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Hodge Polynomials of Singular Hypersurfaces

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1. Introduction and Statement of Results

Let X be an n-dimensional compact complex algebraic manifold and $\mathcal{L}$ a line bundle on X. Let $\mathbf{L} \subset \mathbb{P}(H^0(X,\mathcal{L}))$ be a line in the projectivization of the space of sections of $\mathcal{L}$ (i.e., a pencil of hypersurfaces in X). Assume that the generic element L_t in $\mathbf{L}$ is nonsingular and that L_0 is a singular element of $\mathbf{L}$. The purpose of this paper is to relate the Hodge polynomials of the singular and (respectively) generic member of the pencil—in other words, to understand the difference $\chi_y(L_0) - \chi_y(L_t)$ in terms of invariants of the singularities of L_0. A special case of this situation was considered by Parusiński and Pragacz in [PPr1], who studied the topological Euler characteristic of pencils for which the generic element L_t of the pencil $\mathbf{L}$ is transversal to the strata of a Whitney stratification of L_0. This led the authors of [PPr1] to a calculation of Parusiński's generalized Milnor number (see [P]) of a singular hypersurface and also to a characteristic class version of this formula in [PPr2] for the Chern–Schwartz–MacPherson classes (see [Mac]). In a different vein, the Hodge theory of 1-parameter degenerations was considered in [CLMSh2] (cf. [Di1] for the case when L_0 has only isolated singularities) by using Hodge-theoretical aspects of the nearby and vanishing cycles associated to the degenerating family of hypersurfaces and extending similar Euler characteristic calculations presented earlier in [Di2]. This paper adds an extra layer of complexity by addressing the Hodge-theoretic situation in the context of a pencil of hypersurfaces with nonempty base locus.

Let us first define the invariants to be investigated. A *functorial χ_y-genus* is defined by the ring homomorphism

$$\chi_y\colon K_0(\mathrm{MHS}) \to \mathbb{Z}[y, y^{-1}]; \quad [V] \mapsto \sum_p \dim_{\mathbb{C}} \mathrm{Gr}_F^p(V \otimes_{\mathbb{Q}} \mathbb{C}) \cdot (-y)^p, \qquad (1.1)$$

where $K_0(\mathrm{MHS})$ is the Grothendieck ring (with respect to the tensor product) of the abelian category MHS of rational mixed Hodge structures [De1; De2] and $\mathrm{Gr}_F^p(V \otimes_{\mathbb{Q}} \mathbb{C}) := F^p/F^{p+1}$ $(p \in \mathbb{Z})$ denotes the pth graded part of the (decreasing) Hodge filtration $F^\bullet$ corresponding to the mixed Hodge structure $V \in \mathrm{MHS}$. For $K^\bullet \in D^b(\mathrm{MHS})$ a bounded complex of rational mixed Hodge structures, we set $[K^\bullet] := \sum_i (-1)^i [K^i] \in K_0(\mathrm{MHS})$ and define

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$$\chi_y([K^\bullet]) := \sum_i (-1)^i \chi_y([K^i]). \tag{1.2}$$

Then, for X any complex algebraic variety, we let

$$\chi_y(X) := \chi_y([H^*(X;\mathbb{Q})]) = \sum_j (-1)^j \cdot \chi_y([H^j(X;\mathbb{Q})]) \tag{1.3}$$

be the *Hodge polynomial* of X. Similarly, we define $\chi_y^c(X)$ by using instead the canonical Deligne mixed Hodge structure in the cohomology with compact support $H_c^*(X;\mathbb{Q})$ of X. The specializations of polynomials $\chi_y(X)$ and $\chi_y^c(X)$ for $y = -1$ yield the topological Euler characteristic $e(X)$. Also note that χ_y^c is an additive (motivic) invariant; that is, if Z is a Zariski closed subset of X then

$$\chi_y^c(X) = \chi_y^c(Z) + \chi_y^c(X \setminus Z). \tag{1.4}$$

Finally, we point out that the polynomials $\chi_y(X)$ and $\chi_y^c(X)$ are extensions to the singular setting of Hirzebruch's χ_y-genus of a compact complex algebraic manifold X (cf. [H]). This is a polynomial, defined in terms of the Hodge numbers of X, that includes as special cases the topological Euler characteristic (at $y = -1$), the arithmetic genus (at $y = 0$), and the signature (at $y = 1$).

Before formulating our main results, we need a few definitions and some notation. We begin by recalling standard facts about the incidence variety of a pencil, which plays an essential role in our approach. Let $\mathcal{I} \subset X \times \mathbf{L}$ be the variety defined by the *incidence correspondence*:

$$\mathcal{I} = \{(x,t) \mid t \in \mathbf{L},\ x \in L_t\}. \tag{1.5}$$

We denote the projections of $\mathcal{I}$ on each factor by p_X and $p_{\mathbf{L}}$, respectively; note that both are surjective. Moreover, p_X is one-to-one outside of the base locus of $\mathbf{L}$, and its fibers over any point in the latter are $\mathbb{P}^1$, which $p_{\mathbf{L}}$ maps isomorphically onto $\mathbf{L}$.

