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Fundamental Groups of the Complements to Plane Singular Curves

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1. Introduction. The purpose of these notes is to give an exposition of some recent developments in the study of the fundamental groups of the complements to plane algebraic curves. Historically the subject was very active in the first third of the century due to its role in viewing algebraic surfaces as multiple planes and in the study of the families of plane curves with fixed number of nodes and cusps. The results of this period are presented in Chapter VIII of Zariski's *Algebraic Surfaces* [Z1]. 35 years later, in an appendix to the new edition of this book, D. Mumford remarked,

> "The classification of plane curves C with d nodes and k cusps and the computation of $\pi_1(P^2 - C)$ has unfortunately not been pursued. Zariski's techniques in §3, theorem 1 are closely connected to certain techniques in knot theory. For instance the polynomial $f(x)$, which is determinant appearing in the proof of the theorem 1, §3, is analogous to the Alexander polynomial. If $\Gamma = \pi_1(P^2 - C)$, $\Gamma' = [\Gamma, \Gamma]$, $\Gamma'' = [\Gamma', \Gamma']$ it would be interesting to investigate the structure of Γ'/Γ'' as $Z[\Gamma/\Gamma']$ module..." [Mu].

Here we shall describe the progress made in the directions pointed out in this quotation. The background describing the results known at the time when Zariski's book was written and which weren't developed further, examples, and some generalities are outlined in the rest of this introduction. §2 presents the basics of the techniques of B. Moishezon of braid monodromies, which are closely related to the study of the fundamental groups of the complements of plane curves, especially those which appear in the study of algebraic surfaces via generic projections. §3 surveys the results on Alexander invariants of plane curves. In particular we give examples of the direct computation of Alexander modules for

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certain curves avoiding calculations of the fundamental groups. In §4 we describe M. Nori's theory allowing establishment of the commutativity of $\pi_1(P^2 - C)$ in many cases. Note that we concentrate primarily on the fundamental group of the complements and omit many important developments in the study of plane singular curves since Mumford's appendix, e.g., J. Harris's proof of irreducibility of the families of plane curves with a fixed number of nodes and no cusps ("Severi's conjecture"), deformation theory for the curve with nodes and cusps [Tan, No], possible incompleteness of the characteristic series of complete continuous families of plane curves with cusps and nodes [Wahl], complements to unions of lines [OS], and others.

The problem of finding for which values of n, d, k there exists an irreducible curve of degree n with d nodes and k cusps is largely untouched since Zariski's survey [Z1]. Recall that some restrictions, going back to Lefschetz, come from the fact that if a curve with such n, d, k should exist then the numbers obtained from the Plücker formulas for the degree and the numbers of nodes and cusps of dual curve should be nonnegative. Zariski [Z2, Z3] showed that these restrictions are insufficient to guarantee the existence of a curve. For example, he showed that there is no curve of degree 7 (resp. 8) with 11 (resp. 16) cusps and no nodes. Recently, however, new restrictions on possible values of d and k were obtained based on Yau-Miyaoka inequalities. For example, for a curve of even degree one obtains $\frac{9}{2}d + 8k < n(5n - 6)/2$ by applying Yau-Miyaoka to the desingularization of the double cover of P^2 branched over this curve. This implies the following asymptotic for the maximal number of cusps on a curve of degree n: $\overline{\lim}(k(n)/n^2) < 5/16$. Along different lines Varchenko obtains $\overline{\lim}(k(n)/n^2) < 23/72$.

Methods of construction of singular curves are rather limited. They amount to:

(a) considering branching curves of generic projections of surfaces;

(b) taking generic plane sections of discriminants of linear systems on algebraic manifolds (this includes dual curves);

(c) writing explicit equations;

(d) using deformation theory to prove the existence of curves which can be degenerated into curves of types (a), (b), (c).

The curves of type (a) are discussed in §2. Curves $C_{a,b,c}$ given by equation $f_{ac}^b + f_{bc}^a = 0$ where f_l is a generic form of degree l are an example of curves of type (c). The singularities of $C_{a,b,c}$ are given locally by $x^a + y^b = 0$ and are located at the intersection of the curves $f_{ac} = 0$ and $f_{bc} = 0$. The fundamental groups of the complement to such curves $C_{a,b,c}$ were computed in the following cases:

(1) $a = 2$, $b = 3$, c arbitrary. $\pi_1(P^2 - C_{2,3,c})$ has a presentation on two generators α and β with relations $\alpha^2 = \beta^3$ and $\alpha^{2c} = 1$ ([Z4] for $c = 1$, [Tu] any c).

(2) $c = 1$, a and b are relatively prime. $\pi_1(P^2 - C_{a,b,1})$ is the group on two generators α and β with the relation $\alpha^a = \beta^b = 1$ [Oka].

The curves of type (b) are obtained as follows. Let V be an algebraic manifold and L a line bundle. In many cases the set of elements in the linear system $P(H^0(V,L))$ (projective space associated to $H^0(V,L)$) that have a singular set of zeros form a hypersurface, say $D(V,L)$. A generic section of $D(V,L)$ by a plane H is a curve such that

$$\pi_1(H - H \cap D(V,L)) = \pi_1(P(H^0(V,L)) - D(V,L)).$$

An important special case of this construction is the case when V is a subvariety of P^N and $L = O_{P^N}(1)|_V$. Then $D(V,L)$ is just the dual variety of V, i.e., the set of hyperplanes in P^N tangent to V.

