$ is sufficiently large (see [20] proposition-definition). The intuition behind such a choice of indices is that in the case when X is smooth, the projective and the quotient U/G is compact, and one has $\dim X_G = \dim X + l - g$ and $A_{i+l-g}(X_G) = A^{\dim X - i}(X_G)$ by the Poincare duality.

Let E be an equivariant G-bundle on a quasi-projective variety with the action of G, i.e. the total space of E is endowed with a G-action such that projection $E \to X$ is G-equivariant. Then $E_G = (E \times U)/G \to X_G$ is a vector bundle on X_G and the equivariant Chern class $c_j^G \in A_*^G(X)$ is the Chern class of the vector bundle E_G on X_G. As in the non-equivariant case, one associates the equivariant bundle with the (equivariant) Chern roots $x_i^G \in A_*(X_G)$.

To define the equivariant elliptic class, we note that the map π induced by the projection on the second factor:

$$X_G = (X \times U)/G \xrightarrow{\pi} U/G = BG \tag{18}$$

is a locally trivial fibration with the fiber X.

Definition 4.1. Let X be a *smooth projective* variety with the action of the algebraic group G. The equivariant elliptic genus of (X, G) is the push forward of the equivariant elliptic class, i.e. the class (3) where x_i are the equivariant Chern roots of the tangent bundle of X with its natural G-structure:

$$Ell^G(X) = \pi_*(\mathcal{E}LL(X_G)) \in A_*(BG) \otimes QJac \tag{19}$$

where $\pi : X_G \to BG$ is induced by a projection of $X \times U$ on the second factor and $QJac$ is the ring of the quasi-Jacobi forms, i.e. the ring of functions on $\mathbb{C} \times H$ generated by coefficients of the Taylor expansion in x of a factor in the product (3) (see appendix B)[10].

By the equivariant Riemann–Roch theorem, one can interpret (19) as the character decomposition of the holomorphic Euler characteristic of the G-equivariant bundle (3), where T_X, Ω^1 is endowed with a natural G-structure (see [22]).

In the case when G is a torus T (affine connected commutative algebraic group) of dimension r, the equivariant elliptic class in $A^*(BT, QJac)$ can be viewed as an element of the ring of polynomials in r variables with coefficients in the ring of the quasi-Jacobi forms (see appendix B).

4.2. Equivariant localization

Let T be a torus acting algebraically on a smooth quasiprojective scheme X. Let $\hat{T}$ be the group of characters of T. An identification $T = \mathbb{C}^{*r}$ induces the identification of $\hat{T}$ with a free abelian group generated by the character $t_1, ..., t_r \in \hat{T}$ (such that $t_i(z_1, ..., z_i, ...z_r) = z_i \in \mathbb{C}^*$). The T-equivariant Chow ring of a point, i.e. $A^*(BT)$, as was already mentioned, is isomorphic to the symmetric algebra of the free abelian group $\hat{T}$. More generally, if T acts trivially on X then $A^T_*(X) = A_*(X) \otimes \mathrm{Sym}(\hat{T})$ (here $\mathrm{Sym}(\hat{T})$ is the symmetric algebra with generators $t_1, ..., t_r$; see [21]). For details of the following we refer to [21].

Theorem 4.2. *Let $R_T = \mathrm{Sym}(\hat{T})$, $\mathcal{Q}_T = (R^+_T)^{-1} R_T$ where R^+_T is the semigroup of elements of positive degree and let $i : X_T \to X$ be the embedding of the fixed point set. Then*

$$i_* : A_*(X_T) \otimes \mathcal{Q}_T \to A^T_*(X) \otimes \mathcal{Q}_T \tag{20}$$

is an isomorphism.

If Y and X are smooth, $j : Y \to X$ is a regular embedding of codimension d, N is the normal bundle of Y in X and $\alpha \in A_*(Y)$, one has the self intersection formula $j^* j_*(\alpha) = c_d(N) \cap \alpha$ (see section 6.3, corollary 6.3 [24]). If F is a fixed point set of a torus T acting on a smooth scheme X, then F is smooth and the self intersection formula applied to $i_F : F \times U/G \to X \times U/G$ implies $i^*_F i_{F*}(\alpha) = c^T_d(N) \cap \alpha$. This results in an explicit localization isomorphism:

$$A^T_*(X) \otimes \mathcal{Q}_T \to A_*(X_T) \otimes \mathcal{Q}_T : \quad \beta \to \frac{i^*_F(\beta)}{c^T_d(N)} \tag{21}$$

(here $c^T_d(N)$ denotes the equivariant Chern class of the normal bundle to the fixed point set).

4.3. Push forward of the equivariant elliptic class and equivariant McKay correspondence

The above approach to equivariant intersection theory allows us to deduce directly the equivariant counterparts of the key results about the elliptic genus: the push forward formula of the elliptic class and the McKay correspondence. A different derivation of these properties was given in [52].

Let $\bar{X}$ be a smooth projective variety with a biholomorphic action of a torus T. Let $E = \sum \alpha_i E_i$ be a normal crossings divisor on $\bar{X}$ such that all irreducible components E_i are T-invariant. Then (in notations of section 4.1) $(E_i \times U)/T$ is a divisor on $(\bar{X} \times U)/T$ and

[10] As in (14) one can use any ring of functions containing the coefficients of expansion of the elliptic genera of manifolds in Chern classes.

hence the classes $e_i^T \in A_*^T(\bar{X})$ are well defined. Using (14) we obtain the equivariant elliptic class $\mathcal{E}LL^T(\bar{X}, E) \in A_*^T(\bar{X}, QJac)$.

Theorem 4.3 (Push forward formula). *Let $\bar{X}$ be a smooth projective variety with a torus T acting on $\bar{X}$ via biregular automorphisms. Let E be a T-invariant normal crossings divisor and Z a smooth T-invariant submanifold of $\bar{X}$ transversal to all irreducible components of E. Let $\phi : \bar{X}' \to \bar{X}$ be the blow up of $\bar{X}$ with the center at Z and let E' be the divisor on $\bar{X}'$ such that $\phi^*(K_{\bar{X}} + E) = K_{\bar{X}'+E'}$. Then the action of T on $\bar{X} \setminus Z$ extends to the action on $\bar{X}'$ leaving E' invariant and*

$$\phi_*(\mathcal{E}LL^T(\bar{X}', E')) = \mathcal{E}LL^T(\bar{X}, E) \tag{22}$$

where on the left one has the equivariant elliptic class for the action on $\bar{X}'$ induced by the action of T on $\bar{X} \setminus Z$.

Proof. Let $\pi : \bar{X}_T \to BT$ be a locally trivial fibration defined by the action of T and a representation of T as in section 4.1 (recall that $BT = U/T$ is the quotient space of a Zariski open set U in the representation space with a sufficiently large codimension of the complement to U). Since Z and E_i are T-invariant, one has the embedding of fibrations $Z_T \to \bar{X}_T$, $(E_i)_T \to \bar{X}_T$ of subvarieties of $\bar{X}_T$ corresponding to Z and E_i compatible with the projections on T. Let $\bar{X}'_T = (\bar{X}' \times U)/T$ and $\phi_T : \bar{X}'_T \to \bar{X}_T$ be induced morphisms. $\bar{X}'_T = (\bar{X}' \times U)/T$ can be identified with the blow up of $\bar{X}_T$ along Z_T. This can be seen, for example, from a local description of the blow up as in [51] definition 3.23. Moreover, $E_T = \sum \alpha_i (E_i)_T$, the multiplicity of $(E_i)_T$ along Z_T is the same as the multiplicity β_i of E_i along Z and the codimension of Z_T in $\bar{X}_T$ coincides with the codimension of Z in $\bar{X}$. It follows that $(E'_i)_T$, whose irreducible components are the proper preimages of $(E_i)_T$, and the exceptional locus of ϕ_T all have the same multiplicities as the corresponding components in E' (see [8] p 327 and also theorem 3.3). Therefore, $\phi_T^*(K_{\bar{X}_T} + E_T) = K_{\bar{X}'_T} + E'_T$. Now theorem 3.5 in [9] immediately implies theorem 4.3. □

As in the non-equivariant case, the push forward formula (22) shows us that the following definition is independent of the resolution it uses.

Definition 4.4 (Equivariant singular elliptic class).

Let X be a $\mathbb{Q}$-Gorenstein projective variety with at most klt singularities on which a torus T acts by regular automorphisms. Let $f : \hat{X} \to X$ be an equivariant resolution of its singularities and $\hat{E} = \sum \alpha_k \hat{E}_k$ be a normal crossing divisor on $\hat{X}$ such that $f^*(K_X) = K_{\hat{X}} + \hat{E}$. The equivariant singular elliptic class is defined as

$$\mathcal{E}LL^T_{\mathrm{sing}}(X) = f_*(\mathcal{E}LL^T(\hat{X}, \hat{E})) \tag{23}$$

(it is independent of a choice of equivariant resolution). The equivariant singular elliptic genus is the push forward of $\mathcal{E}LL^T_{\mathrm{sing}}(X)$ to the Chow ring of the point (see definition 19).

In the case when the singular variety is an orbifold with an action of a torus, one has an equivariant version of the orbifold elliptic class related to the equivariant singular elliptic class just described.

Theorem 4.5 (Equivariant version of the McKay correspondence). *Let X be a smooth projective variety with a torus T acting on X via biregular automorphisms. Let Γ be a finite group whose action commutes with the action of T. Then, for any pair $(g, h) \in \Gamma$ of commuting elements, the fixed point locus $X^{g,h}$ is T-invariant, the class obtained by replacing it in* *the elliptic class appearing in (11) the ordinary Chern roots of the bundles V_λ by the equivariant Chern roots of these bundles with a natural T-structure—and called the equivariant orbifold class of (X, T, Γ)—satisfies the following push forward formula*[11]. *If $\psi : X \to X/\Gamma$ is the quotient morphism, then*

$$\psi_*(\mathcal{E}LL^T_{\mathrm{orb}}(X, \Gamma)) = \mathcal{E}LL^T_{\mathrm{sing}}(X/\Gamma). \tag{24}$$

Proof. This follows from the corresponding results in [9] as in the proof of theorem 4.3. Since the actions of T and Γ commute, the torus T acts on $X^\gamma, \gamma \in \Gamma$, the action of Γ on X induces the action of $X_T = (X \times U)/T$ via the action on the first factor and the fixed point set of $\gamma \in \Gamma$ is X^γ_T. Hence $\mathcal{E}LL^T_{\mathrm{orb}}(X, \Gamma) = \mathcal{E}LL_{\mathrm{orb}}(X_T, \Gamma)$. Now the theorem follows from theorem 5.3 in [9] applied to the action of Γ on X_T. □

4.4. Push forward of contributions of components of a fixed point set

The localization map (20) allows us to associate a fixed component F of an action of a torus with an invariant, constructed using the contribution of F in the equivariant elliptic class of X. In the case when X is a smooth projective variety, the sum over all fixed components of these contributions evaluated by the corresponding fundamental classes of the components coincides with the equivariant elliptic genus of X (see [2]). In the case when X is only *quasi*-projective, but a component F is compact, the corresponding contribution is well defined and though by itself it does not have a geometric interpretation, this contribution does play a key role in the definitions of the next section. Here we shall describe the push forward property of the contributions of compact components and its generalization to the orbifold case.

Definition 4.6 (Local contribution of the component of a fixed point set: smooth case). Let X be a smooth quasi-projective variety, T as above, and let E denote a normal crossing divisor with T-invariant irreducible components. Let F be a component of the fixed point set. Assume that F is compact and let $i_F : F \to X$ denote its embedding. Let $c^T_{\mathrm{codim}F}(N_F)$ be the equivariant Chern class of the normal bundle of F in X. Then the local contribution of F in the equivariant elliptic genus of the pair (X, E) is the class[12]:

$$\mathcal{E}LL^T_F(X, E) = \frac{i^*_F \mathcal{E}LL(X, E)}{c^T_{\mathrm{codim}F}(N_F)} \in A_*(F)\{\{q, y\}\} \otimes \mathcal{Q}. \tag{25}$$

Theorem 4.7 (Push forward for the local contribution of the equivariant elliptic genus). *Let X be a smooth quasi-projective variety with the action of a torus T and let $F \subset X$ be a component of the fixed point set which is compact. Denote by $\phi : X' \to X$ the T-equivariant blow up with the T-invariant center $Z \subset F$ and let $\mathcal{F}'_F = \bigcup_{F' \in \mathrm{Irr}\mathcal{F}'} F'$ be the union of the submanifolds F' from the set $\mathrm{Irr}(\mathcal{F}')$ of irreducible components of the fixed point set $\mathcal{F}'$ of the action of T on X' mapped by ϕ onto F. Let E be a T-invariant normal crossing divisor, all components of which are transversal to Z, and let E' be the divisor on X' such that $\phi^*(K_X + E) = K_{X'} + E'$. Then*

[11] Here we consider the full elliptic class, i.e. for each commuting pair g,h one takes the cap product of the class obtained by the expansion of the θ-functions with the fundamental class $[X^{g,h}]$. This cap product is an element of the equivariant Chow ring of $X^{g,h}$. The push forward of this cap product to the Chow ring of a point gives the equivariant orbifold elliptic genus and is an element in the ring of the formal power series in characters of T.