If the intersection of the base locus with the singular locus of any element of the pencil is empty, then $\mathcal{I}$ is a nonsingular variety; otherwise, it has singularities. Indeed, under the empty intersection assumption, let f_1 and f_2 be two generic elements of the pencil written in the local coordinates $(x_1,\dots,x_n)$ of a base point P of the pencil. Then the differentials df_1 and df_2 are independent because, if $df_1 + t_0 df_2 = 0$, then the element of the pencil given by $f_1 + t_0 f_2 = 0$ has a singularity at P. Now using the local coordinates at P in which $f_1 = x_1$ and $f_2 = x_2$, we can view the incidence correspondence as the hypersurface in $\mathbb{C}^n \times \mathbb{C}$ given by the equation $x_1 + tx_2 = 0$, which is nonsingular.

We also note that the fibers of $p_{\mathbf{L}}$ are isomorphic to the corresponding elements of the pencil (and will be denoted by $L_t^{\mathcal{I}}$, or simply L_t if there is no danger of confusion) and that $p_{\mathbf{L}}$ is a locally trivial topological fibration outside a finite set of points in $\mathbf{L}$ containing the point that corresponds to L_0. We will restrict our attention to fibers of $p_{\mathbf{L}}$ near L_0; in other words, we consider the restriction map $p := p_{\mathbf{L}}|_{p_{\mathbf{L}}^{-1}(D_\varepsilon(0))}$ for ε small enough that this restriction is a locally trivial fibration outside the special fiber $L_0 = p_{\mathbf{L}}^{-1}(0)$. Note that p is a proper holomorphic map with compact complex algebraic fibers, the generic fiber being smooth. As remarked previously, the incidence set $\mathcal{I}$ may be singular but is, by definition, a complete intersection of dimension equal to the dimension n of X. Therefore, $\mathbb{Q}_{\mathcal{I}}[n]$ is a perverse sheaf (or, more generally, a mixed Hodge module on $\mathcal{I}$, denoted $\mathbb{Q}^H_{\mathcal{I}}[n]$; see [Sa1; Sa2]). Let $\psi_p\mathbb{Q}_{\mathcal{I}} \in D^b_c(L_0^{\mathcal{I}})$ denote the bounded constructible complex of nearby cycles attached to p. Then $\psi_p\mathbb{Q}_{\mathcal{I}}[n-1]$ is also a perverse sheaf and thus (by Saito's theory) underlies a mixed Hodge module. We consider the shifted complex

$$\mathbf{M}(L_0, p_{\mathbf{L}}) := \psi_p^H \mathbb{Q}^H_{\mathcal{I}}[1], \tag{1.6}$$

where $\mathbb{Q}^H_{\mathcal{I}}$ denotes the "constant" Hodge sheaf and ψ_p^H is the corresponding nearby cycle functor on the level of Saito's mixed Hodge modules (i.e., if rat: $D^b(\mathrm{MHM}(\mathcal{I})) \to D^b_c(\mathcal{I})$ is the forgetful functor associating to a complex in the bounded derived category of mixed Hodge modules the underlying rational constructible complex of sheaves, then $\mathrm{rat} \circ \psi_p^H = \psi_p[-1] \circ \mathrm{rat}$). Hence $\mathbf{M}(L_0, p_{\mathbf{L}})$ is a complex of mixed Hodge modules associated to the pair $(L_0, \mathbf{L})$, which, moreover, is supported only on L_0. We refer to [D2] (see also [Di2]) for the definition of the nearby cycles complex and to [Sa1; Sa2] for the extension of this construction to the category of mixed Hodge modules.

The main result of this paper is the following statement (see also its reformulation in Theorem 2.5).

THEOREM 1.1. *Let $\mathcal{S}$ be a Whitney stratification of L_0 such that the base locus of $\mathbf{L}$ (i.e., $B_{\mathbf{L}} = L_0 \cap L_t$) is a union of strata of $\mathcal{S}$. Then to each stratum $S \in \mathcal{S}$ one can associate a Hodge polynomial invariant $\chi^c_y(S, \mathbf{L})$ such that*

$$\chi_y(L_t) = \sum_{S \in \mathcal{S}} \chi^c_y(S, \mathbf{L}). \tag{1.7}$$

More precisely, $\chi_y(L_t)$ is the total χ_y-genus of the complex $\mathbf{M}(L_0, p_{\mathbf{L}})$ associated to the pair $(L_0, \mathbf{L})$ as in (1.6). *In particular: if the monodromy of the restriction of $\mathbf{M}(L_0, p_{\mathbf{L}})$ to each stratum is trivial (or, more generally, has finite order) and if the corresponding local system extends to the closure of the stratum, then*

$$\chi_y(L_t) = \sum_{S} \chi^c_y(S) \cdot \chi_y(M_S). \tag{1.8}$$

Here M_S is the Milnor fiber (in the incidence variety $\mathcal{I}$ of the pencil) corresponding to a point in the stratum S of L_0. The specialization of (1.8) *for $y = -1$ that yields the equality for Euler characteristics is valid without any monodromy assumption.*

REMARK 1.2. It will follow from the proof of Theorem 1.1 (see also [CLMSh2, eq. (73)]) that each polynomial $\chi^c_y(S, \mathbf{L})$ is actually an alternating sum of Hodge polynomials of S with coefficients in admissible (at infinity) variations of mixed Hodge structures. Such "twisted Hodge polynomials" can be computed by means of Atiyah–Meyer-type formulas, as in [CLMSh1; CLMSh2], in terms of the Deligne extension of the underlying local system on a "good" compactification of S. Of course, in the simple case where the monodromy along each stratum is trivial, the underlying variations are constant and we have the multiplicative formula of (1.8). However, we wish to emphasize that the same phenomenon persists when the monodromy has finite order—provided the underlying local system extends to a (possibly singular) compactification of the stratum (e.g., the stratum closure); see Lemma 2.3. Therefore, (1.7) provides a complete calculation for the Hodge polynomial of L_t in terms of invariants of the singularities of L_0.