If $V = P^1$, $L = O(n)$ then the elements in $P(H^0(V,L))$ can be viewed just as unordered collections of n points in a two-dimensional sphere and

$$P(H^0(P^1, O(h))) - D(P^1, O(n))$$

is the set of collections in which all points are distinct. Therefore

$$\pi_1(P(H^0(P^1, O(n))) - D(P^1, O(n)))$$

is just the braid group of a sphere on n strings (cf. [Bir] and §2). This group is isomorphic to Artin's braid group with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ factored by the normal subgroup generated by the element $\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$. The curve $H \cap D(P^1, O(n))$ has degree $2(n-1)$, and it has $3(n-2)$ cusps and $2(n-2)(n-3)$ nodes. It is the curve which is dual to the generic rational nodal curve of degree n [Z6, DL].

If V is an elliptic curve and D is any divisor of degree n, then this construction applied to $O(D)$ leads to the curve of degree $2n$ with $3n$ cusps and $2n(n-3)$ nodes. The fundamental group of the complement to such a curve is isomorphic to the kernel of the homomorphism of the braid group of the torus (underlying the elliptic curve V) on $H_1(V, Z)$ obtained by sending any braid to the homology class of the sum of the loops traced by the points of D in the motion defined this braid [Z5, DL].

Another case in which the computation of the fundamental group was carried out to the end is $V = P^2$, $L = O_{P^2}(3)$ and $V = P^1 \times P^1$, $L = p_1^* O_{P^1}(1) \otimes p_2^* O_{P^1}(1)$ [DL, L3].

As an example of the use of method (d) above of constructing singular curves, we shall mention Zariski's argument [Z1, p. 223] showing the existence of a curve of degree 6 with 6 cusps not on a conic (an example of a sextic with 6 cusps on a conic is the curve $C_{2,3,1}$ considered above). The curve which is dual to a nonsingular cubic is a sextic with 9 cusps. All these cusps clearly cannot belong to the same conic. The idea is to pick up six cusps not on a conic and to deform the curve to eliminate the remaining 3 cusps. Such a deformation does exist because in a complete family of curves of degree n with k cusps and d nodes the cusps impose independent conditions on the family provided $k < 3n$. (This in turn follows from the fact that in this case the degree of the characteristic series is greater than $2p-2$, where p is the genus of a curve in the system, and hence this series is not special.)

The link between the plane curves with singularities and algebraic surfaces is provided by the existence theorem of Grauert-Remmert-Enriques-Riemann [GR], which implies that given any finite unbranched covering $F: \widetilde{P^2 - C} \to P^2 - C$ (which in turn is uniquely determined by a homomorphism of $\pi_1(P^2 - C)$ into a symmetric group) there exists a normal surface V with a projection $F': V \to P^2$ that is a branched covering of P^2 and such that $F'|_{F'^{-1}(P^2-C)}$ coincides with F. On the other hand, any projective nonsingular surface can be obtained in such a way for example by considering its generic projection on a plane.

Many interesting surfaces can be obtained in such a way (i.e., as multiple planes) even for rather simple branching curves and Galois groups, e.g., to name a few: Enriques surfaces, Campedelli surfaces, and more lately surfaces with $c_1^2 = 3c_2$ [BPV, H]. It would be interesting, however, to understand better this relationship between surfaces and singular curves, for example, to find invariants of surfaces such as Betti and Chern numbers or π_1 in terms of the numerical data of the singular curve such as its degree, the number of singularities of fixed type, their position, and inevitably the fundamental group. Some results of such a type do not require π_1 [H]; others such as the computation of the Betti numbers of cyclic covers do ([Z2, L3] and §3). As pointed out in [M2] (attributed to K. Chakiris) the fundamental group of the complement to a ramification locus may provide a discrete invariant distinguishing different connected components of the moduli spaces of the surfaces of general type. Namely, one can define a map from the set of connected components of the moduli space of surfaces of general type with fixed c_1^2 and c_2 into the set of connected components of a family of plane curves with fixed degree and numbers of nodes and cusps. One can do this by relating to a surface the branching curve of the generic projection of the image of the surface in some P^N under some pluricanonical embedding, say $5K$. It can be verified easily that this map separates components of the moduli space.

2. Braid monodromies and branching loci of generic projections. Recall first the various definitions of Artin's braid group which are useful in the sequel. We shall mention four of them. Let H_1 and H_2 be two parallel planes in $R^3, P_1, \ldots, P_n$ be a set of n distinct points in H_1, and $Q_1, \ldots, Q_n$ be projections of $P_1, \ldots, P_n$ on H_2 along the direction perpendicular to the planes H_1 and H_2. A braid is a union of nonintersecting segments connecting $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ or rather the class of isotopy of such a union leaving $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ fixed (Figure 1). If H_3 is a plane under the lower plane among H_1 and H_2, say H_2, parallel to H_1 and H_2, then any braid between H_1 and H_2 defines the braid between H_1 and H_3 by extending paths between H_1 and H_2 by segments of straight lines perpendicular to H_2 and H_3. This allows us to identify braids between different planes. The composition of the braids between planes H_1 and H_2 and braids between H_2 and H_3 is defined by attaching together portions of the space between H_1 and H_2 and between H_2 and H_3. In such a way