[12] The ring in this formula can be taken to be $A_*(F, QJac) \otimes \mathcal{Q}$.

$$\phi_* \sum_{F'\in \mathrm{Irr}(\mathcal{F}')} \frac{i^*_{F'}\mathcal{E}LL^T(X',E')}{c^T_{\mathrm{Codim}F'\subset X'}(N_{F/X'})} = \frac{i^*_F\mathcal{E}LL^T(X,E)}{c^T_{\mathrm{Codim}F\subset X}(N_{F/X})}. \tag{26}$$

Proof. Let $\bar{X}$ be a compactification of X and $\bar{X}'$ be the blow up of $\bar{X}$ at $Z\subset X\subset \bar{X}$. Let $\mathcal{F}'=\bigcup_{F'\in \mathrm{Irr}(\mathcal{F}')}F'$ (respectively $\mathcal{F}=\bigcup_{F\in \mathrm{Irr}\mathcal{F}}F$) be the submanifold of $\bar{X}'$ of fixed points of action of T on $\bar{X}'$ (respectively $\bar{X}$) and $i_{\mathcal{F}'}:\mathcal{F}'\to\bar{X}'$ (respectively $i_{\mathcal{F}}:\mathcal{F}\to\bar{X}$) be their embeddings. The push forward formula of theorem 4.3 can be rewritten as:

$$i_{\mathcal{F}*}(i_{\mathcal{F}*}^{-1}\phi_* i_{\mathcal{F}'*})i^{-1}_{\mathcal{F}'*}\mathcal{E}LL^T(\bar{X}',E') = \mathcal{E}LL^T(X,E). \tag{27}$$

Now using the description of the inverse of i_* given in (21) and $(\phi|_{\mathcal{F}'})_* = i_{\mathcal{F}*}^{-1}\phi_* i_{\mathcal{F}'*}$ we obtain

$$\phi_* \frac{i^*_{\mathcal{F}'}\mathcal{E}LL^T(\bar{X}',E')}{c^T_{\mathrm{top}}(N_{\mathcal{F}'/\bar{X}'})} = \frac{i^*_{\mathcal{F}}\mathcal{E}LL^T(X,E)}{c^T_{\mathrm{top}}(N_{\mathcal{F}/\bar{X}})}. \tag{28}$$

The fixed point set $\mathcal{F}'$ (respectively $\mathcal{F}$) is a disjoint union of smooth irreducible components and hence $A^*(\mathcal{F}')=\oplus_{F'\in \mathrm{Irr}(\mathcal{F}')}A^*(F')$ (a similar direct sum decomposition for $\mathcal{F}$) where the summation is over the set $\mathrm{Irr}(\mathcal{F}')$ of irreducible components of $\mathcal{F}'$ (respectively $\mathcal{F}$). The split is given by the projections $i^*_{F'}:A^*(\mathcal{F}')\to A^*(F')$ (respectively $i^*_F:A^*(\mathcal{F})\to A^*(F)$) where $i_{F'}:F'\to\mathcal{F}'$ is the embedding of an irreducible component into the disjoint union (and the same for F). The map $\phi|_{\mathcal{F}'*}$ respects the above direct sum decomposition with $\phi|_{\mathcal{F}'*}^{-1}(A^*(F))=\oplus_{F'\in \mathrm{Irr}(\mathcal{F}')}A^*(F')$. This implies (26). □

4.5. *Contributions of components of a fixed point set into an orbifold elliptic genus*

Let X be a smooth quasi-projective variety, let T be a torus acting on X effectively and let Γ be a finite group acting upon X (all actions are via biholomorphic automorphisms). We shall assume that the action of Γ commutes with the action of T, i.e. for all $t\in T,\gamma\in\Gamma$ and any $x\in X$ one has $\gamma t\cdot x = t\cdot\gamma x, \gamma, t\in Aut(X)$. This implies that Γ leaves invariant the fixed point set X^T of the torus T, each fixed point set $X^g, g\in\Gamma$ is T-invariant and that T acts on the quotient X/Γ. We denote by T^{eff} the quotient of T, which acts *effectively* on X/Γ.

If F is a connected component of X^T and F^γ is a component of the fixed point set of an element $\gamma\in\Gamma$ acting upon F, then the restriction of the cotangent (or tangent) bundle Ω^1_X of X on F^γ has the canonical structure of an equivariant T-bundle. If V is an eigenbundle of this T-action on $\Omega^1_X|_{F^\gamma}$, then since we assume that actions of Γ and T commute, V is invariant under the action of γ as well.

If $rkV=1$, then as in section 3.2, for $\gamma\in\Gamma$ we let $\lambda(\gamma)$ denote the logarithm $\frac{1}{2\pi\sqrt{-1}}\log\in[0,1)$ of the value on γ of the character of the action on V of the subgroup $\langle\gamma\rangle$ of Γ generated by γ. We assign the subscript λ to such a line bundle V, put $x_\lambda=c^T_1(V_\lambda)\in A^2_T(F^\gamma)$ and count the class x_λ with the multiplicity equal to the multiplicity of the character $\gamma\to\exp(2\pi\sqrt{-1}\lambda(\gamma))$ in the bundle $\Omega^1_X|_{F^\gamma}$. A similar collection of equivariant Chern classes arises from the normal bundles to the fixed point sets $F^g\cap F^h$ of the pairs g,h commuting the elements in Γ.

Definition 4.8. Let $F\subset X$ be a connected *compact* component of the fixed point set of an action of T and for a commuting pair $g,h\in\Gamma$, let $F^{g,h}$ denote the submanifold of F consisting of the points fixed by both g and h. We associate a connected component of $F^{g,h}$ and a rank one T-eigenbundle V of $\Omega^1_X|_{F^{g,h}}$ with the characteristic class in the ring $A^*_T(X,\mathbb{C})[[q,y]]$ given by:

$$\Phi^T_{F^{g,h}}(x^T,g,h,z,\tau,\Gamma)=\frac{\theta(\frac{x^T}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\frac{x^T}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h))}\mathrm{e}^{2\pi\mathrm{i}z\lambda(h)} \tag{29}$$

where x^T is an equivariant Chern class of V.

Below we also denote by $Cong(\Gamma)$ the set of conjugacy classes of Γ, $C(g), g\in\Gamma$ will denote the centralizer of g, Λ is the set of (g,h)-eigenbundles of the tangent bundle to X restricted to $F^{g,h}$ and $\Lambda_{F^{g,h}}$ will be the collection of (g,h)-eigenbundles of $N_{F^{g,h}\subset F}$, such that $\lambda(g)=\lambda(h)=0$.

Definition 4.9. The contribution of $F\in X^T$ to the T-equivariant orbifold elliptic genus of (X,Γ) is the sum:

$$\mathcal{E}LL^{T^{\mathrm{eff}}}_F(X,\Gamma,u,z,\tau)=\sum_{\{g\}\in\mathrm{Cong}(\Gamma)}\frac{1}{|C(g)|}\sum_{h\in C(g)}(\prod_{\lambda\in\Lambda_{F^{g,h}}}x_\lambda)\prod_{\lambda\in\Lambda}\Phi^{T^{\mathrm{eff}}}_{F^{g,h}}(x_\lambda,g,h,z,\tau,\Gamma)[F^{g,h}] \tag{30}$$

where $N_{F^{g,h}\subset F}$ is the normal bundle for $F^{g,h}$ in F and all equivariant Chern classes are expressed in terms of the characters of T^{eff}.

The motivation of this definition is the following: the Orbifold elliptic genus (12) is a sum over the pairs of commuting elements in Γ of classes in the Chow ring (which are combinations of the Chern classes x_λ of bundles V_λ in (10)) evaluated on the fundamental class of $X^{g,h}$ (see proof of theorem 4.3 in [8]). In the case when X is projective, the localization formula (see (21)) applied to the equivariant version of the orbifold elliptic genus replaces each summand in (12) by the sum over the components F of X^T of the pullbacks to the $F^{g,h}=F\cap X^{g,h}$ classes (12) divided by the equivariant top Chern class of the normal bundle to $F^{g,h}$ in $X^{g,h}$. Definition 4.9 is the sum over (g,h) of the contribution from one individual component F.

Example 4.10 (Trivial sector of the contribution described in definition 4.9 for χ_y-genus). The specialization to the case $q=0$ of the term corresponding to the pair $g=h=1$ (i.e. the trivial sector) gives the following local contribution of the component F of the fixed point set of action of T on X to the Γ-orbifold χ_y-genus:

$$\chi_y(X,\Gamma,g=h=1)^{T^{\mathrm{eff}}}_F=\prod\nolimits_\lambda(y^{-\frac{1}{2}}x_{\lambda(T)}\frac{1-y\mathrm{e}^{-x_{\lambda(T)}}}{1-\mathrm{e}^{-x_{\lambda(T)}}})\cdot\prod\nolimits_\lambda(y^{-\frac{1}{2}}\frac{1-y\mathrm{e}^{-x^n_{\lambda(N)}}}{1-\mathrm{e}^{-x^n_{\lambda(N)}}}) \tag{31}$$

where $x_{\lambda(T)}$ are the Chern roots of the tangent bundle to F (appearing in the first product) and $x^n_{\lambda(N)}$ are the equivariant Chern roots of the normal bundle to F (contributing to the second factor in (31)). Indeed, in the sector $g=h=1$ in (30), we have only one term, which is a specialization of class Φ given in the definition 4.8.

The contributions in the orbifold elliptic genus corresponding to the compact components of the fixed point set described in definition 4.9 satisfy the following McKay correspondence, localized at F, proof of which can be obtained in the same way as proof of the theorem 4.7. A more general case, providing a local equivariant version for pairs as in [9] can be obtained similarly.

Theorem 4.11. *Let X be a smooth quasi-projective variety, T be a torus and Γ be a finite group both acting on X via biholomorphic automorphisms so that their actions commute, i.e. $\gamma \cdot tv = t \cdot \gamma v, \gamma \in \Gamma, t \in T, v \in X$. Let $\phi : \tilde{X} \to X/\Gamma$ be a crepant resolution of singularities of the quotient X/Γ (if it exists). As above, denote by T^{eff} the quotient of T by the finite group which acts effectively on X/Γ. Let $F \subset X/\Gamma$ be a component of the fixed point set of T^{eff} and $\mathcal{F}'$ be the collection of components of the fixed point set of T such that $\phi(F') \subset F, F' \in \mathcal{F}'$. Then*

$$\sum_{F' \in \mathcal{F}'} \mathcal{E}LL_{F'}^{T^{\mathrm{eff}}}(\tilde{X}) = \mathcal{E}LL_{F}^{T^{\mathrm{eff}}}(X, \Gamma). \tag{32}$$

In the next section, we consider explicit examples of the calculations of the contributions of fixed components of $\mathbb{C}^*$-actions on the GIT quotients by the actions of tori on bundles over quasi-projective varieties. They will provide ample illustration of the theorem 4.11.

5. Elliptic genus of phases

This section discusses applications of the local contributions of compact components of the fixed point sets introduced in the previous section in the special case when the action of $T = \mathbb{C}^*$ takes place on a GIT quotient of the total space of a vector bundle by an action of the reductive group. This action of T is canonical in the sense that it is induced from the action of T on the total space of the vector bundle by dilations $v \to t \cdot v, t \in \mathbb{C}^*$. This is an extension of the framework of examples considered by Witten in [56]. Following this work, in [40] we called our GIT quotient *phases* as well. We also attach an elliptic genus to such a framework and describe its orbifoldization when additional symmetries are present. We show that this extends the well-known elliptic genera of the Landau–Ginzburg and σ-models.