An important consequence of the proof of Theorem 1.1 is an identity comparing the Hodge polynomials $\chi_y(L_0)$ and $\chi_y(L_t)$ (see Theorem 2.5 for the precise formulation). For example, suppose that all strata of a stratification $\mathcal{S}$ as just described are simply connected; then

$$\chi_y(L_0) = \chi_y(L_t) - \sum_S \chi_y^c(S) \cdot \chi_y([\tilde{H}^*(M_S; \mathbb{Q})]), \tag{1.9}$$

where the summation runs over only the singular strata of $\mathcal{S}$ (i.e., strata S satisfying $\dim(S) < \dim(L_0)$). This result is in the spirit of the Parusiński–Pragacz works [PPr1; PPr2] and should be regarded as a Hodge-theoretic extension to arbitrary pencils of the formula from [PPr1]. As explained in Section 3, this formula imposes strong restrictions on the type of singularities that the pencil's singular fiber can have.

REMARK 1.3. A characteristic class version of the Parusiński–Pragacz formula for the generalized Milnor number was obtained in [PPr2] (see also [S1; S2; Y]) by studying the Milnor class of a complex hypersurface—that is, the difference between the Fulton–Johnson class [FJ] and the Chern–MacPherson class [Mac]. As in this paper, the key tools used in [PPr2] are the complexes of nearby and vanishing cycles together with a specialization argument. That being said, the difference between the Hodge polynomials of the singular and the generic fiber of a 1-parameter family $\{X_t\}$ of compact complex hypersurfaces can be expressed as the degree of a certain Hodge-theoretic Milnor class, which is defined as the difference between the motivic Hirzebruch class of X_0 (cf. [BrSY]) and the Hirzebruch class of its virtual tangent bundle in the ambient smooth variety. This generalized Milnor class associated to a divisor in a complex algebraic manifold was studied in [CMSSh], but it is not immediately clear how the results there can be adapted to our setup of pencils with nonempty base locus. This problem will be addressed elsewhere.

Let us elaborate on the computational aspects of Theorem 1.1. Note that the stalk of the cohomology sheaf $\mathcal{H}^\bullet(\mathrm{rat}(\mathbf{M}(L_0, p_{\mathbf{L}})))$ at any point $B \in L_0$ is a graded algebra, which we denote by $\mathcal{H}^\bullet(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$. More explicitly, it follows by construction that

$$\mathcal{H}^\bullet(\mathbf{M}(L_0, p_{\mathbf{L}}))_B = H^\bullet(M_S; \mathbb{Q}); \tag{1.10}$$

here, as before, M_S is the Milnor fiber in $\mathcal{I}$ corresponding to the stratum S of L_0 (or of $L_0^{\mathcal{I}}$) containing the point B. As regards the last sentence of Theorem 1.1, we note that the Euler characteristic of M_S can be computed via the following version of A'Campo's formula.

PROPOSITION 1.4. *Let $\pi_{\mathcal{I}} \colon \tilde{\mathcal{I}} \to \mathcal{I}$ be the restriction to the proper preimage of $\mathcal{I}$ of an embedded resolution $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$ of singularities of the triple $(X \times \mathbf{L}, \mathcal{I}, L_0)$. In other words, $\tilde{\mathcal{I}}$ is an embedded resolution of singularities of $\mathcal{I} \subset X \times \mathbf{L}$, and the components $E_{\tilde{\mathcal{I}},k}$ of the exceptional locus of $\pi_{\mathcal{I}}$ are the intersections of the components E_k of the exceptional locus of $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$ with $\tilde{\mathcal{I}}$; moreover, the proper transform $\tilde{L}_0 \subset \tilde{\mathcal{I}}$ of L_0 and the components $E_{\tilde{\mathcal{I}},k} \subset \tilde{\mathcal{I}}$ of the exceptional locus $E_{\tilde{\mathcal{I}}} = \bigcup E_{\tilde{\mathcal{I}},k}$ of $\pi_{\mathcal{I}}$ form a normal crossings divisor in $\tilde{\mathcal{I}}$.*

Let m_{E_k} be the multiplicity of the pullback of $p_{\mathbf{L}} \colon X \times \mathbf{L} \to \mathbf{L}$ along $E_k \subset \widetilde{X \times \mathbf{L}}$. Let D_B be a ball in a germ of a smooth subspace of X that is transversal at $B \in X$ to the stratum of L_0 containing B. Then the Euler characteristic of $\mathcal{H}^\bullet(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ is given by

$$\sum e((E_{\mathcal{I},k} - E_{\mathcal{I},k} \cap \tilde{L}_0) \cap \pi_{\mathcal{I}}^{-1}(D_B)) \cdot m_{E_{\mathcal{I},k}}. \tag{1.11}$$