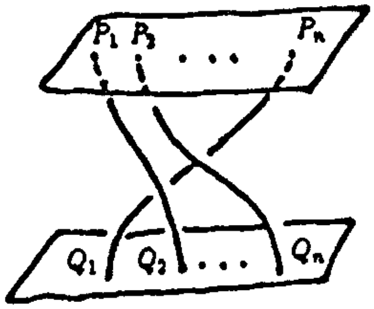


FIGURE 1

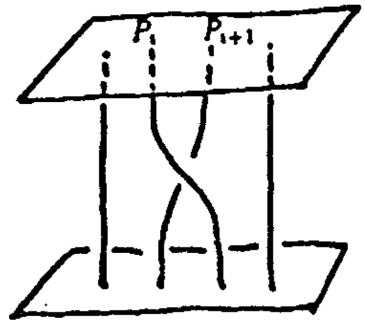


FIGURE 2

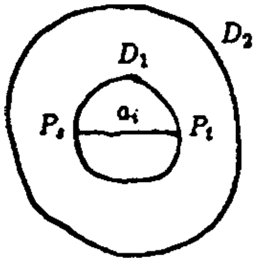


FIGURE 3

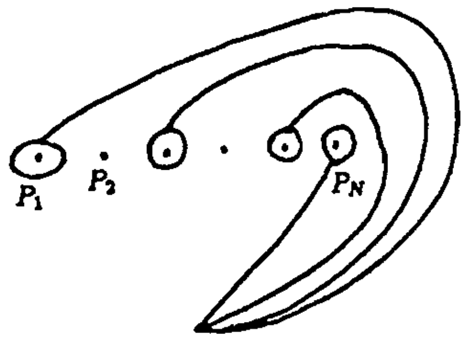


FIGURE 4

one obtains a group B_n which has as generators elements X_i ($i = 1, \ldots, n-1$) corresponding to the braids interchanging P_i and P_{i+1} given in Figure 2. The complete set of relations among X_i's is given by [Ar].

$$(1) \qquad X_iX_j = X_jX_i \quad \text{for } |i-j| \geq 2, \qquad X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}.$$

This presentation can be taken as a definition of the braid group B_n.

The braid group B_n is isomorphic also to the group of isotopy classes of orientation-preserving homeomorphisms of the plane which are identity outside of a fixed disk and which map into itself a fixed set of n points $P_1, \ldots, P_n$ inside the disk. The identification of the group of isotopy classes of such homeomorphisms with the braid group B_n introduced above depend on a choice of a system of $(n-1)$ paths $\alpha_1, \ldots, \alpha_{n-1}$ in the plane such that (a) each of the points $P_1, \ldots, P_n$ belongs to one of the paths, and (b) any two paths have no points in common except maybe for one of the points $P_1, \ldots, P_n$. To each path from such a system one can associate the following homeomorphism of the plane. Take a metric in the plane such that the path connecting points, say P_s and P_t, is a diameter of a circle D_1 of radius 1 and such that the concentric circle D_2 of radius 2 contains no points P_i for $i \neq s, t$ (see Figure 3).

Then the homeomorphism associated to α is Dehn's half twist about the boundary of the circle $D_{3/2}$ of the radius 3/2 concentric with D_1 and D_2, i.e., the homeomorphism which is the identity outside of D_2, which is the rotation by π inside D_1 and which is the rotation by $r\pi$ on the circle of radius r concentric with

D_1 and D_2. One can show that these homeomorphisms generate the group of isotopy classes of homeomorphisms of the type described above, and the relations among them are given by (1).

Finally, the braid group can be described as a subgroup of the group of automorphisms of a free group F which take any generator of F into a conjugate of a generator and which leave invariant the product of chosen generators of free group, i.e.,

$$B_n = \left\{ \varphi \in \operatorname{Aut} F_n(x_1, \dots, x_n) | \varphi x_i = T_i x_i T_i^{-1},\ \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \varphi(x_i) \right\},$$

where $F_n(x_1, \dots, x_n)$ is the free group on symbols $x_1, \dots, x_n$ and T_i are certain words in $F_n(x_1, \dots, x_n)$. The isomorphism $B_n \to \operatorname{Aut} F_n$ is given by assigning to a generator X_i from presentation (1) the automorphism of $F_n(x_1, \dots, x_n)$.

(2) $\qquad X_i \colon (x_1, \dots, x_i, x_{i+1}, \dots, x_n) \to (x_1, \dots, x_i x_{i+1} x_i^{-1} x_i, \dots, x_n).$

Geometrically this action is the action of the braid group interpreted as group of the classes of homeomorphisms as above on the punctured plane with the generators of $\pi_1(H - \bigcup_{i=1}^{N} P_i)$ chosen as in Figure 4.