5.1. Phases

We will start with a very special example of a phase considered by Witten ([56]), in which we calculate the contribution of the component of the fixed point set, not in the elliptic genus but rather in the χ_y-genus (which is the limit $q \to 0$ of the elliptic genus). The 'advantage' of the χ_y-genus of course is that it is a Laurent polynomial, rather than a more general holomorphic function. In this example, we work with the χ_y-genus directly, i.e. we perform the localization of the χ_y-genus rather than the elliptic genus. Already, in the case of the Landau–Ginzburg phase, this calculation results in the Arnold–Steenbrink spectrum of weighted homogeneous singularity, providing interpretation of the latter using equivariant cohomology.

Example 5.1. Let $w_1, ..., w_n, D$ be a collection of positive integers. Consider the $G = \mathbb{C}^*$-action on $\mathbb{C} \times \mathbb{C}^n$ given by

$$\lambda(s, z_1, ..., z_n) = (\lambda^{-D} s, \lambda^{w_1} z_1, ..., \lambda^{w_n} z_n). \tag{33}$$

The quotient of the subset in $\mathbb{C} \times \mathbb{C}^n$ given by $s \neq 0$ is the orbifold W/μ_D, where W is a vector space, $\dim_{\mathbb{C}} W = n$ and μ_D is the group of roots of the unity of degree D acting via multiplication on W. The group $T^{\mathrm{eff}} = \mathbb{C}^*$ acts on $\mathbb{C} \times \mathbb{C}^n$ via $t(s, z_1, ..., z_n) \to (ts, z_1, ..., z_n)$ and this action induces the effective action of T^{eff} on W/μ_D. The effective $\mathbb{C}^*$-action on W, which induces this action of T^{eff} on W/μ_D, is multiplication by $r \in \mathbb{C}^*$ where $r^{-D} = t$ (r and $r\exp(\frac{2\pi\sqrt{-1}l}{D}), l \in \mathbb{Z}$ induce the same automorphisms of W/μ_D), i.e. W is acted upon by

$T = \mathbb{C}^*$, with the D-fold cover of the group T^{eff} acting on W/μ_D. In particular infinitesimal characters of the normal bundle at the fixed point $\mathcal{O}$ of the action of T on W (i.e. the origin) in terms of the characters of T^{eff} are $x_i = -\frac{w_i u}{D}$, where u is the infinitesimal character of T.

It follows from (31) that the trivial sector of the local contribution of $\mathcal{O}$ to the orbifold χ_y-genus is given by:

$$\prod_i y^{-\frac{1}{2}} \left(\frac{w_i u}{D} \frac{1 - y e^{-\frac{w_i u}{D}}}{1 - e^{\frac{-w_i u}{D}}} \right) \cdot \frac{1}{\frac{w_i u}{D}}. \tag{34}$$

For a special value of u given by $u = 2\pi\sqrt{-1}z$ one obtains:

$$\prod_i y^{\frac{-1}{2}} \frac{1 - y^{(1 - \frac{w_i}{D})}}{1 - y^{\frac{-w_i}{D}}} = \prod_i y^{\frac{-1}{2}} \frac{y^{\frac{w_i}{D}} - y}{y^{\frac{w_i}{D}} - 1} \tag{35}$$

which coincides with the generating function of the spectrum, as calculated in (see [47]).

Now we consider the general case for which the example 5.1 is an illustration.

Definition 5.2 (See [40]). Let $\mathcal{E}$ be the total space of a vector bundle E on a smooth quasi--projective manifold X. Let G be a reductive algebraic group acting by the biholomorphic authomorphism on $\mathcal{E}$. Let κ be a linearization of this G-action satisfying the conditions of proposition 3.1 in [40][13]. The phase of the G-action on $\mathcal{E}$ corresponding to the linearization κ is the GIT quotient $\mathcal{E}//_\kappa G = \mathcal{E}^{ss}/G$ endowed with the $\mathbb{C}^*$-action induced by the $\mathbb{C}^*$-action given by dilations $t(v) = t \cdot v, t \in \mathbb{C}^*, v \in \mathcal{E}$.

A phase is called Landau–Ginzburg if this GIT quotient is an orbifold that is biholomorphic to a quotient of $\mathbb{C}^n$ by a finite subgroup of $GL_n(\mathbb{C})$.

A phase is called a σ-model (respectively Calabi–Yau) if this GIT quotient is biholomorphic to the total space of a vector bundle (respectively the canonical bundle) over a compact orbifold.

The change of linearization κ of the G-action on $\mathcal{E}$ may result in a birationally equivalent GIT-quotient. More specifically, if $NS^G(\mathcal{E})$ denotes the equivariant Neron–Severi group (in the case when $\mathcal{E}$ is an affine space this is just the group Char$G$ of characters of G), then there is a partition of $NS^G(\mathcal{E}) \otimes \mathbb{Q}$ into a union of cones, such that GIT-quotients are biregular for linearizations within a cone and $\mathcal{E}//_\kappa G$ acquires a change when κ belongs to the boundary of a cone or is moving into the adjacent one. For a general discussion of the changes of GIT-quotients we refer to [49] or [16] and to [40] for the particular case of the total spaces of bundles as in the definition 5.2.

GIT-quotients are often singular but we will be interested in the cases when they are biholomorphic to the global quotients of a smooth manifold which we call the *uniformization* of a global quotient.

Definition 5.3. A smooth quasi-projective variety $\bar{X}$ together with the action of a finite group Γ is called the uniformization of a phase $\mathcal{E}//_\kappa G$ if

1. there exists a biholomorphic isomorphism $\mathcal{E}//_\kappa G \to \bar{X}/\Gamma$
2. there is an action of the 1-dimensional complex torus T on X, and a finite degree covering the map $\pi : T \to T^{\text{eff}}$ of the 1-dimensional torus T^{eff} acting on $\mathcal{E}//_\kappa G$ via dilations (see definition 5.2), such that the quotient map $\phi : \bar{X} \to \bar{X}/\Gamma = \mathcal{E}//_\kappa G$ is equivariant, i.e. $\phi(t \cdot x) = \pi(t) \cdot \phi(x), t \in T$.

[13] Which implies that the $\mathbb{C}^*$-action by dilations is well defined on the GIT quotient.

The following is an illustration of the definitions 5.2 and 5.3, with an example borrowed from [56].

Example 5.4. The quotient in example 5.1 is a special case of the quotients considered in definition 5.2 with $X = \mathbb{C}^n$, $\mathcal{E} = \mathbb{C}^{n+1}$ being the total space of the trivial line bundle and $G = \mathbb{C}^*$ acting on $\mathcal{E}$ via (33). In this case $Char(\mathbb{C}^*) \otimes \mathbb{Q} = \mathbb{Q}$, there are two cones and for a pair of linearizations κ_1, κ_2 from distinct cones, the corresponding semi-stable loci are:

$$(\mathbb{C}^{n+1})^{ss}_{\kappa_1} = \mathbb{C} \times (\mathbb{C}^n - 0) \subset \mathbb{C}^{n+1} \quad (\mathbb{C}^{n+1})^{ss}_{\kappa_2} = (\mathbb{C} - 0) \times \mathbb{C}^n \subset \mathbb{C}^{n+1}. \tag{36}$$

In the simplest case, when $w_i = 1$, $D = n$, the corresponding GIT quotients are respectively the total space $[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]$ of the canonical bundle over $\mathbb{P}^{n-1}$ and the quotient W/μ_n, $W = \mathbb{C}^n$ by the group of roots of unity of degree n acting diagonally. As was mentioned in the discussion of example 5.1, the dilations $t \cdot (s, z_1, ..., z_n) = (ts, z_1, ..., z_n)$ induce on $\mathbb{C}^n/\mu_n$ the action $t \cdot [(z_1, ..., z_n) \bmod \mu_n] = t(1, z_1, .., z_n) \bmod \mathbb{C}^* = (t, z_1, ..., z_n) \bmod \mathbb{C}^* = (t^{-\frac{1}{n}} z_1, ..., t^{-\frac{1}{n}} z_n) \bmod \mu_n$. This action is effective on the quotient W/μ_n. Denote by $\pi : \lambda \to t = \lambda^{-n}$ the (cyclic) covering map of one-dimensional tori $\mathbb{C}^*_\lambda \to \mathbb{C}^*_t$ and let $\phi : \mathbb{C}^n \to \mathbb{C}^n/\mu_n$ be the quotient map. Assume that $\mathbb{C}^*_\lambda$ is acting on $\mathbb{C}^n$ via multiplication of the coordinates by λ, and $\mathbb{C}^*_t$ acts on $\mathbb{C}^n/\mu_n$ as above. Then $\phi(\lambda v) = t \cdot \phi(v)$, and therefore we have a uniformization in the sense of the definition 5.3. Hence we have an LG phase. The quotient, which is the total space $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, is the σ-model (in fact the CY) phase. Here the GIT quotient is smooth, dilations on $\mathbb{C} \times \mathbb{C}^n$ induce on $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ the multiplication by elements of $\mathbb{C}^*$, which is an effective action and does not require uniformization.

5.2. Elliptic genus of a phase

Next we shall define the elliptic genus of a phase for which the fixed point set of $\mathbb{C}^*$-action induced by dilations has a compact component.

Definition 5.5 (Elliptic genus of a phase). Let $X, G, \mathcal{E}, \kappa$ be as in definition 5.2. Assume that $\mathcal{E}//_\kappa G$ admits uniformization $\widetilde{\mathcal{E}//_\kappa G}$, i.e. $\widetilde{\mathcal{E}//_\kappa G}/\Gamma = \mathcal{E}//_\kappa G$ for an action of a finite group Γ and that one has the action of $T = \mathbb{C}^*$ on $\widetilde{\mathcal{E}//_\kappa G}$ such that the quotient map $\widetilde{\mathcal{E}//_\kappa G} \to \mathcal{E}//_\kappa G$ is equivariant for the $\mathbb{C}^*$-action on $\mathcal{E}//_\kappa G$ induced by dilations on $\mathcal{E}$. Let $F \subset \widetilde{\mathcal{E}//_\kappa G}$ be a compact component of a fixed point set of T-action on $\widetilde{\mathcal{E}//_\kappa G}$. Consider the local contribution of the component F to the T-equivariant orbifold elliptic genus

$$Ell^{T^{\rm eff}}_{\rm orb}(\widetilde{\mathcal{E}//_\kappa G}, \Gamma, u, z, \tau) \tag{37}$$

given by (30) in definition 4.9, where u is the infinitesimal character of the action of the maximal, effectively acting quotient $T^{\rm eff}$. Then, the elliptic genus of the phase $(X, G, \mathcal{E}, \kappa)$ relative to the component F, denoted as $Ell(\mathcal{E}//_\kappa G, F, z, \tau)$, is defined *as the restriction of the local contribution (37) on the diagonal* $u = z$ of $\mathbb{C} \times \mathbb{C} \times \mathcal{H}$:

$$Ell(\mathcal{E}_\kappa//G, F, z, \tau) = Ell^{T^{\rm eff}}_{\rm orb}(\widetilde{\mathcal{E}//_\kappa G}, \Gamma, F, z, z, \tau). \tag{38}$$

More generally, the same definition can be used in cases when $\widetilde{\mathcal{E}//_\kappa G}$ has Kawamata log-terminal singularities and when $Ell(\widetilde{\mathcal{E}//_\kappa G}, \Gamma)$ is well defined as the orbifold elliptic genus of a pair obtained via a resolution of singularities and taking into account the divisor determined by the discrepancies of the resolution (see [9]).

In the next theorem, we shall describe a class of phase transitions in which one can apply equivariant McKay correspondence to obtain the invariance of elliptic genus in such transitions.

Theorem 5.6 (Invariance of elliptic genus in phase transitions). *Let $\mathcal{E}//_{\kappa_1}G = X_1 = \bar{X}_1/\Gamma$, $\mathcal{E}//_{\kappa_2}G = X_2 = \bar{X}_2/\Gamma, \tilde{X}_1, \tilde{X}_2, \Gamma$ be as in 5.3. Assume that $\psi : X_1 \to X_2$ is a K-equivalence, i.e. $\psi^*(K_{X_2}) = K_{X_1}$. Then*

$$\sum_{P_i} Ell(\mathcal{L}//_{\kappa_1}, F_i) = Ell(\mathcal{L}//_{\kappa_2}, F) \tag{39}$$

where F_i is a collection of fixed point sets which ψ takes into F.