Proof. The proof follows by standard arguments used in the proof of A'Campo's formula for the Euler characteristic of the monodromy of the generic fiber of a base point free pencil. We apply such arguments to the restriction of the pullback of $p_{\mathbf{L}}$ to $\widetilde{X \times \mathbf{L}}$ on an appropriate subspace of the latter. More precisely, let H be a germ of a smooth submanifold in X containing $B \in L_0$, and let D_B be a small ball about B in H. Then the proper preimage of D_B in $\tilde{\mathcal{I}}$ is a resolution of its preimage in $\mathcal{I}$. (Indeed, a small neighborhood of B in X can be decomposed as $D_S \times D_B$, where D_S is a neighborhood of B in the stratum S and where the map $\tilde{\mathcal{I}} \to \mathcal{I}$ is a locally trivial fibration over D_S with fiber the total preimage of D_B in $\mathcal{I}$; hence this total preimage is also smooth.) Let $t \in D_\varepsilon(0) \subset \mathbf{L}$ with ε sufficiently small that the fibers L_t are transversal to D_B for $t \neq 0$ and yield a fibration of the preimage of D_B in $\mathcal{I}$ with one degenerate fiber $L_0 \cap D_B$. Now we apply A'Campo's formula to the morphism of the proper preimage $\tilde{D}_B$ of D_B in $\tilde{\mathcal{I}}$ that is the restriction on the latter of the pullback of $p_{\mathbf{L}}$ on $\tilde{\mathcal{I}}$. The components of the exceptional locus of $\pi_{\mathcal{I}}|_{\tilde{D}_B}$ are the intersections of the components of the exceptional locus of $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$ (i.e., $E_{\tilde{\mathcal{I}},k}$. By transversality, the multiplicity of $p_{\mathbf{L}}$ along E_k and along its restriction on $\tilde{D}_B$ along $E_{\mathcal{I},k}$ are the same. Hence (1.11) follows from A'Campo's result [A] because, as pointed out earlier, the Euler characteristic of $\mathcal{H}^\bullet(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ coincides with the Euler characterisitc of the Milnor fiber in the direction transversal to the stratum. □

REMARK 1.5. It follows from (1.10) and Saito's theory (but see also [N1; N2]) that $\mathcal{H}^\bullet(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ also carries canonical mixed Hodge structures. This property will be needed in the proof of Theorem 1.1.

Let us now illustrate the identity (1.8) with a concrete example.

EXAMPLE 1.6. Let $X = \mathbb{P}^3_{\mathbb{C}}$ and let L_0 be the union of a nonsingular hypersurface V_{d-1} of degree $d-1$ and a transversal hyperplane $H = V_1$. Let $\mathbf{L}$ be the pencil generated by V_d and $V_{d-1} \cup H$, and assume that V_d is transversal to $V_{d-1} \cap H$. Then the stratification of the singular locus $V_{d-1} \cap H$ of $V_{d-1} \cup H$ consists of its intersection with the base point locus of the pencil containing $V_{d-1} \cup H$ and V_d (i.e., $V_{d-1} \cap H \cap V_d$) and the complement to this intersection in $V_{d-1} \cap H$.

For the reader's convenience, we first recall the computation of Euler characteristics (i.e., $y = -1$ in equation (1.8)). The contribution of the stratum $V_{d-1} \cap H - V_{d-1} \cap H \cap V_d$ is 0, the contribution of each point in $V_{d-1} \cap H \cap V_d$ is 1 (provided V_d is transversal to $V_{d-1} \cap H$), and the contribution of each nonsingular point of $V_{d-1} \cup H$ is 1. Since the Euler characteristic of a nonsingular hypersurface of degree d in $\mathbb{P}^3$ is $d^3 - 4d^2 + 6d$, it follows that the Euler characteristics of strata $V_{n-1} - V_{n-1} \cap H$ and $H - V_{d-1} \cap H$ are $(d-1)^3 - 4(d-1)^2 + 6(d-1) - 3(d-1) + (d-1)^2$ and $3 - 3(d-1) + (d-1)^2$, respectively. Then the identity (1.8) for $y = -1$ verifies as

$$\begin{aligned} d^3 - 4d^2 + 6d = {} & (d-1)^3 - 4(d-1)^2 + 6(d-1) - 3(d-1) \\ & + (d-1)^2 + 3 - 3(d-1) + (d-1)^2 + d(d-1). \end{aligned}$$

Let us next discuss the case of Hodge polynomials, assuming for simplicity that $d = 4$. Following [H], we have

$$\chi_y(V_4) = 2 - 20y + 2y^2, \qquad \chi_y(V_3) = 1 - 7y + y^2,$$
$$\chi_y(V_1) = 1 - y + y^2, \qquad \chi_y(V_3 \cap V_1) = 0.$$

From the additivity of the χ_y^c-polynomial we obtain

$$\chi_y^c(V_3 - V_3 \cap V_1) = 1 - 7y + y^2, \qquad \chi_y^c(V_1 - V_3 \cap V_1) = 1 - y + y^2,$$
$$\chi_y^c(V_3 \cap V_1 - V_3 \cap V_4 \cap V_1) = -12.$$