Now we are going to define the braid monodromy of a plane algebraic curve. We shall denote it by C and d will denote its degree. We'll be concerned primarily with the affine portion of this curve but we assume that C is transversal to the line L at infinity. Let $\Pi \colon P^2 - L = \mathbf{C}^2 - \mathbf{C}$ be a linear projection, $\pi = \Pi|_C$, and let $S = \{p_1, \dots, p_N\}$ be the subset of $\mathbf{C}$ consisting of points the π-preimage of which contains less than d elements. Points of S correspond to the members of the pencil of lines defined by the projection Π which either pass through singularities of C or are tangent to C. (Lines corresponding to S we call the singular elements of the pencil.) Let us fix a system of generators $g_1, \dots, g_N$ ($N = \operatorname{card} S$) of $\pi_1(\mathbf{C} - S, p)$ for some $p \in \mathbf{C}$. A trivialization of the locally trivial fibration $\Pi \colon \mathbf{C}^2 - C - \Pi^{-1}(S) \to \mathbf{C} - S$ over a path representing g_1 defines a homeomorphism of $\Pi^{-1}(p)$, which can be chosen to be identity outside of a sufficiently large disk. A choice of a system of paths in $\Pi^{-1}(p)$, as in the aforementioned definition of the braid group as a group of homeomorphisms, defines therefore the homomorphism $\theta \colon \pi_1(\mathbf{C} - S) \to B_2$. This homomorphism θ is called the braid monodromy defined by C (and the choices made all along). Particularly important are the so-called well-ordered systems of generators of $\pi_1(\mathbf{C} - S)$ [M1]. Let us fix a system of paths $\tilde{\gamma}_i$ ($i = 1, \dots, N$) connecting p with the points $P_1, \dots, P_N \in S$ such that any two have only p as a common point. Let D_i be small disks about P_i. Then we define the path γ_i corresponding to P_i as $\gamma_i = (\tilde{\gamma}_i - \tilde{\gamma}_i \cap D_i) \cup \partial D_i$ which runs in a counterclockwise direction. If D_0 is a small disk about p, then the counterclockwise direction defines an ordering of points of $\partial D_i \cap \tilde{\gamma}_i$ ($i = 1, \dots, N$) (cf. Figure 4). Such a system γ_i ($i = 1, \dots, N$) (depending on a choice of $\tilde{\gamma}_i$) is called a well-ordered system of generators of $\pi_1(\mathbf{C} - S, p)$. For any two well-ordered systems of generators $\gamma_1, \dots, \gamma_N$ and $\gamma'_1, \dots, \gamma'_N$, one can find an element $\gamma \in \pi_1(\mathbf{C} - S, p)$ such that $\gamma_1, \dots, \gamma_N)$ can be obtained from $\gamma\gamma'_1\gamma^{-1}, \gamma\gamma'_2\gamma^{-1}, \ldots, \gamma\gamma'_N\gamma^{-1}$ by means of the transformations (2). Indeed $\prod_{i=1}^N \gamma_i$ and $\prod_{i=1}^N \gamma'_i$ represent homotopic unbased loops in $\mathbf{C} - S$ (each is homotopic to a circle containing the set S). Hence they are conjugate in $\pi_1(\mathbf{C} - S, p)$, i.e., $\prod_{i=1}^N \gamma_i = \prod_{i=1}^N \gamma\gamma'_i\gamma^{-1}$ for some γ. Moreover γ_i and $\gamma\gamma'_{s(i)}\gamma^{-1}$ are conjugate for some permutation s of indices $1, \ldots, N$. Therefore the automorphism of the free group $\pi_1(\mathbf{C} - S, p)$, taking $\gamma_i \to \gamma\gamma'_{s(i)}\gamma^{-1}$ is the action of a braid on $\pi_1(\mathbf{C} - s, p)$, and the claim follows.

The collection of elements $\theta(\gamma_i)$ in the braid group constructed using the braid monodromy θ for some well-ordered system of generators of $\pi_1(\mathbf{C} - S, p)$ is not arbitrary and it is of great interest to find conditions which $\theta(\gamma_i)$ satisfy. One of them is that $\theta(\gamma_i)$ belong to a semigroup in B_d generated by conjugates to the generators X_i of B_d (cf. [M2, Ru]) (which is called the semigroup of positive braids). Indeed a small perturbation of a given curve produces a nonsingular curve C_t such that the lines of the pencil defined by projection have at most simple tangency points with the curve C_t locally given by $x = y^2$ relative to the projection $(x, y) \to x$. Each singular point of C splits into several points in which the lines of the pencil are tangent to C. Suppose that the ith singular point of C splits into n_i such points. Moreover each element $\theta(\gamma_i)$ is replaced by elements $\theta(\gamma_{i,j})$ ($j = 1, \ldots, n_i$), each of which is conjugate to the braid corresponding to the simple tangency point of a line from the pencil. The latter braid is conjugate to a standard generator of B_d because in the case when C is given by $x = y^2$ and the projection by $(x, y) \to x$ the corresponding braid is just the standard generator interchanging a pair of points. On the other hand, for C_t sufficiently close to C we have $\theta(\gamma_i) = \prod_{j=1}^{n_i} \theta(\gamma_{i,j})$ and the claim follows.