5.3. Quotients of phases by the action of a finite group

The constructions of mirror symmetry in the toric or weighted homogeneous case (see [3, 4]) suggest considering the orbifoldization of phases with respect to finite groups. Even the very first construction of the mirror symmetric of the Calabi–Yau quintic in $\mathbb{P}^4$ (see [11]) was obtained via orbifoldization. The orbifoldization of the elliptic genus of Calabi–Yau and Landau–Ginzburg models was proposed in [4, 34]. Here we discuss the orbifoldization of arbitrary phases including hybrid ones.

Let X be a quasi-projective manifold with an action for a reductive group G and let Γ be a finite subgroup of the group of biregular automorphisms of X which normalizes G, i.e. for any $\gamma \in \Gamma$, $g \in G$ one has $\gamma g \gamma^{-1} \in G$. We say that Γ normalizes a linearization κ of the G-action on X, if the action of Γ on X lifts to the action on the total space of the ample line bundle L_κ underlying κ so that this lift normalizes the action of G on the total space of L_κ. This assumption implies that Γ acts on the semi-stable locus

$$X^{ss} = \{x \in X | \exists s \in \Gamma(X, L_\kappa^{\otimes m})^G, s(x) \neq 0\}. \tag{40}$$

Here the action of either G or Γ on $\Gamma(X, L_\kappa^{\otimes m})$ is given by $(gs)(x) = gs(g^{-1}x)$ (g is an element of either G or Γ). Indeed, $(\gamma \cdot s)(\gamma(x)) = \gamma s(x) \neq 0$ if $s(x) \neq 0$. The action of Γ on X_κ^{ss} in turn defines its action on $X//_\kappa G$.

First we shall consider the orbifoldization of the elliptic genus (i.e. defining the elliptic genus of the corresponding orbifold) in the case when the GIT quotient $\mathcal{E}//_\kappa G$ is smooth.

Definition 5.7 (Orbifoldization of smooth phases). Let $X, \mathcal{E}, G, \kappa$ be as in definition 5.2, Γ be a finite group of automorphisms of bundle $\mathcal{E} \to X$ normalizing linearization κ and $\mathcal{E}//_\kappa G$ be the phase corresponding to $X, \mathcal{E}, \kappa$ endowed with the action of Γ induced from the action on the G-semistable locus in $\mathcal{E}$ corresponding to κ. If $\mathcal{E}//_\kappa G$ is smooth and F is a compact component, the fixed point set of the $\mathbb{C}^*$ action on $\mathcal{E}//_\kappa G$ induced by dilations $\lambda(v) = \lambda \cdot v, v \in \mathcal{E}, \lambda \in \mathbb{C}^*$, then the Γ-orbifoldized elliptic genus of this phase corresponding to F is the contribution (4.9) of component F to the $\mathbb{C}^*$-equivariant Γ-orbifold elliptic genus of $\mathcal{E}//_\kappa G$.

More generally, in the case when $(\mathcal{E}//_\kappa G)$ is an orbifold, assume further that it is a global quotient admitting (Y, Γ, T) as uniformization in the sense of definition 5.3 and that there is a finite group Δ of automorphisms of Y, containing Γ as a normal subgroup[14], with the action

[14] In particular Δ acts on the quotient Y/Γ.

of Δ commuting with the action of T. We want to describe the Δ/Γ-orbifold elliptic genus attached to $\mathcal{E}//_\kappa G = Y/\Gamma$ for the action induced by the action of Δ.

Let F_Y be the preimage in the uniformization of a component of the fixed point set $F \subset (\mathcal{E}//_\kappa G)$. Then the Δ-orbifoldized contribution of F is the sum over all connected components Q in F_Y of Δ-orbifoldized contributions of components Q to the equivariant elliptic genus of Y, as described in definition 4.9. More precisely, let $Q^{g,h}$ be the fixed point set of the pair of commuting elements $g, h \in \Delta$ acting on Q, $V_\lambda \subset T_Y|_{Q^{g,h}}$ be the eigenbundle of the subgroup $<g,h>$ of Δ generated by (g, h), and Λ is the full set of such eigenbundles in $T_Y|_{Q^{g,h}}$, $\Lambda_{g,h} = \{\lambda\} \subset \Lambda$ is the set of eigenbundles in the normal bundle to $Q^{g,h}$ in Q such that $\lambda(g) = \lambda(h) = 0$. Since we assume that the actions of Δ and T commute, the bundles V_λ are the eigenbundles of T as well. Let x_λ^T be T-equivariant Chern classes of V_λ written in terms of the characters of T^{eff}, which is the quotient of T acting effectively on the orbifold $\mathcal{E}//_\kappa G = Y/\Gamma$.

Definition 5.8. The Δ-orbifoldized contribution of component Q to the equivariant elliptic genus of $\mathcal{E}_\kappa//G$ is given as follows:

$$\frac{1}{|\Delta|}\sum_{gh=hg, g,h\in\Delta}\Big(\prod_{\lambda\in\Lambda_{Q^{g,h}}} x_\lambda\Big)\prod_{\lambda\in\Lambda_Q}\Phi^{T^{\mathrm{eff}}}_{Q^{g,h}}(x_\lambda, g, h, z, \tau, \Delta)[Q^{g,h}] \tag{41}$$

where

$$\Phi^{T^{\mathrm{eff}}}_{Q^{g,h}}(x^T_\lambda, g, h, z, \tau, \Delta) = \frac{\theta(\frac{x^T_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x^T_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))}\mathrm{e}^{2\pi \mathrm{i} z\lambda(h)}.$$

The next and final section contains examples showing how these definitions yield the invariants of the Calabi–Yau and Landau–Ginzburg models, which have already appeared in the literature, as well as the explicit examples of some hybrid models.

6. The calculations of the elliptic genera of phases and their specializations

6.1. Elliptic case: the weighted projective spaces and LG models

The following is a continuation of the examples 5.1 and 5.4 and gives an explicit form of the elliptic genera of corresponding phases and their specializations. We shall start with the case of the GIT quotient from example 5.4, i.e. example 5.1 with $w_i = 1, D = n$.

Proposition 6.1. *Consider the $\mathbb{C}^*$-action on $\mathbb{C} \times \mathbb{C}^n$ given by:*

$$\lambda(s, z_1, ..., z_n) = (\lambda^{-n}s, \lambda z_1, ..., \lambda z_n).$$

There are two GIT quotients corresponding to the linearizations $\psi(\lambda) = \lambda^r$ with $r > 0$ (called the σ-model phase) biholomorphic to the total space $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ of the canonical bundle of $\mathbb{P}^{n-1}$ and $r < 0$ (called the Landau–Ginzburg phase) biholomorphic to $\mathbb{C}^n/\mu_n$.

1. *The trivial sector of the elliptic genus of the Landau–Ginzburg phase is given by*

$$(-1)^n\Big(\frac{\theta(z(1-\frac{1}{n}))}{\theta(\frac{z}{n})}\Big)^n. \tag{42}$$

2. *The elliptic genus of the Landau–Ginzburg phase is given by:*

$$\frac{1}{n}\sum_{0\leqslant a,b<n}\left(-\frac{\theta((1-\frac{1}{n})z+\frac{(a-b\tau)}{n})}{\theta(\frac{z}{n}+\frac{(a-b\tau)}{n})}\right)^n e^{2\pi \mathrm{i}bz}. \tag{43}$$

3. *The elliptic genus of the σ-model phase is given by:*

$$\left(x\frac{\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})}\right)^{n-1}\left(\frac{\theta(\frac{nx}{2\pi\mathrm{i}})}{\theta(\frac{nx}{2\pi\mathrm{i}}-z)}\right)[\mathbb{P}^{n-1}] \tag{44}$$

and coincides with the elliptic genus of the smooth hypersurface of degree n in $\mathbb{P}^{n-1}$.

4. *(LG-CY correspondence) The elliptic genera (44) and (43) of σ and the LG models respectively coincide.*

Proof. The calculation of GIT quotients was already made in example 5.4. The uniformization is given by $W\to W/\mu_n$ with the $\mathbb{C}^*$-action given by dilations of W. The normal bundle of the fixed point, i.e. the origin $\mathcal{O}$ is the direct sum of lines with the equivariant Chern class being $\frac{u}{n}$ where u is the infinitesimal character of $\mathbb{C}^*$ acting effectively on W/μ_n). Hence the contribution of the origin $\mathcal{E}LL_{\mathcal{O}}^{\mathbb{C}^*}(W,\mu_n)$ to the equivariant elliptic genus is given by

$$\frac{1}{n}\sum_{0\leqslant a,b<n}\left(-\frac{\theta(\frac{1}{2\pi\mathrm{i}}\frac{u}{n}-z+\frac{(a-b\tau)}{n})}{\theta(\frac{1}{2\pi\mathrm{i}}\frac{u}{n}+\frac{(a-b\tau)}{n})}\right)^n e^{2\pi\mathrm{i}bz}$$

which for $u=2\pi\mathrm{i}z$ gives (43). For $a=b=1$, one obtains (42).

In the case of the σ-model, the $\mathbb{C}^*$-action is the action via the dilations on the fibers of the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. The tangent bundle of $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ restricted to the fixed point set, i.e. the zero section, obtains contributions from the tangent bundle to $\mathbb{P}^{n-1}$ and from $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. The equivariant Chern polynomial of the tangent bundle to $\mathbb{P}^{n-1}$ is $(1+x)^n$ and the equivariant Chern class of $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ is $\frac{-nx+u}{2\pi\mathrm{i}}$. Hence the contribution of the fixed point set is:

$$\left(x\frac{\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})}\right)^{n-1}\left(\frac{\theta(-\frac{nx}{2\pi\mathrm{i}}+\frac{u}{2\pi\mathrm{i}}-z)}{\theta(-\frac{nx}{2\pi\mathrm{i}}+\frac{u}{2\pi\mathrm{i}})}\right)[\mathbb{P}^{n-1}].$$

Since $\theta(z)$ is an odd function, for $u=2\pi\mathrm{i}z$ we obtain (47). Since the Chern roots of a hypersurface V^n_{n-2} of degree n in $\mathbb{P}^{n-1}$ are found from relation $c(V_{n-2})=\frac{(+x)^n}{(1+nx)}|_{V^n_{n-2}}$, it follows that the elliptic genus of the hypersurface is given by

$$\left(x\frac{\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})}\right)^{n-1}\left(\frac{\theta(\frac{nx}{2\pi\mathrm{i}})}{nx\cdot\theta(\frac{nx}{2\pi\mathrm{i}}-z)}\right)[V^n_{n-2}].$$

The latter coincides with (47) since $[V^n_{n-2}]=nx\cap[\mathbb{P}^{n-1}]$.

The LG/CY correspondence follows from the McKay correspondence since the contraction $[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]\to\mathbb{C}^n/\mu_n$ is a crepant morphism. □

In the case where the action in example 5.1 has arbitrary weights, we obtain the following:

Proposition 6.2. *Consider the $\mathbb{C}^*$-action on $\mathbb{C}\times\mathbb{C}^n$ with weights $w_1,...,w_n$ ($w_i\in\mathbb{Z}$ pairwise relatively prime) and degree $D\in\mathbb{Z}_{>0}$ given by (33):*

$$\lambda(s,z_1,....,z_n)=(\lambda^{-D}s,\lambda^{w_1}z_1,....,\lambda^{w_n}z_n).$$

There are two GIT quotients corresponding to the linearizations $\psi(\lambda) = \lambda^r$ with $r > 0$ (called the σ-model phase) and $r < 0$ (called the Landau–Ginzburg phase) respectively.