Moreover, the contribution of the fibers of the nearby cycles over $V_3 \cap V_1 - V_3 \cap V_1 \cap V_4$ amounts to $1 + y$. This corresponds to the Hodge polynomial of the Milnor fiber of a node singularity because the monodromy is trivial. (In local coordinates near a base point, the pencil has the form $xy + zt = 0$ for $xy = 0$ and $z = 0$ the local equations of the reducible and irreducible fibers, respectively; the germ of the stratum of the singular member of the pencil is given by $x = y = 0$ and the monodromy around the base point $x = y = z = 0$ of the pencil is given by $z = \exp(2\pi i\theta)$, which is the same as the monodromy action on the cohomology of the Milnor fiber $xy = s$.) Since the contribution of a point in the intersection $V_3 \cap V_1 \cap V_4$ is 1, the identity

$$\begin{aligned} \chi_y(V_4) = {} & \chi_y^c(V_3 - V_3 \cap V_1) + \chi_y^c(V_1 - V_1 \cap V_3) \\ & + \chi_y^c(V_3 \cap V_1 - V_3 \cap V_1 \cap V_4) \cdot \chi_y([(\psi_p \mathbb{Q}_{\mathcal{I}})_{x \in V_3 \cap V_1 - V_3 \cap V_1 \cap V_4}]) \\ & + \chi_y^c(V_3 \cap V_1 \cap V_4) \cdot \chi_y([(\psi_p \mathbb{Q}_{\mathcal{I}})_{x \in V_3 \cap V_1 \cap V_4}]) \end{aligned}$$

becomes the following relation representing (1.8):

$$2 - 20y + 2y^2 = (1 - 7y + y^2) + (1 - y + y^2) + (-12) \cdot (1 + y) + 12.$$

2. Proof of Theorem 1.1

Recall our setting: L_0 is a singular hypersurface of the n-dimensional compact complex algebraic manifold X, and we fix a Whitney stratification of L_0; we assume that L_0 is the singular member of a pencil $\mathbf{L}$ of hypersurfaces with nonsingular generic member L_t, and (if necessary) we refine the stratification of L_0 so that the base locus of $\mathbf{L}$ (i.e., $B_{\mathbf{L}} = L_0 \cap L_t$) is a union of strata; call the new stratification $\mathcal{S}$. The idea of the proof is to lift all the computations at the level of the incidence variety $\mathcal{I}$ of $\mathbf{L}$. More precisely, by using the incidence variety $\mathcal{I}$ of the pencil (see (1.5) for the definition) we construct a 1-parameter family, denoted $\{L_t^{\mathcal{I}}\}$, of compact complex algebraic manifolds degenerating onto the singular variety $L_0^{\mathcal{I}}$. We denote by p the projection map onto a disk $\Delta \subset \mathbb{C}$ (so $L_0^{\mathcal{I}} = p^{-1}(0)$), and we observe that the domain of p is locally a complete intersection of pure dimension n. Furthermore, we note (as in Section 1) that the fibers $L_t^{\mathcal{I}}$ (resp., $L_0^{\mathcal{I}}$) of this family are in fact isomorphic to the generic member L_t (resp., singular member L_0) of the given pencil $\mathbf{L}$. To simplify the exposition, in what follows we omit the index $\mathcal{I}$ when working on the incidence variety.

Consider the nearby cycles $\psi_p \mathbb{Q}_{\mathcal{I}}$ and the vanishing cycles $\phi_p \mathbb{Q}_{\mathcal{I}}$ associated to the 1-parameter family, and note that the following identifications hold:

$$H^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\psi_p \mathbb{Q}_{\mathcal{I}})_x, \qquad \tilde{H}^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\phi_p \mathbb{Q}_{\mathcal{I}})_x. \tag{2.1}$$

Here M_x denotes the Milnor fiber of p at $x \in L_0$ in the incidence variety $\mathcal{I}$ of the pencil, and $\mathcal{H}^*(\cdot)_x$ denotes the stalk cohomology at x (we assume that the vanishing/nearby sheaf complex is constructible with respect to the chosen stratification). In particular, these groups inherit canonical mixed Hodge structures because the nearby and vanishing cycles lift to Saito's category of mixed Hodge modules (cf. [Sa2] or see [N1; N2]).

There is a long exact sequence of mixed Hodge structures (see e.g. [N1; N2] or use that the nearby and vanishing cycles lift to the category of mixed Hodge modules):

$$\cdots \to H^j(L_0;\mathbb{Q}) \to \mathbb{H}^j(L_0;\psi_p\mathbb{Q}) \to \mathbb{H}^j(L_0;\phi_p\mathbb{Q}) \to \cdots, \tag{2.2}$$

with $\mathbb{H}^j(L_0;\psi_p\mathbb{Q})$ carrying the *limit mixed Hodge structure* defined on the cohomology of the canonical fiber L_∞ of the 1-parameter degeneration p (see e.g. [PeSt, Sec. 11.2]) and where $\mathbb{H}^*$ denotes the hypercohomology groups of the corresponding complex of sheaves. The existence of the limit mixed Hodge structure is also a consequence of Saito's theory, since

$$\mathbb{H}^j(L_0;\psi_p\mathbb{Q}) = \mathrm{rat}(H^j(k_*\psi_p^H \mathbb{Q}_{\mathcal{I}}^H[1])) \tag{2.3}$$

for $k\colon \mathcal{I} \to pt$ the constant map. Moreover, a consequence of the definition of the limit mixed Hodge structure is that

$$\dim_{\mathbb{C}} F^p H^j(L_\infty;\mathbb{C}) = \dim_{\mathbb{C}} F^p H^j(L_t;\mathbb{C}) \tag{2.4}$$

(cf. [PeSt, Cor. 11.25]), where L_t is the generic fiber of the family (and of p). Therefore,

$$\chi_y(L_\infty) := \chi_y([\mathbb{H}^\bullet(L_0;\psi_p\mathbb{Q})]) = \chi_y(L_t). \tag{2.5}$$

The rest of the proof follows from Lemmas 2.1, 2.2, and 2.3.