Another restriction on collections of elements $\theta(\gamma_i)$ is given by the following:

PROPOSITION 1 [M1, Ch]. *$\prod_{i=1}^N \theta(\gamma_i)$ is equal to the positive generator of the center of B_d, i.e., to $\Delta^2 = \Pi(X_1 \cdots X_{d-1})^d$. Note that the product in the statement is well defined because the system of paths γ_i is ordered.*

PROOF. Let us consider a deformation of a given curve C into a curve U which is a union of lines passing through a point. This can be done, for example, by considering C as a section of a cone by a plane and then by moving this plane into the one passing through the vertex of the cone. The element $\prod_{i=1}^N \theta(\gamma_i)$ in the braid group can be viewed as the monodromy along a loop containing the whole set S of singular points of the chosen projection of C, and clearly this monodromy is unchanged in the deformation of C into U. Furthermore, we can assume that U is given the equation $x^d = y^d$. Let $x = \exp(2\pi i\varphi)$. If $\varphi = 0$ we obtain in the fibre of the pencil d points located on the unit circle. Now the increase of φ by $1/d$ amounts to rotation of this circle by $2\pi/d$, which corresponds to the braid $X_1 \cdots X_{d-1}$. Therefore the change of between 0 and 1 amounts to one full twist of the unit circle mentioned above, i.e., corresponds to $\Delta^2 = \Pi(X_1 \cdots X_{d-1})^d$. Q.E.D.

It is convenient to consider the collection $\theta(\gamma_i)$ as a factorization of the work Δ^2, and that is actually what Moishezon calls the braid monodromy. (Chisini

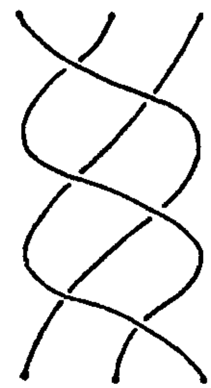


FIGURE 5. Δ^2 for $d = 3$.

and his school were calling this situation a helicoidal braid (i.e., Δ^2) separated into pieces corresponding to the singular points of the pencil by the diaphragms [Ch].)

Braid monodromy determines the fundamental group of $\mathbf{C}^2 - C$. Moreover one can recover from it even the homotopy type of $\mathbf{C}^2 - C$. To describe how to do this, recall that to any presentation of a group G with generators $e_1, \ldots, e_n$ and relations $R_1, \ldots, R_N$ one can associate a 2-dimensional complex with single 0-cell, 1-cells corresponding to the generators $e_1, \ldots, e_n$, 2-cells corresponding to N relations of the chosen presentation and such that the attaching map of ith cell ($i = 1, \ldots, N$) takes its boundary into the class R_i in $\pi_1(S^1 \vee \cdots \vee S^1)$. Note also that the decomposition of γ_i as union $(\gamma_i - \tilde{\gamma}_i \cap D_i) \cup \partial D_i \cup (\tilde{\gamma}_i - \tilde{\gamma}_i \cap D_i)^{-1}$ corresponds to the splitting of $\theta(\gamma_i)$ as $Q_i \beta_i Q_i^{-1}$, where Q_i is a trivialization of $\mathbf{C}^2 - C|_{\gamma_i}$ composed with fixed identification of $\Pi^{-1}(\tilde{\gamma}_i - \partial D_i)$ with $\Pi^{-1}(p)$ (for example from a decomposition $\mathbf{C} = \mathbf{C} + \mathbf{C}$), β_i represents the local monodromy near P_i. Usually one takes a system of generators of $\pi_1(\Pi^{-1}(P_i))$ and generators of the corresponding braid group in such a way that the β_i's have simple form reflecting the type of singularities at the point corresponding to P. For example, for simple tangency point (resp. node, resp. cusp) for appropriate choices can be made equal to one (resp. square, resp. cube) of a generator (cf. Figure 6).

THEOREM 2 [L4]. *Let C be a curve in $\mathbf{C}^2$ with the braid monodromy $\prod_{i=1}^{N} Q_i \beta_i Q_i^{-1} = \Delta^2$. Let m_i denote the multiplicity of the singular point of C corresponding to P_i. (Notations are as in the text above.) Assume that the braid β_i acts nontrivially on $e_{j_1}, \ldots, e_{j_{m_i}}$ and acts trivially on the rest of the generators of $\pi_1(\Pi^{-1}(\tilde{\gamma}_i \cap \partial D_i))$. Then $\mathbf{C}^2 - C$ has the homotopy type of a 2-dimensional complex corresponding to the following presentation of $\pi_1(\mathbf{C}^2 - C)$: $\{e_1, \ldots, e_d | Q_i \beta_i(e_j) = Q_i(e_j),\ j = j_1, \ldots, j_{m_i - 1}\}$.*

For example, using the computations [M1] described below one obtains that the complement to the affine portion of a sextic with six cusps on a conic has the homotopy type of the wedge of 13 copies of a 2-dimensional sphere with the complement to the trefoil knot in S^3.

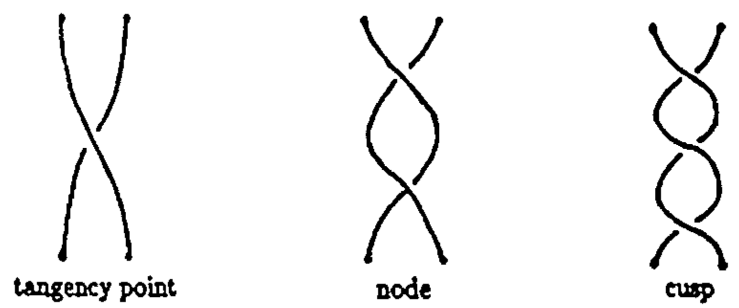


FIGURE 6

Computation of braid monodromy at this point is a rather painful process. In works [M1, M2, M3] certain techniques are developed for this in the case of the curves which are branching loci of generic projections of surfaces. Numerical invariants of such surfaces are easy to find. We have the following:

PROPOSITION 3. *Let $V \subset \mathbf{C}P^N$ be a nonsingular surface, $p\colon V \to \mathbf{C}P^2$ a generic projection, and $B \subset \mathbf{C}P^2$ the branching curve of p. Let c_1^2 and c_2 be Chern numbers of V and let H be the class of hyperplane section. Then B has degree $3H^2 - (c_1, H)$, the number of cusps of B is equal to $12H^2 - 9(c_1, H) + 2c_1^2 - c_2$, and the number of nodes is $\frac{1}{2}(((c_1,H)^2)^2 - 6(c_1,H)\cdot H^2 + 9(H^2)^2) + 30(c_1,H) - 6c_1^2 + 2c_2 - 42H^2$.*

PROOF. Clearly $K_V = p^*(K_{\mathbf{P}^2}) + R$, where $R \subset V$ is the ramification locus. $B = p(R)$, and R is the normalization of B. Hence $\deg B = (H, R) = (K_V + 3H)H$ and the first formula follows. Next recall that the set of cusps of B is the image of the Boardman stratum $\Sigma^{1,1}$ and that the homology class of $\Sigma^{1,1}$ is given by $(c_1^2 + c_2)(p^*T_{\mathbf{P}^2} - T_V) \cup [V]$ (cf. [Ro]). This implies the formula for the number of cusps. Finally we have

$$\chi(R) = \chi(B) + d = \deg B(3 - \deg B) + 2k + 2d.$$

Using the adjunction formula $\chi(R) = -R(R + K_V)$ and the formula for the class of R used above, we arrive at the expression for the number of nodes d.

For example, we obtain, as the branching curve of the generic projection of the image F_n of P^2 under the map defined by the linear system $H^0(P^2, O(n))$, a curve of degree $3n(n-1)$ with $12n^2 - 27n + 15$ cusps and

$$\frac{3}{2}(n-1)(n-2)(3n^2+3n-8)$$

nodes. For the branching curve of the generic projection of the surface $\chi_{a,b}$, which is the image of the quadric $P^1 \times P^1$ under the map defined by the linear system $H^0(P^1 \times P^1, p_1^* O_{P^1}(a) \otimes p_2^* O_{P^1}(b))$ (where p_1, p_2 are projections of $P^1 \times P^1$ onto its factors), we have the degree equal to $6ab - 2a - 2b$, $k = 24ab - 18a - 18b + 12$, and $d = 4(a+b)^2 - 24ab(a+b) + 36a^2b^2 + 60(a+b) - 84ab - 40$ (cf. [MT]). In the case of a nonsingular surface V_n in P^3 the branching curve has degree $n(n-1)$, $k = n(n-1)(n-2)$, and $d = (1/2)(n-1)(n-2)(n-3)$.

It is of great interest to characterize the branching curves of generic projections. The following restriction was pointed out in [M1]: for such a curve one has the following congruences: $k \equiv 0 \pmod 3$ and $d \equiv 0 \pmod 4$. (This follows immediately from Proposition 2 using $c_1^2 + c_2 \equiv 0 \pmod{12}$.)

The braid monodromies of the branching curves of surfaces $V_n, F_n, X_{a,b}$ are computed in [M1, M2, M3]. For example, for V_3 one has as the branching curve a sextic with 6 cusps on the conic, and the corresponding braid monodromy is

$$\Delta^2 = [Z_{24} Z_{13} Z_{56} Z_{35} Z_{12}^3 Z_{34}^3 Z_{56}^3]^2 Z_{24} Z_{13} Z_{50} Z_{35}, \tag{3}$$

where $Z_{ij} = X_{j+1} \cdots X_{i+1} X_i (X_{j+1} \cdots X_{i+1})^{-1}$. The corresponding fundamental group (of the complement in the affine plane to the branching curve of the generic projection of V) is the braid group B_n. The results on the braid monodromies of the branching curves of projections of F_n and $X_{a,b}$, or even on π_1 of their complements, are too cumbersome to quote here. Note, however, that this computation led to the proof in [MT] that the Galois covering of P^2, corresponding to the full symmetric group, branched over the branching curve of the generic projection of $X_{a,b}$, is simply connected for a and b relatively prime, and has positive index for $a > 4$, $b > 4$.

3. Alexander modules. An Alexander module of a plane algebraic curve is an invariant of the fundamental group of its complement. It turns out that the first homology group of the cyclic covering of P^2 depends only on the Alexander module of the branching curve. On the other hand, the Alexander module can be found in a simple manner in terms of the degree, the local type, and the position of the singularities of the curve. In particular, we shall show below how this allows us to derive information on the fundamental group of the complement to a curve just from the geometry of the set in the plane of singular points of the curve.

The idea of the definition of the Alexander module essentially suggested by Mumford [Mu] (cf. Introduction). However, for a number of technical reasons, it is more convenient to work with the affine portion of the curve. One of the reasons is to allow coverings of P^2 of arbitrary degree. If C is an irreducible curve of degree n and L is a line in infinity, then $H_1(P^2 - C, Z) = Z/nZ$ while $H_1(P^2 - C - L, Z) = Z$. The compactified cyclic covering of degree m of $P^2 - C - L$ has canonical map onto P^2 branched over C and for certain m over L. Let $\Gamma = \pi_1(P^2 - C - L)$ (it depends in general on a choice of L), $\Gamma' = [\Gamma, \Gamma]$ is the commutator subgroup, and $\Gamma'' = [\Gamma', \Gamma']$. For irreducible C we have $\Gamma/\Gamma' = Z$.