1. *The trivial sector of the elliptic genus of the Landau–Ginzburg phase is*

$$\prod_j \frac{\theta(\frac{w_j z}{D} - z)}{\theta(\frac{w_j z}{D})}. \tag{45}$$

2. *The elliptic genus of the Landau–Ginzburg phase is given by:*

$$\frac{1}{D} \sum_{0 \leqslant a,b < D} \prod_{i=1}^{i=n} \frac{\theta((\frac{w_i}{D} - 1)z + \frac{w_i(a-b\tau)}{D})}{\theta(\frac{zw_i}{D} + \frac{w_i(a-b\tau)}{D})} \mathrm{e}^{2\pi\sqrt{-1}\frac{bw_j z}{D}}. \tag{46}$$

3. *Let $\Gamma = \mu_{w_1} \times .... \times \mu_{w_n}$ be the product of a group of roots of unity acting coordinatewise on $\mathbb{P}^{n-1}$. Then with $x \in H^2(\mathbb{P}^{n-1}, \mathbb{Z})$ being the positive generator and with notations used in (12) the elliptic genus of the σ-model phase is given by*

$$\frac{1}{|\Gamma|} \sum_{gh=hg} (\prod_{\lambda(g)=\lambda(h)=0} x_\lambda) \prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i} z \lambda(h)} \frac{\theta(\frac{Dx}{2\pi \mathrm{i}})}{\theta(\frac{Dx}{2\pi \mathrm{i}} - z, \tau)} [(\mathbb{P}^{n-1})^{g,h}]. \tag{47}$$

4. *(LG-CY correspondence) If $\sum_1^n w_i = D$, then the elliptic genus of the LG model is equal to the orbifold elliptic genus of the hypersurface of degree D in the weighted projective space $\mathbb{P}(w_1, ..., w_n)$, i.e. the Γ-orbifoldized elliptic genus of the hypersurface of degree D in $\mathbb{P}^{n-1}$ is invariant under the action of the group Γ.*

Proof. The semistable loci corresponding to two linearizations of the $\mathbb{C}^*$ action (33) are $s \neq 0$ and $\sum_i |z_i^2| \neq 0$. The quotient of the first locus is the quotient of $\mathbb{C}^n$ by the action of μ_D and gives the Landau–Ginzburg phase. Parts 1 and 2 follow directly from definition 5.5 using the uniformization W as used in 5.1.

The quotient of the second locus has a projection onto $\mathbb{C}^n \setminus 0/\mathbb{C}^*$ with the action on $\mathbb{C}^n \setminus 0$ being the restriction of the action (33). Hence this GIT quotient can be identified with the orbifold bundle over a weighted projective space. Using its presentation as a quotient of the total space of $\mathcal{O}_{\mathbb{P}^n}(-D)$ by the action of $\mu_{w_1} \times ..... \times \mu_{w_n}$ we obtain a uniformization of this phase. The c_1^T of the normal bundle to the fixed point set in the uniformization is $-Dx + u$ where u is the infinitesimal character, and the claim follows from definition 5.5. The rest of the calculation is a direct generalization of those in the proposition 6.1. □

Remark 6.3. Though without a Calabi–Yau condition, the equality of the elliptic genus of the LG model and the σ-model fails, the McKay correspondence for the pairs (see [9]) still provides an expression for the elliptic genus of the LG model as the elliptic genus of a pair.

6.2. Specialization of the elliptic genus $q \to 0$

Proposition 6.2 has, as an immediate consequence, the following relation between the spectrum of the weighted homogeneous singularities and the χ_y genus of the corresponding hypersurfaces.

Proposition 6.4.

1. *(The trivial sector of the LG models) The specialization* $q \to 0$ *of the elliptic genus of the LG phase corresponding to the action (33) is given by*

$$\lim_{q\to 0}\mathcal{E}LL(LG) = y^{\frac{-n}{2}}\prod_{j=1}^{n}\frac{y^{\frac{w_j}{D}}-y}{y^{\frac{w_j}{D}}-1} \tag{48}$$

(where $y = \exp(2\pi \mathrm{i} z)$*).*

2. *(The relation between the trivial sector of the LG model and the spectrum) Let* $\{q_l\}, q_l \in \mathbb{Q}$ *be the Steenbrink spectrum of the isolated singularity of a weighted homogeneous polynomial* $s = f(z)$ *with weights* w_i *and degree D,*[15] *i.e.* q_l *is the collection of* μ*, where* μ *is the Milnor number of f, the rational number* q_l *such that* $\exp(2\pi \mathrm{i} q_l)$ *is an eigenvalue of the monodromy acting on the graded component* $Gr_F^p(H^{n-1}(X_{\infty,f}))$ *of the Hodge filtration of the limited mixed Hodge structure on the cohomology of the Milnor fiber of* $f = 0$ *(with the multiplicity of* q_l *being equal to the dimension of the eigenspace) and the integer part* $[q_l]$ *being equal to* $n-p-1$ *(respectively* $n-p$*) if* $\exp(2\pi \mathrm{i} q_l) \neq 1$ *(respectively* $\exp(2\pi \mathrm{i} q_l) = 1$*). Let*

$$\Xi(y) = y^{\frac{-n}{2}}\sum_{l=1}^{\mu} y^{q_l}. \tag{49}$$

Then

$$\lim_{q\to 0}\mathcal{E}LL(LG)(y) = (-1)^n \Xi(y). \tag{50}$$

3. *(The orbifoldized-*χ_y *genus of the LG model) The specialization of the elliptic genus of the Landau–Ginzburg model is given by:*

$$\frac{1}{D}y^{-\frac{n}{2}}\sum_{0\leqslant a<D}\Big(\prod\frac{y^{\frac{w_j}{D}}-y\omega_D^{-aw_j}}{y^{\frac{w_j}{D}}-\omega_D^{-aw_j}} + \sum_{1\leqslant b<D}(y^{\frac{b}{D}})^D\Big) \tag{51}$$

where $\omega_D = \mathrm{e}^{\frac{2\pi\sqrt{-1}}{D}}$.

4. *In the case* $w_j = 1$*,* $j = 1,\ldots,n, D = n$ *(i.e. the Calabi–Yau condition is satisfied) the specialization* $q = 0$ *has the form:*

$$\frac{1}{n}y^{-\frac{n}{2}}\sum_{k=0}^{n-1}\Big[\Big(\frac{1-y^{1-\frac{1}{n}}\mathrm{e}^{\frac{2\pi\sqrt{-1}k}{n}}}{1-y^{-\frac{1}{n}}\mathrm{e}^{\frac{2\pi\sqrt{-1}k}{n}}}\Big)^n + \sum_{b=1}^{n-1}(y^{\frac{b}{n}})^n\Big]. \tag{52}$$

Proof. The trivial sector of the χ_y-genus of the LG model was already derived directly in example 5.1. Now we shall obtain it as the $q \to 0$ limit of the trivial sector of the elliptic genus given in part 1 of the proposition 6.2. Part 1 of proposition 6.4 follows from:

$$\lim_{\tau\to i\infty}\frac{\theta(u(z)-z,\tau)}{\theta(u(z),\tau)}\mathrm{e}^{2\pi\sqrt{-1}cz} = \frac{\sin\pi(u(z)-z)}{\sin(\pi u(z))}\mathrm{e}^{2\pi\mathrm{i}\sqrt{-1}cz}$$

$$= y^{-\frac{1}{2}}\frac{1-y\mathrm{e}^{-2\pi\sqrt{-1}u(z,0)}}{1-\mathrm{e}^{-2\pi\sqrt{-1}u(z,0)}}y^c \tag{53}$$

[15] I.e. a polynomial $f(z_1,\ldots,z_n)$ such that $sf(z)$ is invariant for the action (33).

where $u(z)$ is a linear function of z and, as above, $y = e^{2\pi\sqrt{-1}z}$. (53) implies that the factor corresponding to $\frac{w_i}{D}$ in (45) has $y^{-\frac{1}{2}}\frac{y^{\frac{w_i}{D}}-y}{y^{\frac{w_i}{D}}-1}$ as the limit and (48) follows. Part 2, as was mentioned in 5.1, is a consequence of [47].

The specialization of a summand in (46) with $b \neq 0$ gives $y^{\frac{b}{D}}$, while each factor in the summand with $b = 0$ becomes $y^{-\frac{1}{2}}\frac{y^{\frac{w_i}{D}}-y\omega_D^{-aw_i}}{y^{\frac{w_i}{D}}-\omega_D^{-aw_i}}$. Applying (53) to (46) one obtains:

$$\frac{1}{D}y^{-\frac{n}{2}}\sum_{0\leqslant a<D}(\prod\frac{y^{\frac{w_i}{D}}-y\omega_D^{-aw_i}}{y^{\frac{w_i}{D}}-\omega_D^{-aw_i}}+\sum_{1\leqslant b<D}(y^{\frac{b}{D}})^D). \tag{54}$$

This implies 3 while 4 follows from it immediately. □

6.3. Specialization $q \to 0, y = 1$

Such a specialization leads to numerical invariants of the phases.

Corollary 6.5.

1. *The specialization $q = 0$, $y = 1$ of the untwisted section of the LG model is given by*

$$\mathcal{E}LL(LG)(q=0,y=1)=\prod\nolimits_j(1-\frac{D}{w_j}) \tag{55}$$

i.e. up to sign it coincides with the Milnor number of the weighted homogeneous singularity with weights $w_1, ..., w_n$ and degree D.

2. *The specialization $q = 0, y = 1$ of the elliptic genus of the LG model in case 4 of proposition 6.4 gives the orbifoldized Euler characteristic of the LG model*[16]*:*

$$\frac{1}{D}[(1-D)^D+D^2-1] \tag{56}$$

and coincides with the Euler characteristic of the smooth hypersurface of degree n in $\mathbb{P}^{n-1}$ (LG/CY correspondence for the Euler characteristic; recall that for $w_i = 1$ the CY condition is $n = D$).

Proof. The contributions of either the trivial or remaining sectors follow from (52) and

$$\lim_{y\to 1}\frac{1-y^{1-\frac{1}{D}}e^{\frac{2\pi\sqrt{-1}k}{D}}}{1-y^{-\frac{1}{D}}e^{\frac{2\pi\sqrt{-1}k}{D}}}=\begin{cases}(1-D) & k=0\\ 1 & k\neq 0.\end{cases} \tag{57}$$

In fact, the specialization of proposition 6.4 part 4 gives $\frac{1}{D}((1-D)^n+(D-1)+D(D-1))$, with the first and second summands corresponding to the first summand in the bracket with $k = 0$ and $k \geqslant 1$ respectively (since for $k > 0$, each factor in the product is equal to 1). The claim about matching the Euler characteristic of the LG model and smooth hypersurface can be seen directly, i.e. without use of the McKay correspondence, as in 4 in proposition 6.2, using the following formula (see [27]) for the Euler characteristic of a smooth $(N-2)$-dimensional hypersurface of degree D:

[16] Or the 'orbifoldized Milnor number'.

$$e(V_{N-2}^D) = \frac{(1-D)^N + ND - 1}{D}. \tag{58}$$

□

6.4. The orbifoldization of phases by the action of finite groups

In this section we illustrate the orbifoldization of the elliptic genus of phases as defined in section 5.3.

Example 6.6. Consider the σ-model phase corresponding to the action (33) with $w_i = 1$, $i = 1,....,n$ and linearization with the semistable locus $\mathbb{C} \times (\mathbb{C}^n \setminus 0)$. The GIT quotient $\mathbb{C} \times \mathbb{C}^n \setminus 0//_\kappa \mathbb{C}^*$ is the total space of the line bundle $[\mathcal{O}_{n-1}(-D)]$. Let $\Gamma \subset SL_n(\mathbb{C})$ be a finite subgroup, which we consider as acting on $\mathcal{E} = \mathbb{C} \times \mathbb{C}^n$ via $\gamma(s, v) = (s, \gamma \cdot v), \gamma \in \Gamma$. The orbifoldization of the contribution of the only fixed component of the $\mathbb{C}^*$ action by dilations, which is the zero section of $\mathcal{O}_{n-1}(-D)$, is given by the same formula as (47) but in which Γ is an arbitrary subgroup of $SL_n(\mathbb{C})$ viewed as acting on the total space of bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-D)$. As in the proof of part 3 proposition 6.2, we see that the orbifoldization of the σ-model phase is the Γ-orbifoldized elliptic genus of the hypersurface of degree D in $\mathbb{P}^{n-1}$.

Example 6.7. Next we shall consider the Γ-quotients of the LG models in the sense of section 5.3. First let us look at the LG model corresponding to the case $w_1 = ... = w_n = D = 1$ and its orbifoldization by the cyclic group $\Gamma = \mu_D$ generated by the exponential grading operator $J_W = (......, \exp(2\pi\sqrt{-1}\frac{w_i}{D}), ...)$. The GIT quotient corresponding to this LG phase is $\mathbb{C}^n$, i.e. we have the orbifoldization of the smooth phase, and the elliptic genus of such orbifoldization coincides with the elliptic genus of the LG models with $\mathbb{C}^*$-action (33), as specified in definition 5.7.