LEMMA 2.1 (Additivity of the χ_y^c-polynomial). *Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Then, for any bounded complex of mixed Hodge modules $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(Z)$*

such that the underlying rational complex $\mathrm{rat}(\mathcal{M}^\bullet)$ *is constructible with respect to* $\mathcal{S}$, *we have*

$$\chi_y([\mathbb{H}^\bullet_c(Z;\mathcal{M}^\bullet)]) = \sum_{S\in\mathcal{S}} \chi_y([\mathbb{H}^\bullet_c(S;\mathcal{M}^\bullet)]). \tag{2.6}$$

Proof. See, for example, [CLMSh2, Cor. 3]. □

LEMMA 2.2 (Trivial monodromy). *In the notation of Lemma 2.1, let* $\mathcal{F}^\bullet$ *denote the rational constructible complex associated to* $\mathcal{M}^\bullet$. *Assume, moreover, that the local systems* $\mathcal{H}^j(\mathcal{F}^\bullet)|_S$ *are constant on* S *for each* $j\in\mathbb{Z}$ *(e.g.,* $\pi_1(S)=0$*). Then*

$$\chi_y([\mathbb{H}^\bullet_c(S;\mathcal{F}^\bullet)]) = \chi^c_y(S)\cdot\chi_y([\mathcal{F}^\bullet_x]), \tag{2.7}$$

where $[\mathcal{F}^\bullet_x] := [i_x^*\mathcal{F}^\bullet] = [\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x] \in K_0(\mathrm{MHS})$ *is the complex of mixed Hodge structures induced by the pullback of* $\mathcal{M}^\bullet$ *over the point* $x\in S$ *under the inclusion* $i_x\colon\{x\}\hookrightarrow S$.

Proof. See, for example, [CLMSh2, Prop. 3]. For the case of coefficients in geometric variations, see [DiLe, Thm. 6.1]. □

If each local system $\mathcal{H}^j(\mathcal{F}^\bullet)|_S$ has a finite-order monodromy and is the restriction of a local system defined on a compactification of the stratum, then by our next lemma one has a similar multiplicative formula.

LEMMA 2.3 (Finite order monodromy extending to a compactification). *Let* S *be a connected complex algebraic manifold of dimension* n, *and let* $\mathcal{V}$ *be a local system on* S *underlying an admissible variation of mixed Hodge structures with quasi-unipotent monodromy at infinity. Assume that the monodromy representation of* $\mathcal{V}$ *is of finite order and that* $\mathcal{V}$ *extends as a local system to some (possibly singular) compactification* $\bar{S}$ *of* S. *Then the twisted Hodge polynomial*

$$\chi^c_y(S;\mathcal{V}) := \chi_y([H^*_c(S;\mathcal{V})])$$

is computed by the multiplicative formula

$$\chi^c_y(S;\mathcal{V}) = \chi^c_y(S)\cdot\chi_y([\mathcal{V}_x]) \tag{2.8}$$

for $[\mathcal{V}_x]\in K_0(\mathrm{MHS})$ *the class of the fiber of* $\mathcal{V}$ *at some point* $x\in S$.

Proof. We begin by remarking that an easy consequence of Saito's theory of mixed Hodge modules [Sa2] is that the groups $H^i_c(S;\mathcal{V})$ carry canonical mixed Hodge structures. Let W be a resolution of singularities of $\bar{S}$, which is an isomorphism over S, such that $D := W\setminus S$ is a simple normal crossing divisor. Denote by $\bar{\mathcal{V}}$ the pullback to W of the extension of $\mathcal{V}$ on $\bar{S}$.

Now observe that the local system $\bar{\mathcal{V}}$ underlies an admissible variation of mixed Hodge structures on W. Indeed, since both S and W are smooth, if $j\colon S\hookrightarrow W$ is the inclusion map then the intermediate extension (cf. [BBeD]) $j_{!*}(\mathcal{V}[n])$ of $\mathcal{V}$ from S to W is given by

$$j_{!*}(\mathcal{V}[n]) = \mathrm{IC}_W(\mathcal{V}) \simeq \mathrm{IC}_W(\bar{\mathcal{V}}) \simeq \mathrm{IC}_W\otimes\bar{\mathcal{V}} \simeq \mathbb{Q}[n]\otimes\bar{\mathcal{V}} \simeq \bar{\mathcal{V}}[n], \tag{2.9}$$

where IC denotes an intersection cohomology sheaf complex. This yields the identification

$$\bar{\mathcal{V}} = j_{!*}\mathcal{V}. \tag{2.10}$$

In addition, for $\mathcal{V}^H$ the smooth mixed Hodge module on S defined by $\mathcal{V}$ (cf. [Sa2]), the isomorphisms in (2.9) can be lifted to the level of algebraic mixed Hodge modules. Hence $j_{!*}\mathcal{V}^H[n]$ is a smooth mixed Hodge module on W and so $\bar{\mathcal{V}} = \mathrm{rat}(j_{!*}\mathcal{V}^H)$ underlies an admissible variation of mixed Hodge structures.