DEFINITION [L1, L3]. The Alexander module $A(C, Z)$ of C relative to L is Γ'/Γ'' considered as module over $Z[\Gamma/\Gamma']$, where the action of Γ/Γ' is obtained from the exact sequence

$$0 \to \Gamma'/\Gamma'' \to \Gamma/\Gamma'' \to \Gamma/\Gamma' \to 0.$$

$Z[\Gamma/\Gamma']$ is actually just the ring $Z[t,t^{-1}]$ of the Laurent polynomials with the integral coefficients. We will be concerned with the rationalized Alexander module $A(C,Q) = A(C,Z) \otimes Q$, though a number of results have been obtained over Z/pZ [L2]. For an irreducible curve C, $A(C,Q)$ is a $Q[t,t^{-1}]$-torsion module (cf. [L1]) and hence is isomorphic to $\bigoplus_{i=1}^{l} Q[t,t^{-1}]/(\lambda_i)$ for some integer l, where (λ_i) denotes the principal ideal generated by the Laurent polynomial λ_i. The order of $A(C,Q)$ as a $Q[t,t^{-1}]$ module, i.e., $\Delta(C) = \prod_{i=1}^{l} \lambda_i$, is called the Alexander polynomial of C.

Another description of the Alexander module comes from the fact that it can be identified with the fundamental group of the infinite cyclic cover of $P^2 - C - L$ and Γ'/Γ'' is just H_1 of this cover. The action of Γ/Γ' on Γ'/Γ'' is the usual action of the group of deck transformations on the homology of the covering.

The relation between $\pi_1(P^2 - C \cup L)$ and the homology of a desingularization of a cyclic m-fold cover X_m of P^2 branched along C and L can be described as follows. The rank of $H_1(X_m, \mathbf{C})$ is just the sum over i of the number of common roots of λ_i and $t^m - 1$ [L1]. The proof can be obtained by showing that the first Betti number of the m-fold unbranched cover of $P^2 - C \cup L$ is greater by 1 than the first Betti number of a desingularization of X_m and by deriving the formula for the homology of an m-fold cyclic unbranched cover in a way similar to [MI].

Next we shall describe the dependence of the Alexander module on the degree of the curve and on the local structure of the singularities. For each singular point p_i $(i = 1, \ldots, N)$ of C let S_i be the boundary of a small ball about the point p_i and let S_∞ denote the boundary of a sufficiently small tubular neighborhood of L in P^2. For each branch C_{ij} of the curve C at p_i let $\sim$ denote the infinite cyclic cover of the corresponding space relative to the map $\pi_1(\widetilde{S_i - S_{ij}}) \to Z$ defined by appropriate linking number. Then the local Alexander module corresponding to C_{ij} (resp. the Alexander module at infinity) is $H_1(\widetilde{S_i - S_i \cap C_{ij}}, Z)$ (resp. $H_1(\widetilde{S_\infty - S_\infty \cap C}, Z)$) considered as a module over the ring of Laurent polynomials. We'll denote these Alexander modules by $A_{C_{ij}}(C,Z)$ and $A_\infty(C,Z)$ respectively. The order $\Delta_{C_{ij}}(C)$ of $A_{C_{ij}}(C,Q)$ is just the characteristic polynomial of the local monodromy of the singularity C_{ij}. For example, if C_{ij} near p_i is given locally by $x^p + y^q = 0$, then $\Delta_{C_{ij}}(C) = (t^{pq}-1)(t-1)/(t^p-1)(t^q-1)$. If C is transversal to L, then the order $\Delta_\infty(C)$ of $A_\infty(C,Z)$ is $(t^n-1)^{n-2}(t-1)$. The analysis of the proof of the theorem from [L1] leads to the following

THEOREM 4. *The maps* $\bigoplus_{i=1}^{N} A_{C_{ij}}(C,Z) \to A(C,Z)$ *and* $A_\infty(C,Z) \to A(C,Z)$ *induced by inclusions are surjective.*

This implies that the Alexander polynomial of a curve divides the product of the local Alexander polynomials of all branches of all singularities of C as well as the Alexander polynomial of C at infinity. For example, if all singularities of C are either nodes or locally given by $x^p + y^q = 0$, then all roots of the Alexander polynomial are roots of unity of degree pq. On the other hand, roots of the Alexander polynomial of a curve of degree n are roots of unity of degree n. This, combined with the fact that the Alexander polynomial (defined up to a unit in the ring of Laurent polynomials) can be normalized in such a way that $\Delta(1) = 1$, implies that if $pq \nmid n$ then $A(C,Q) = 0$. Using this with aforementioned formula for the first Betti number of cyclic covers and specializing this to the case $p = 2, q = 3$ we obtain Zariski's theorem that the cyclic branched cover of P^2 is regular (i.e. irregularity $h^{1,0}$ is zero) unless both the degree of the cover and the degree of the curve are divisible by 6 [Z2].