Example 6.8. Now we shall look at the orbifoldization of the arbitrary LG phase. Let $\Delta \subset SL_n(\mathbb{C})$ be a finite subgroup containing the exponential grading operator $J_W = (......, \exp(2\pi\sqrt{-1}\frac{w_i}{D}, ...)$[17] and such that J_W belongs to its center. These conditions imply that one can use the space W as a uniformization of the LG phase with the Δ-action, such that for the cyclic group $\Gamma = \mu_D$ generated by J_W one has $W/\Gamma = \mathbb{C} \times \mathbb{C}^n//_\kappa \mathbb{C}^*$. Now, the definition 5.8 yields the following expression for the orbifoldized LG phase:

$$\frac{1}{|\Delta|} \sum_{g,h \in \Delta, gh=hg} \prod_\lambda \frac{\theta((\frac{w_i}{D} - 1)z + \lambda(g) - \lambda(h)\tau)}{\theta(\frac{zw_i}{D} + \lambda(g) - \lambda(h)\tau)} e^{2\pi i \lambda(h) z}. \tag{59}$$

The specialization $q \to 0$ of the orbifoldized phases goes as follows:

Proposition 6.9. *With the notations as above, the elliptic genus of the LG phase orbifoldized by a group Δ for $q \to 0$ specializes to*

$$\sum_{\{h\} \in \mathrm{Conj}(\Delta), X^h} y^{\sum_{\lambda, \lambda(h) \neq 0}(-\frac{1}{2} + \lambda(h))} \frac{1}{|C(g)|} \sum_{g \in \mathrm{Cent}_\Delta(h)} \prod_{\lambda, \lambda(h)=1} y^{-\frac{1}{2}} \frac{1 - y^{(1-\frac{w_j}{D})}\exp(2\pi i \lambda(g))}{1 - y^{(-\frac{w_j}{D})}\exp(2\pi i \lambda(g))} \tag{60}$$

(here X^h is the maximal subspace of $\mathbb{C}^n$ fixed by a representative of the conjugacy class). In the case when Δ is abelian, one has:

[17] For a discussion of the origins of this condition see [6], corollary 2.3.5.

$$\frac{1}{|G|}\sum_{\{h\}\in\Gamma,X^h} y^{\sum_{\lambda,\lambda(h)\neq 0}(-\frac{1}{2}+\lambda(h))}\prod_{\lambda,\lambda(h)=1}\frac{y^{\frac{1}{2}}-y^{(\frac{w_j}{D})-\frac{1}{2}}\exp(2\pi\mathrm{i}\lambda(g))}{1-y^{(\frac{w_j}{D})}\exp(2\pi\mathrm{i}\lambda(g))}. \tag{61}$$

Remark 6.10. The expression (60) coincides with the one given in [4] and the expression (61) coincides with the one given in theorem 6 in [19].

Proof. The term $\Phi^T(x,g,h,z,\tau,\Gamma)$ for $q\to 0$ has as a limit:

$$y^{\frac{-1}{2}+\lambda(h)} \text{ if } \lambda(h)\neq 0 \text{ resp. } y^{\frac{-1}{2}}\frac{1-y\mathrm{e}^{x+2\pi\mathrm{i}\lambda(g)}}{1-\mathrm{e}^{x+2\pi\mathrm{i}\lambda(g)}}. \tag{62}$$

For an action corresponding to the weighted homogeneous polynomials with weights w_i and degree D, the equivariant Chern class of action of $\mathbb{C}^*$ is $\frac{tw_i}{D}$. This implies the proposition. □

6.5. Hybrid models

Here we shall consider types of phases which are neither σ-models or LG, called *hybrid* models (see [13, 56]).

6.5.1. Complete intersection. Sigma models corresponding to the complete Calabi–Yau intersections have hybrid counterparts rather than LG phases appearing in the case of hypersurfaces. See [13, 58] for an alternative treatment of the complete intersections via the hybrid models.

Definition 6.11 (Phases of the complete intersection). Consider the action of

$$\lambda(p_1,...,p_r,z_1,...,z_n)=(\lambda^{-q_1}p_1,...,\lambda^{-q_k}p_r,\lambda z_1,....,\lambda z_n). \tag{63}$$

One of the GIT quotients is the total space of the bundle $\oplus\mathcal{O}_{\mathbb{P}^{n-1}}(-q_i)$ (corresponding to the linearization in one of the cones in $\mathrm{Char}(\mathbb{C}^*)\times\mathbb{Q}$) having as a semistable locus $\mathbb{C}^r(p_1,...,p_r)\times(\mathbb{C}^n(z_1,....,z_n)\setminus 0))$. For linearizations in the second cone, the semistable locus is $(\mathbb{C}^r\setminus 0)\times\mathbb{C}^n$. The GIT quotient of the total space of the direct sum of the n line bundles over the weighted projective space $\mathcal{O}_{\mathbb{P}(q_1,....,q_r)}(-1)^{\oplus n}/\mu_D$, where $D=gcd(q_1,..,q_r)$ and μ_D is the group of roots of unity of degree D acting diagonally on the fibers of this direct sum. The contribution to the equivariant elliptic genus of the component of the fixed point set in the first case is given by

$$[\frac{x\theta(\frac{x}{2\pi\sqrt{-1}}-z,\tau)}{\theta(\frac{x}{2\pi\sqrt{-1}},\tau)}]^n\cdot\prod_{i=1}^r\frac{(x\theta(\frac{-q_ix}{2\pi\sqrt{-1}}+t-z,\tau)}{\theta(\frac{-q_ix}{2\pi\sqrt{-1}+t},\tau))^n}[\mathbb{P}^{n-1}] \tag{64}$$

where t is the generator of the equivariant cohomolgy $H^*_{\mathbb{C}^*}(p)$ of a point. For $t=z$ one obtains the elliptic genus of the smooth complete intersection of hypersurfaces of degree $q_1,...,q_r$ in $\mathbb{P}^{n-1}$.

In the second case, one has a hybrid model (see [56]). The GIT quotient $((\mathbb{C}^r\setminus 0)\times\mathbb{C}^n)/\mathbb{C}^*$ is a fiber space with the orbifold $\mathbb{C}^n/\mu_D$ as a fiber, and its base being the weighted projective space with the orbifold structure given by viewing $\mathbb{P}^{r-1}(q_1,....,q_r)$ as a quotient of $\mathbb{P}^{r-1}$ by the action of the abelian group $\Gamma=\oplus_i\mu_{q_i}$. The uniformization is given by taking the quotient of the total space $[\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)]$ of the split vector bundle on $\mathbb{P}^{r-1}$ by the action of Γ, such that the projection on $\mathbb{P}^{n-1}$ is compatible with $\mathbb{P}^{r-1} \to \mathbb{P}^{r-1}/\Gamma = \mathbb{P}(q_1, ..., q_r)$. The fixed point set of the action of $\mathbb{C}^*$ on the GIT-quotient induced by action $t(p_1, ..., p_r, z_1, ..., z_n) \to (tp_1, .., tp_r, z_1, ..., z_n)$ is $\mathbb{P}^{r-1}(q_1, .., q_r)$, and for the induced $\mathbb{C}^*$-action on $[\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)]$ it is the zero section of this bundle. Hence for each pair (g, h) of elements of Γ, the contribution of the $\mathbb{C}^*$-fixed point to the summand of the orbifold elliptic genus $\mathcal{E}LL^{\mathbb{C}^*}_{\mathbb{P}^{r-1}}([\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)], \Gamma)$ corresponding to (g, h) will have two factors. One comes from a restriction of the tangent bundle

$$T_{[\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)]}|_{\mathbb{P}^{r-1 g,h}} \tag{65}$$

to the subspace of $\mathbb{P}^{r-1}$ fixed by both g, h. The latter coincides with $T_{\mathbb{P}^{r-1}}|_{\mathbb{P}^{r-1 g,h}}$. This contribution is the summand $\mathcal{E}LL_{\mathrm{orb}}(\mathbb{P}^{r-1}, \Gamma)^{g,h}$ of the elliptic class

$$\mathcal{E}LL_{\mathrm{orb}}(\mathbb{P}^{r-1}, \Gamma) = \frac{1}{|\Gamma|} \sum_{g,h} \mathcal{E}LL_{\mathrm{orb}}(\mathbb{P}^{r-1}, \Gamma)^{g,h}$$

corresponding to the pair g, h since $\mathbb{P}^{r-1}$ is the fixed point set of the $\mathbb{C}^*$-action. The quotient $T_{[\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)]}|_{\mathbb{P}^{r-1 g,h}}/T_{\mathbb{P}^{r-1}}|_{\mathbb{P}^{r-1 g,h}}$ is just $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)^n|_{\mathbb{P}^{r-1 g,h}}$. The total space of this bundle acted upon by the group $<g, h>$ is considered as the automorphism group of $[\mathcal{O}_{\mathbb{P}^{r-1}}(-1)^{\oplus n}]$. It also supports the $\mathbb{C}^*$-action by dilation. The corresponding equivariant contribution of this part of $T_{[\mathcal{O}^{\oplus n}_{\mathbb{P}^{r-1}}(-1)]}|_{\mathbb{P}^{r-1}}$ is

$$\left(\frac{\theta(\frac{x}{2\pi \mathrm{i}} + \frac{u}{D} - z + \lambda(g) - \tau\lambda(h))}{\theta(\frac{x}{2\pi \mathrm{i}} + \frac{u}{D} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i}\lambda(h)z}\right)^n$$

where λ is the character of $\langle g, h \rangle$ acting on this eigenbundle (term $\frac{u}{D}$ reflects that the contribution is written in terms of the character of $\mathbb{C}^*/\mu_D$ acting effectively on the fibers). The resulting elliptic genus of the hybrid model can described as

$$\frac{1}{|\Gamma|} \sum_{g,h} \mathcal{E}LL_{\mathrm{orb}}(\mathbb{P}^{r-1}, \Gamma))^{g,h} \cdot \left(\frac{\theta(\frac{x}{2\pi \mathrm{i}} + (\frac{1}{D} - 1)z + \lambda(g) - \lambda(h)\tau)}{\theta(\frac{x}{2\pi \mathrm{i}} + \frac{z}{D} + \lambda(g) - \lambda(h)\tau)} \mathrm{e}^{2\pi \mathrm{i}\lambda(h)z}\right)^n [\mathbb{P}^{r-1}]^{g,h}. \tag{66}$$

Note that this expression in the case $r = 1$ becomes the elliptic genus of the LG-model since $x = 0$, $\Gamma = \mu_D$ and for $g = \mathrm{e}^{2\pi \mathrm{i}\frac{a}{D}}$, $h = \mathrm{e}^{2\pi \mathrm{i}\frac{b}{D}}$ one has $\lambda(g) = \frac{a}{D}$, $\lambda(g) = \frac{b}{D}$.

6.5.2. Hypersurfaces in the products of projective spaces. This material is discussed in [56], section 5.5. Consider the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^m$ given by

$$(\lambda, \mu)(p, x_1, ..., x_n, y_1, ..., y_m) = (\lambda^{-n}\mu^{-m}p, \lambda x_1, ..., \lambda x_n, \mu y_1, ..., \mu y_m). \tag{67}$$

There are three cones in $Char((\mathbb{C}^*)^2) \otimes \mathbb{Q}$ corresponding to linearizations with a constant GIT with semistable loci respectively:

$$\{\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^m\}_{ss} = \begin{cases} \mathbb{C} \times (\mathbb{C}^n \setminus 0) \times (\mathbb{C}^m \setminus 0), & \text{Calabi} - \text{Yau phase} \\ \mathbb{C}^* \times (\mathbb{C}^n \setminus 0) \times (\mathbb{C}^m), & \text{hybrid phase} \\ \mathbb{C}^* \times (\mathbb{C}^n) \times (\mathbb{C}^m \setminus 0), & \text{hybrid phase} \end{cases} \tag{68}$$

with the GIT quotients being respectively:

$$\begin{cases} [p_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(-n) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{m-1}}(-m)], \\ [\oplus\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^m]/\mu_m, \\ [\oplus\mathcal{O}_{\mathbb{P}^{m-1}}(-1)^n]/\mu_n. \end{cases} \tag{69}$$

The respective elliptic genera are:

$$\begin{cases} [\frac{x\theta(\frac{x}{2\pi\sqrt{-1}}-z,\tau)}{\theta(\frac{x}{2\pi\sqrt{-1}},\tau)}]^n[\frac{y\theta(\frac{x}{2\pi\sqrt{-1}}-z,\tau)}{\theta(\frac{y}{2\pi\sqrt{-1}},\tau)}]^n[\frac{\theta(\frac{nx+my}{2\pi\sqrt{-1}},\tau)}{\theta(\frac{nx+my}{2\pi\sqrt{-1}}-z,\tau)}][\mathbb{P}^{n-1}\times\mathbb{P}^{m-1}] \\ \frac{1}{n}\sum_{0\leqslant a,b<n}(\frac{\theta(-m\frac{x}{2\pi\mathrm{i}}+(\frac{1}{n}-1)z+\frac{a-b\tau}{n},\tau)}{\theta(-m\frac{x}{2\pi\mathrm{i}}+\frac{1}{n}z+\frac{a-b\tau}{n},\tau)}\mathrm{e}^{\frac{2\pi\mathrm{i}bz}{n}})^n(\frac{x\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})})^m[\mathbb{P}^{\,m-1}] \\ \frac{1}{m}\sum_{0\leqslant a,b<m}(\frac{\theta(-\frac{ny}{2\pi\mathrm{i}}+(\frac{1}{m}-1)z+\frac{a-b\tau}{m},\tau)}{\theta(-\frac{ny}{2\pi\mathrm{i}}+\frac{1}{m}z+\frac{a-b\tau}{m},\tau)}\mathrm{e}^{\frac{2\pi\mathrm{i}bz}{m}})^m(\frac{x\theta(\frac{y}{2\pi\mathrm{i}}-z)}{\theta(\frac{y}{2\pi\mathrm{i}})})^n[\mathbb{P}^{\,n-1}]. \end{cases} \tag{70}$$

The expression in the upper row represents the elliptic genus of the Calabi–Yau hypersurface of bidegree (n,m) in $\mathbb{P}^{n-1}\times\mathbb{P}^{m-1}$.