By the additivity of the χ_y^c-polynomial, we clearly have

$$\chi_y^c(S;\mathcal{V}) = \chi_y(W;\bar{\mathcal{V}}) - \chi_y(D;\bar{\mathcal{V}}|_D); \tag{2.11}$$

here, by the inclusion–exclusion principle,

$$\chi_y(D;\bar{\mathcal{V}}|_D) = \sum_{i_0<\cdots<i_k} (-1)^k \chi_y(D_{i_0}\cap\cdots\cap D_{i_k};\bar{\mathcal{V}}|_{D_{i_0}\cap\cdots\cap D_{i_k}}) \tag{2.12}$$

for D_i the irreducible components of the divisor D. Of course, identities similar to (2.11) and (2.12) hold for the usual Hodge polynomials with trivial coefficients (corresponding to the constant variation $\mathbb{Q}$). Since D is a simple normal crossing divisor on W, it follows that the intersections of its components are algebraic submanifolds. So, in order to prove (2.8), it suffices to show the following: *If X is a* compact *complex algebraic manifold and if $\mathcal{V}$ is an admissible variation of mixed Hodge structure on X with finite order monodromy, then*

$$\chi_y(X;\mathcal{V}) = \chi_y(X)\cdot\chi_y([\mathcal{V}_x]). \tag{2.13}$$

This claim can be proved by using the Atiyah–Meyer-type results from [CLMSh2]. Indeed, if

$$\chi_y(\mathcal{V}) := \sum_p [\mathrm{Gr}^p_{\mathcal{F}}(\mathcal{V}\otimes_{\mathbb{Q}}\mathcal{O}_X)]\cdot(-y)^p \in K^0(X)[y,y^{-1}]$$

is the K-theory χ_y-characteristic of $\mathcal{V}$ (with $\mathcal{F}^\bullet$ the corresponding filtration on the flat bundle $\mathcal{V}\otimes_{\mathbb{Q}}\mathcal{O}_X$), then by [CLMSh2] we have

$$\chi_y(X;\mathcal{V}) = \int_{[X]} \mathrm{ch}^*(\chi_y(\mathcal{V}))\cup\tilde{T}^*_y(TX), \tag{2.14}$$

where $\tilde{T}^*_y(TX)$ is the unnormalized Hirzebruch class of (the tangent bundle of) X that appears in the generalized Hirzebruch–Riemann–Roch theorem [H]. Recall that in this case

$$\chi_y(X) = \int_{[X]} \tilde{T}^*_y(TX).$$

The claim in (2.13) follows if we can show that the bundles $\mathrm{Gr}^p_{\mathcal{F}}(\mathcal{V}\otimes_{\mathbb{Q}}\mathcal{O}_X)$, $p\in\mathbb{Z}$, are flat. Because flatness is a local property, it is enough that we check for it on a finite cover. For this we make use of the finite monodromy assumption. Indeed, there is a finite cover $p\colon X'\to X$ on which the pullback of the local system $\mathcal{V}$ becomes constant. By rigidity, the pullback variation underlying this local system is constant; hence the Hodge filtration (and its graded pieces) for the associated flat bundle $p^*\mathcal{V}\otimes_{\mathbb{Q}}\mathcal{O}_{X'}$ is by trivial bundles. Since these are all pullbacks of the corresponding bundles from X, the claim follows. □

REMARK 2.4. Our methods for proving Theorem 1.1 (e.g., our use of the limit mixed Hodge structure and of Atiyah–Meyer-type formulas) cannot be used to consider more general Hodge-theoretic invariants such as the Hodge–Deligne E-polynomial, which also takes into account the weight filtrations.

An important reformulation of Theorem 1.1 is the following generalization of some results of [CLMSh2]. It explicitly compares the Hodge polynomials of the singular and the generic fiber in a pencil of hypersurfaces (cf. the Parusiński–Pragacz formula [PPr1] in the case of Euler characteristics).

THEOREM 2.5. *With assumptions and notation as in Theorem 1.1, we obtain the following relation between the Hodge polynomials of* L_0 *and of* L_t:

$$\chi_y(L_0) = \chi_y(L_t) - \sum_S \chi_y^c(S; \tilde{\mathbf{M}}(L_0, p_{\mathbf{L}})) \tag{2.15}$$

for $\tilde{\mathbf{M}}(L_0, p_{\mathbf{L}}) := \phi_p^H \mathbb{Q}_{\mathcal{I}}[1]$ *the complex of mixed Hodge modules corresponding to Deligne's constructible complex of vanishing cycles on the incidence variety* $\mathcal{I}$. *Here the summation runs over only singular strata—that is, over strata* $S \in \mathcal{S}$ *such that* $\dim(S) < \dim(L_0)$.

In particular: *if for any* $S \in \mathcal{S}$ *the variations of mixed Hodge structures* $\mathcal{H}^i(\tilde{\mathbf{M}}(L_0, p_{\mathbf{L}}))|_S$, $i \in \mathbb{Z}$, *are constant* (*e.g.*, $\pi_1(S) = 0$) *or have finite-order monodromy representations that extend to the closure* $\bar{S}$ *of the stratum, then*

$$\chi_y(L_0) = \chi_y(L_t) - \sum_S \chi_y^c(S) \cdot \chi_y(\tilde{H}^{\bullet}(M_S; \mathbb{Q})). \tag{2.16}$$

The corresponding Euler characteristic formula (*i.e., for* $y = -1$) *holds without any restrictions on the monodromy along the singular strata of* L_0.

Proof. The underlying rational constructible complex of sheaves for $\tilde{\mathbf{M}}(L_0, p_{\mathbf{L}})$ is the complex $\phi_p \mathbb{Q}$ of the vanishing cycles associated with the 1-parameter family $\{L_t\}$ on the incidence variety. It is supported only on the singular locus of the singular fiber L_0.