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Appendix A. Theta functions

The Jacobi theta function $\theta(z,\tau), z\in\mathbb{C}, \tau\in\mathbb{H}$ is an entire function on $\mathbb{C}\times\mathbb{H}$, where $\mathbb{H}$ is the upper half plane[18] defined as the product:

$$\theta(z,\tau)=q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1-q^l)\prod_{l=1}^{l=\infty}(1-q^l\mathrm{e}^{2\pi\mathrm{i}z})(1-q^l\mathrm{e}^{-2\pi\mathrm{i}z}) \tag{A.1}$$

where $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$.

Its transformation law is as follows:

$$\theta(\frac{z}{\tau},-\frac{1}{\tau})=-\mathrm{i}\sqrt{\frac{\tau}{\mathrm{i}}}\mathrm{e}^{\frac{\pi\mathrm{i}z^2}{\tau}}\theta(z,\tau) \tag{A.2}$$

$$\theta(z+1,\tau)=-\theta(z,\tau),\quad \theta(z+\tau,\tau)=-\mathrm{e}^{-2\pi\mathrm{i}z-\pi\mathrm{i}\tau}\theta(z,\tau).$$

The derivative $\theta'(0,\tau)$ appears in expansion $\theta(z,\tau)=\theta'(0,\tau)z+\frac{1}{2}\theta''(0,\tau)z^2+....$ and satisfies:

$$\theta'(0,\tau)=\eta^3(\tau),\ \text{ where }\ \eta(\tau)=q^{\frac{1}{24}}\prod(1-q^n). \tag{A.3}$$

(Dedekind's) $\eta(\tau)$-function transforms as follows:

$$\eta(-\frac{1}{\tau})=(\frac{\tau}{\mathrm{i}})^{\frac{1}{2}}\eta(\tau). \tag{A.4}$$

It follows that

$$\frac{\theta(\frac{z}{\tau},-\frac{1}{\tau})}{\theta'(0,-\frac{1}{\tau})}=\frac{\mathrm{e}^{\frac{\pi\mathrm{i}z^2}{\tau}}}{\tau}\frac{\theta(z,\tau)}{\theta'(0,\tau)}. \tag{A.5}$$

Let

[18] $\theta_1(z,\tau)$ or $\theta_{1,1}(z,\tau)$ are other common notations.

$$\Upsilon(x,\tau) = (1-\mathrm{e}^{-x})\prod_{n=1}^{\infty}\frac{(1-q^n\mathrm{e}^{x})(1-q^n\mathrm{e}^{-x})}{(1-q^n)^2} \tag{A.6}$$

and

$$\Phi(x,\tau) = \mathrm{e}^{\frac{x}{2}}\Upsilon(x,\tau) = (\mathrm{e}^{\frac{x}{2}}-\mathrm{e}^{-\frac{x}{2}})\prod_{n=1}^{\infty}\frac{(1-q^n\mathrm{e}^{x})(1-q^n\mathrm{e}^{-x})}{(1-q^n)^2}$$

(see [29] p 170 and [7] p 456)[19]. Hence

$$\Phi(x,\tau) = 2\sinh(\frac{x}{2})\prod_{n=1}^{\infty}\frac{(1-q^n\mathrm{e}^{x})(1-q^n\mathrm{e}^{-x})}{(1-q^n)^2}$$

(see [29] p 117), i.e.

$$\Phi(x,\tau) = \frac{\mathrm{i}\theta(\frac{x}{2\pi\mathrm{i}},\tau)}{\eta^3(\tau)}$$

(see [7] p 461).

The Weierstrass σ-function is defined by

$$\sigma(z,\tau) = z\prod\nolimits_{\omega\neq 0,\omega\in\mathbb{Z}+\mathbb{Z}\tau}(1-\frac{z}{\omega})\mathrm{e}^{\frac{z}{\omega}+\frac{1}{2}(\frac{z}{\omega})^2} \tag{A.7}$$

(see [12] p 52), which can be used to describe $\Phi(z,\tau)$, where $z=\frac{x}{2\pi\sqrt{-1}}$ (see [29] p 145, corollary 5.3)[20].

$$\Phi(z,\tau) = \exp(4\pi^2 G_2(\tau)z^2)\sigma(z,\tau) = \exp(-\frac{e_2(\tau)}{2}z^2)\sigma(z,\tau). \tag{A.8}$$

Here the quasi-modular forms $G_2(\tau)$ and $e_2(\tau)$ are given by

$$G_2(\tau) = -\frac{1}{24}+\sum_{n=1}^{\infty}(\sum_{d|n}d)q^n = -\frac{1}{8\pi^2}e_2(\tau) \quad \text{where } \ e_2(\tau) = \sum_{n,(m,n)\neq(0,0)}\sum_{m}\frac{1}{(m+n\tau)^2}. \tag{A.9}$$

We also consider the following product expansion (see [53] chapter 4 section 3):

$$\phi(z,\tau) = x\prod\nolimits_{\mathrm{e}}'(1-\frac{z}{w}) \tag{A.10}$$

(related to (A.7); the product is taken over the elements w of the lattice $W=\{1,\tau\}$, the subscript e designates the Eisenstein ordering of factors and $'$ indicates omission of $(0,0)\in W$. $\phi(z,\tau)$ admits the following product formula in q (see (15) ibid)

$$\phi(z,\tau) = \frac{1}{2\pi\sqrt{-1}}\frac{(\mathrm{e}^{\pi\sqrt{-1}z}-\mathrm{e}^{-\pi\sqrt{-1}z})\prod_{n\geqslant 1}(1-q^n\mathrm{e}^{2\pi\sqrt{-1}z})(1-q^n\mathrm{e}^{-2\pi\sqrt{-1}z})}{\prod_{n\geqslant 1}(1-q^n)^2}$$

$$= \frac{1}{2\pi\sqrt{-1}}\Phi(x,\tau) = \frac{1}{2\pi}\frac{\theta(z,\tau)}{\eta^3(\tau).} \tag{A.11}$$

[19] In [7], Hirzebruch's $\Upsilon(z,\tau)$-function is denoted as $\Phi(z,\tau)$; the notation $\Phi(z,\tau)$ is the one used in the appendix to the [29] corollary 5.3 p 145.

[20] I.e. in terms of $x=2\pi\sqrt{-1}z$, for which the lattice is $2\pi\sqrt{-1}(\mathbb{Z}+\mathbb{Z}\tau)$, one has $\Phi(x,\tau)=\sigma(x,\tau)\exp(-G_2(\tau)x^2)$. [28, 29] use this notation while we selected traditional notations (in particular consistent with [53]).

Appendix B. Quasi-Jacobi forms

Recall the following:

Definition B.1 (See [23, 39]). The meromorphic Jacobi form of index $t \in \frac{1}{2}\mathbb{Z}$ and weight k for a finite index subgroup of the Jacobi group $\Gamma_1^J = SL_2(\mathbb{Z}) \propto \mathbb{Z}^2$ is defined as meromorphic in the elliptic variable z function χ on $\mathbb{H} \times \mathbb{C}$ having the expansion $\sum c_{n,r} q^n \zeta^r$ in $q = \exp(2\pi\sqrt{-1}\tau)$ and satisfying the following functional equations:

$$\chi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k \mathrm{e}^{\frac{2\pi \mathrm{i} tcz^2}{c\tau + d}} \chi(\tau, z) \tag{B.1}$$

$$\chi(\tau, z + \lambda\tau + \mu) = (-1)^{2t(\lambda+\mu)} \mathrm{e}^{-2\pi \mathrm{i} t(\lambda^2\tau + 2\lambda z)} \chi(\tau, z) \tag{B.2}$$

for all elements $[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0]$ and $[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (a, b)]$ in Γ. A meromorphic Jacobi form is called a weak Jacobi form if

(a) it is holomorphic in $\mathbb{H} \times \mathbb{C}$ and
(b) it has a Fourier expansion $\sum c_{n,r} q^n \zeta^r$ in $q = \exp(2\pi\sqrt{-1}\tau)$ in which $n \geqslant 0$.

The functional equation (B.2) implies that the Fourier coefficients $c_{n,r}$ depend on r mod $2m$ and $\Delta = 4nm - r^2$ (*the discriminant*). A weak Jacobi form is called the Jacobi form (respectively the cusp form) if the coefficients $c_{n,r}$ with $\Delta < 0$ (respectively $\Delta \leqslant 0$) are vanishing[21].

Remark B.2. Presentation (2) provides the Fourier expansion of the elliptic genus having non-negative powers of q (i.e. it yields a weak Jacobi form) while the powers of y can be negative.

The algebra of the Jacobi forms is the bi-graded algebra $J = \oplus J_{t,k}$. and the algebra of the Jacobi forms of index zero is the sub-algebra $J_0 = \oplus_k J_{0,k} \subset J$.

We shall need below the following real analytic functions:

$$\lambda(z, \tau) = \frac{z - \bar{z}}{\tau - \bar{\tau}}, \quad \mu(\tau) = \frac{1}{\tau - \bar{\tau}}. \tag{B.3}$$

Their transformation properties are as follows:

$$\lambda(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)\lambda(z, \tau) - 2icz \tag{B.4}$$

$$\lambda(z + m\tau + n, \tau) = \lambda(z, \tau) + m$$

$$\mu(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2\mu(\tau) - 2ic(c\tau + d). \tag{B.5}$$

Definition B.3. An almost meromorphic Jacobi form of weight k, index zero and depth (s, t) is a (real) meromorphic function in $\mathbb{C}\{q^{\frac{1}{l}}, z\}[z^{-1}, \lambda, \mu]$, with λ, μ given by (B.4), i.e. the polynomial in λ, μ with complex meromorphic functions as coefficients which

[21] Mentioning that this condition on the Fourier expansion is applicable in the holomorphic case only, and the restriction $n \geqslant 0$ was inadvertently omitted in [39].

(a) satisfies the functional equations in the definition B.1 of the Jacobi forms of weight k and index zero, and
(b) which has a degree of at most s in λ and at most t in μ.

The quasi-Jacobi form of weight k, index zero and depth (s, t) is the term of bi-degree $(0, 0)$ in λ, μ of an almost meromorphic Jacobi form of weight k and depth (s, t). The algebra of the quasi-Jacobi forms is bi-graded filtered algebra generated by the filtered algebra of the quasi-Jacobi forms and the algebra of the Jacobi forms (which have a depth $(0, 0)$ and have trivial filtration).

Example B.4. 1. Two-variable Eisenstein series (see [53, 39]). Consider the following functions meromorphic in z:

$$E_n(z,\tau) = \sum_{a,b\in\mathbb{Z}^2} \left(\frac{1}{z + a\tau + b}\right)^n \quad n \in \mathbb{Z}, n \geqslant 1. \tag{B.6}$$

These series are absolutely convergent for $n \geqslant 3$ and yield meromorphic Jacobi forms of weight n and index 0. For $n = 1, 2$ one obtains a meromorphic function using an Eisenstein summation (see [53]) which are quasi-Jacobi forms of index 0, weight $n = 1, 2$ and depth $(1, 0)$ for $n = 1$ and $(0, 1)$ for $n = 2$ (see [39]). $E_2 - e_2$ is a Jacobi form (here $e_2(\tau)$ is a quasi-modular form, which is the one-variable Eisenstein series). The products

$$\hat{E}_n(z,\tau) = E_n(z,\tau)\left(\frac{\theta(z,\tau)}{\theta'(0,\tau)}\right)^n \quad (n \neq 2) \quad \hat{E}_2 = (E_2(z,\tau) - e_2(\tau))\left(\frac{\theta(z,\tau)}{\theta'(0,\tau)}\right)^2 \tag{B.7}$$

are holomorphic quasi-Jacobi forms (Jacobi forms for $n \geqslant 2$.

The structure of the algebra of quasi-Jacobi forms generated by the forms (B.7) is as follows:

Theorem B.5. *The algebra $QJac_{0,*}$ (or simply QJac) of the quasi-Jacobi forms of weight zero and index $\frac{d}{2}, d \in \mathbb{Z}^{\geqslant 1}$ is a polynomial algebra with the generators $\hat{E}_n, n = 1, 2, 3, 4$. The algebra $Jac_{0,*}$ of the Jacobi forms of weight zero and index $\frac{d}{2}$ (or Jac) is a polynomial algebra in three generators $\hat{E}_2, \hat{E}_3, \hat{E}_4$. The algebra QJac is isomorphic to the algebra of the complex cobordisms Ω^U modulo, the ideal I generated by $X_1 - X_2$ where X_1, X_2 are K-equivalent. The algebra Jac is isomorphic to the algebra Ω^{SU} of the complex cobordisms of the manifolds with a trivial first Chern class modulo, which is the ideal $I \cap \Omega^{SU}$.*

Remark B.6.

1. Different generators of the algebra *Jac* are described in [26].
2. The term 'quasi-Jacobi forms' used in [45] in a slightly more narrow sense than in [39] and above, where the author was apparently unaware of [39]. The quasi-Jacobi forms considered in [45] belong to the algebra generated by the function:

$$\frac{\theta(z,\tau)}{\eta^3(\tau)}, \frac{\partial \log\left(\frac{\theta(z,\tau)}{\eta^3(\tau)}\right)}{\partial z}, e_2(\tau), e_4(\tau), \wp(z,\tau), y\frac{[\mathrm{d}\wp(z,\tau)}{\mathrm{d}y} \tag{B.8}$$

($y = \exp(2\pi\sqrt{-1}z)$) are in the algebra of the meromorphic quasi-Jacobi forms as defined in (B.3) (see also [39]). Indeed, $\frac{\partial(\frac{\theta(z,\tau)}{\eta^3(\tau)})}{\partial z} = E_1(z,\tau)$ (which follows from appendix A; see also [53] chapter IV, section 3 (15)) and also $\wp(z,\tau) = E_2 - e_2, y\frac{\mathrm{d}\wp(z,\tau)}{\mathrm{d}y} = -2E_3(z,\tau)$ and the modular functions are clearly part of the algebra described in definition B.3.

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References

[1] Ando M, Hopkins M and Strickland N 2001 Elliptic spectra, the Witten genus and the theorem of the cube *Inventiones Math.* **146** 595–687
[2] Atiyah M and Bott R 1967 A Lefschetz fixed point formula for elliptic complexes. I *Ann. Math.* **86** 374–407
[3] Batyrev V 1994 Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties *J. Algebr. Geom.* **3** 493–535
[4] Berglund P and Henningson M 1995 Landau–Ginzburg orbifolds, mirror symmetry and the elliptic genus *Nucl. Phys.* B **433** 311–32
[5] Borcherds R 1986 Vertex algebras, Kac–Moody algebras, and the Monster *Proc. Natl Acad. Sci. USA* **83** 3068–71
[6] Borisov L 2013 Berglund–Hübsch mirror symmetry via vertex algebras *Commun. Math. Phys.* **320** 73–99
[7] Borisov L and Libgober A 2000 Elliptic genera of toric varieties and applications to mirror symmetry *Inventiones Math.* **140** 453–85
[8] Borisov L and Libgober A 2003 Elliptic genera of singular varieties *Duke Math. J.* **116** 319–51
[9] Borisov L and Libgober A 2005 McKay correspondence for elliptic genera *Ann. Math.* **161** 1521–69
[10] Bott R and Taubes C 1989 On the rigidity theorems of Witten *J. Am. Math. Soc.* **2** 137–86
[11] Candelas P, de la Ossa X, Green P and Parkes L 1991 A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory *Nucl. Phys.* B **359** 21–74
[12] Chandrasekharan K 1985 *Elliptic Functions* (*Grundlehren der Mathematischen Wissenschaften* vol 281) (Berlin: Springer)
[13] Chiodo A and Nagel J 2015 The hybrid Landau–Ginzburg models of Calabi–Yau complete intersections (arXiv:1506.02989)
[14] Dijkgraaf R, Moore G, Verlinde E and Verlinde H 1997 Elliptic genera of symmetric products and second quantized strings *Commun. Math. Phys.* **185** 197–209
[15] Dixon L J, Harvey J A, Vafa C and Witten E 1985 Strings on orbifolds *Nucl. Phys.* B **261** 678–86
Dixon L J, Harvey J A, Vafa C and Witten E 1985 Strings on orbifolds (II) *Nucl. Phys.* B **274** 285–314
[16] Dolgachev I 2003 *Lectures on Invariant Theory* (*London Mathematical Society Lecture Note Series* vol 296) (Cambridge: Cambridge University Press)
[17] Dong C and Mason G 1996 Vertex operator algebras and Moonshine: a survey *Progress in Algebraic Combinatorics* (*Advanced Studies in Pure Mathematics* vol 24) (Tokyo: Mathematical Society of Japan) (Fukuoka, 101136)
[18] Dong C, Liu K and Ma X 2002 On orbifold elliptic genus *Orbifolds in Mathematics and Physics* (*Contemporary Mathematics* vol 310) (Providence, RI: American Mathematical Society) pp 87–105 (Madison, WI, 2001)
[19] Ebeling W and Takahashi A 2013 Variance of the exponents of orbifold Landau–Ginzburg models *Math. Res. Lett.* **20** 51–65
[20] Edidin D and Graham W 1998 Equivariant intersection theory *Inventiones Math.* **131** 595–634
[21] Edidin D and Graham W 1998 Localization in equivariant intersection theory and the Bott residue formula *Am. J. Math.* **120** 619–36
[22] Edidin D and Graham W 2000 Riemann–Roch for equivariant Chow groups *Duke Math. J.* **102** 567–94

[23] Eichler M and Zagier D 1985 *The Theory of Jacobi Forms* (*Progress in Mathematics* vol 55) (Boston, MA: Birkhüser)
[24] Fulton W 1984 *Intersection Theory* (*Ergebnisse der Mathematik und ihrer Grenzgebiete*) (Berlin: Springer)
[25] Goresky M and MacPherson R 1984 Problems and bibliography on intersection homology *Intersection Cohomology* ed A Borel *et al* (Boston, MA: Birkhüser) pp 221–33
[26] Gritsenko V 1999 Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms *Algebra Anal.* **11** 100–25
[27] Hirzebruch F 1966 *Topological Methods in Algebraic Geometry* 3rd edn (New York: Springer)
[28] Hirzebruch F 1988 Elliptic genera of level N for complex manifolds *Differential Geometrical Methods in Theoretical Physics (Como, 1987)* (*Nato Science Series C: Mathematical and Physical Sciences* vol 250) (Dordrecht: Kluwer) pp 37–63
[29] Hirzebruch F, Berger T and Jung R 1992 Manifolds and modular forms. With appendices *Aspects of Mathematics* vol E20, ed N-P Skoruppa and P Baum (Braunschweig: Vieweg)
[30] Hirzebruch F and Höfer T 1990 On the Euler number of an orbifold *Math. Ann.* **286** 255–60
[31] Höhn G 1991 *Komplexe elliptische Geschlechter und S1-quivariante Kobordismustheorie (Diplomarbeit, Bonn, Germany)* http://baby.mathematik.uni-freiburg.de/papers/
[32] Hopkins M 1995 Topological modular forms, the Witten genus, and the theorem of the cube *Proc. of the Int. Congress of Mathematicians (Zurich, 1994) vol* 1, 2 (Basel: Birkhuser) pp 554–65
[33] Kachru S and Tripathy A 2017 The Hodge-elliptic genus, spinning BPS states, and black holes *Comm. Math. Phys.* **355** 245–59
[34] Kawai T, Yamada Y and Yang S K 1994 Elliptic genera and $N = 2$ superconformal field theory *Nucl. Phys.* B **414** 191–212
[35] Kontsevich M 1995 *Homological Algebra of Mirror Symmetry Proc. of the Int. Congress of Mathematicians (Zurich, 1994) vol* 1, 2 (Basel: Birkhuser) pp 120–39
[36] Krichever I 1990 Generalized elliptic genera and Baker–Akhiezer functions *Math. Notes* **47** 132–42
[37] Landweber P 1988 Elliptic genera: an introductory overview *Elliptic Curves and Modular Forms in Algebraic Topology* (*Lecture Notes in Mathmatics* vol 1326) (Berlin: Springer) pp 1–10 (Princeton, NJ, 1986)
[38] Landweber P (ed) 1988 *Elliptic Curves and Modular Forms in Algebraic Topology* (*Lecture Notes in Mathematics* vol 1326) (New York: Springer)
[39] Libgober A 2011 Elliptic genera, real algebraic varieties and quasi-Jacobi forms *Topology of Stratified Spaces* (*Mathematical Sciences Research Institute Publications* vol 58) (Cambridge: Cambridge University Press) pp 95–120
[40] Libgober A 2015 Elliptic genus of phases of $N = 2$ theories *Commun. Math. Phys.* **340** 939–58
[41] Liu K 1996 On elliptic genera and theta-functions *Topology* **35** 617–40
[42] Malikov F, Schechtman V and Vaintrob A 1999 Chiral de Rham complex *Commun. Math. Phys.* **204** 439–73
[43] McKay J 1980 Graphs, singularities, and finite groups *The Santa Cruz Conf. on Finite Groups (Univ. California, Santa Cruz, California 1979) (Proc. Symp. Pure Mathmatics* vol 37) (Providence, RI: American Mathematical Society) pp 183–6
[44] Mumford D, Fogarty J and Kirwan F 1994 *Geometric Invariant Theory* (*Ergebnisse der Mathematik und ihrer Grenzgebiete* vol 34) 3rd edn (Berlin: Springer)
[45] Oberdieck G 2018 Gromov–Witten invariants of the Hilbert schemes of points of a $K3$ surface *Geom. Topol.* **22** 323–43
[46] Schellekens A N and Warner N P 1986 Anomaly cancellation and self-dual lattices *Phys. Lett.* B **181** 339–43
[47] Steenbrink J 1977 Mixed Hodge structure on the vanishing cohomology. Real and complex singularities *Proc. 9th Nordic Summer School/NAVF Symp. Mathmatics (Oslo, 1976)* (Sijthoff and Noordhoff, Alphen aan den Rijn) pp 525–63
[48] Stong R 1968 *Notes on Cobordism Theory* (Princeton, NJ: Princeton University Press)
[49] Thaddeus M 1996 Geometric invariant theory and flips *J. Am. Math. Soc.* **9** 691–723
[50] Totaro B 2000 Chern numbers for singular varieties and elliptic homology *Ann. Math.* **151** 757–91
[51] Voisin C 2007 *Hodge Theory and Complex Algebraic Geometry.* I (*Cambridge Studies in Advanced Mathematics* vol 76) (Cambridge: Cambridge University Press)
[52] Waelder R 2008 Equivariant elliptic genera and local McKay correspondences *Asian J. Math.* **12** 251–84

[53] Weil A 1999 *Elliptic Functions According to Eisenstein and Kronecker* (*Classics in Mathematics*) (Berlin: Springer) (Reprint of the 1976 original)
[54] Witten E 1987 Elliptic genera and quantum field theory *Commun. Math. Phys.* **109** 525–36
[55] Witten E 1994 On the Landau–Ginzburg description of $N = 2$ minimal models *Int. J. Mod. Phys.* A **9** 4783–800
[56] Witten E 1993 Phases of $N = 2$ theories in two dimensions *Nucl. Phys.* B **403** 159–222
[57] Witten E 1992 Mirror manifolds and topological field theory *Essays on Mirror Manifolds* (Hong Kong: International press) pp 120–58
[58] Filipazzi S and Rota F 2017 An example of Berglund–Hübsch mirror symmetry for a Calabi–Yau complete intersection (arXiv: 1711.02812)