The identity in (2.15) follows from the functoriality of the χ_y-genus, the fact that the long exact sequence (2.2) is a sequence of mixed Hodge structures, and the additivity of the χ_y^c-genus of Lemma 2.1. Under the trivial (resp., finite-order) monodromy assumption, Lemma 2.2 (resp., Lemma 2.3) and the identification in (2.1) yield (2.16). □

REMARK 2.6. We wish to emphasize that formulas (1.8) and (2.16), which were obtained in the case of simple monodromy situations at each stratum, admit reformulations expressed entirely in terms of the Hodge polynomials of closures of strata in the singular fiber of the pencil. This makes it easier to identify each of these formulas as the degree-0 part of a hypothetical corresponding characteristic class formula for the motivic Hirzebruch classes of [BrSY]; similar considerations were employed by Parusiński and Pragacz in [PPr1; PPr2]. Indeed, for a given stratum S of a Whitney stratification as in Theorem 1.1, we may inductively define

$$\hat{\chi}_y(\bar{S}) := \chi_y(\bar{S}) - \sum_{P<S} \hat{\chi}_y(\bar{P}), \tag{2.17}$$

where the summation is over (boundary) strata $P \subset \bar{S} \setminus S$. By the additivity of the χ_y^c-genus and since L_0 is compact, it follows easily that we actually have

$$\hat{\chi}_y(\bar{S}) = \chi_y^c(S) = \chi_y(\bar{S}) - \chi_y(\bar{S} \setminus S). \tag{2.18}$$

Hence (1.8), for example, can be now rewritten as

$$\chi_y(L_t) = \sum_S \hat{\chi}_y(\bar{S}) \cdot \chi_y(M_S), \tag{2.19}$$

and similarly for (2.16).

3. Examples and Applications

We conclude this paper with another example and further remarks on the applicability of our results.

3.1. Quadrics

Consider a quadric Q_0 given by the equation

$$f_r(x_0, \dots, x_n) = 0, \tag{3.1}$$

where f_r is a quadratic form of rank r. Then the singular locus $\mathrm{Sing}(Q_0)$ is a linear space of dimension $n - r$.

Perhaps the easiest way to calculate the Hodge polynomial $\chi_y(Q_0)$ is to use the fibration of its resolution $\tilde{Q}_0$, which is the proper preimage of Q_0 in the blowup of $\mathbb{P}^n$ at $\mathrm{Sing}(Q_0) \subset \mathbb{P}^n$. More precisely, we have the following fibration with fiber $\mathbb{P}^{n-r+1}$ and base a nonsingular quadric Q_{ns}^{r-2} of dimension $r-2$:

$$\tilde{Q}_0 \xrightarrow{\mathbb{P}^{n-r+1}} Q_{\mathrm{ns}}^{r-2}. \tag{3.2}$$

The exceptional locus of the resolution (3.2) is a Q_{ns}^{r-2}-fibration over $\mathbb{P}^{n-r}$. Hence:

$$\begin{aligned}\chi_y(Q_0) &= \chi_y(Q_{\mathrm{ns}}^{r-2})(-1)^{n-r+1}y^{n-r+1} + \chi_y(\mathbb{P}^{n-r})\\ &= \left(\sum_{i=0}^{i=r-2}(-1)^i y^i + \frac{(1+(-1)^r)}{2}(-y)^{r-2}\right)(-1)^{n-r+1}y^{n-r+1}\\ &\quad + \sum_{i=0}^{i=n-r}(-1)^i y^i.\end{aligned}$$

This calculation can also be performed using our results from Theorem 1.1 or Theorem 2.5 with strata of the stratification that is suitable for applying Theorem 1.1 to the pencil generated by Q_0 and the generic quadric Q_1 consisting of

$$\begin{aligned}S_1 &= Q_0 - \mathrm{Sing}(Q_0) - Q_1 \cap Q_0,\\ S_2 &= \mathrm{Sing}(Q_0) - \mathrm{Sing}(Q_0) \cap Q_1,\\ S_3 &= \mathrm{Sing}(Q_0) \cap Q_1.\end{aligned} \tag{3.3}$$

(Note that S_3 is a generic quadric in $\mathbb{P}^{n-r}$.) We leave the details of this calculation as an exercise for the interested reader.

3.2. Miscellanea

Similar calculations can be done for other singular hypersurfaces of low degree. For example, using Theorem 2.5 and the classification of cubic surfaces with 1-dimensional singular locus given in [BruW], one can obtain χ_y-polynomials of all singular cubic surfaces in $\mathbb{P}^3$ with nonisolated singularities. The possibilities consist of (i) irreducible surfaces that are cones over nodal and cuspidal plane cubics and (ii) surfaces given by

$$F : x_0^2 x_2 + x_1^2 x_3, \qquad G : x_0^2 x_2 + x_0 x_1 x_3 + x_1^3. \tag{3.4}$$

For a cubic in any dimension, the singular locus of codimension 1 is a linear space becuase a transversal plane section is an irreducible cubic and so has only one singularity. It is not difficult to solve explicit formulas for the χ_y-polynomials in this case as well.

We remark that the images of generic projections $X^n \to \mathbb{P}^{n+1}$ provide an interesting class of hypersurfaces with codimension-1 singular locus. The numerology of singularities is given in [K]. Our Theorem 1.1 can be used to compute Hodge-theoretical invariants of strata.

Finally, note that the relation between the Euler characteristic of a singular curve in $\mathbb{P}^2$ and its smoothing places an important restriction on the number of singular points of a curve. Similarly, Theorem 1.1 yields restrictions on data of singular strata in higher dimensions.

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