Another corollary from Theorem 4 is the semisimplicity of $A(C,Q)$. Indeed in this case each $A_{C_{ij}}(C,Z)$ is semisimple because of semisimplicity of monodromy in the unibranched case [Le, A'C].

Next we shall describe the relation between the Alexander module and the geometry of the set of singularities of C. Assume that all singularities locally are given by $x^p + y^q = 0$ or are the nodes (the more general case is discussed in [L3]). Recall that for a finite set N of points in P^2 the superabundance $s_N(k)$ of N relative to the curves of degree k is the difference betweeen the actual and expected dimensions of the space of curves of degree k containing N (the latter one is just $(k+1)(k+2)/2 - \operatorname{card}(N)$). In other words $s_N(k) = \dim H^1(P^2, I_N(k))$, where $I_N(k)$ is the ideal sheaf of N. We can assume that $pq|n$ because otherwise, as noted earlier, $A(C,Q) = 0$.

THEOREM 5. *If C is an irreducible curve of degree n, all singularities which are either nodes or locally given by $x^p + y^q = 0$ for a fixed pair of integers such that $pq|n$, then*

$$A(C,R) = \bigoplus_{S_{Sing}(n-3-(1-1/p-1/q)n)} R[t,t^{-1}]/((t^{pq}-1)(t-1)/(t^p-1)(t^q-1))$$

where Sing is the set of singularities of C.

According to the Cayley-Bacharach theorem [GH] the superabundance of a complete intersection of two curves of degrees a and b relative to the curves of degree $a+b-3$ is equal to 1. For Turpin's and Oka's (cf. Introduction) curves one obtains that the Alexander modules over R are isomorphic respectively to $R[t,t^{-1}]/(t^2-t+1)$ and $R[t,t^{-1}]/((t^{pq}-1)(t-1)/((t^p-1)(t^q-1))$.

Somewhat different approaches to this subject were taken in works [Ran, Esn, and Ko]. R. Randell showed that the Alexander polynomial relative to a generic line in infinity of a curve given by the equation $f(x,y,z) = 0$ can be found as the characteristic polynomial of the monodromy of the two-dimensional singularity at the origin of $f(x,y,z)$. As a corollary he obtained the cyclotomicity of the Alexander polynomial, the normalization $\Delta(1) = 1$, and the fact that the degree of the Alexander polynomial is equal to the first Betti number of the covering of P^2 of degree equal to the degree of the branching curve which it contains in the above discussion. Kohno [Ko] described the Alexander polynomial in purely algebraic fashion from the cohomology groups of the de Rham complex constructed using a flat connection defined in terms of $f(x,y,z)$. H. Esnault [Es] computed, purely algebraically using logarithmic complex, the mixed Hodge structure on the Milnor fibre of the two-dimensional singularity at the origin of

$f(x,y,z)$. Her approach allows her to find, among other things, the first Betti number of the covering of P^2 of degree equal to the degree of the branching curve, thus recovering the phenomenon of dependence of this Betti number on the position of singularities.

4. Abelian fundamental groups. The first systematic work on finding plane curves for which the fundamental group of the complement is abelian was done by S. Abhyankar in the late 1950s [Ab; Z1, Appendix 1 to Chapter VIII]. His approach included the treatment of the complements to the divisors on a nonsingular simply connected surface (actually on any simply connected nonsingular algebraic variety, which is the same as far as the fundamental group is concerned) rather than only the case of curves on P^2. Abhyankar also worked with the algebraic fundamental group and introduced geometric methods in the subject using intersection theory, Bertini's theorem, etc. He showed that if a plane curve has a small number of nodes and cusps relative to its degree then the fundamental group of its complement in P^2 is abelian. His inequalities weren't sharp however. Further improvements were made in [AM, Ed, Pr]. The next step was made by W. Fulton and P. Deligne, who found the long-sought proof of the statement (known as Zariski's conjecture) that the complement to a plane curve with nodes as the only singularities has abelian fundamental group (the case not always covered by Abhyankar inequalities). Their proof (for algebraic and topological fundamental groups respectively) is based on the connectedness theorem and couldn't be extended to arbitrary surfaces. Shortly after that, M. Nori [N] brought a new circle of ideas which we are going to describe. Like Abhyankar he works with an arbitrary nonsingular surface, but does not assume that they are simply connected or projective.

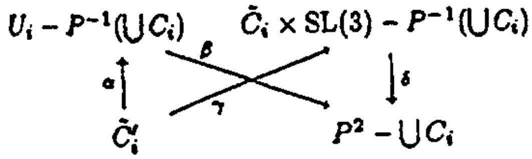
THEOREM 6. *Let D and E be (possibly reducible) curves on a nonsingular projective surface X. Assume that D has only nodes as singularities and that $C^2 > 2r(C)$ for any irreducible component C of the curve D, where $r(C)$ is the number of nodes on C. Then $N = \operatorname{Ker} \pi_1(X - D \cup E) \to \pi_1(X - E)$ is abelian.*

In particular, this implies that $\pi_1(P^2 - D)$ is abelian for any nodal curve D because for any irreducible curve C the maximal number of nodes is $\frac{1}{2}(\deg C - 1)(\deg C - 2)$ and $C^2 = (\deg C)^2 > (\deg C - 1)(\deg C - 2) > 2r(C)$.

To illustrate the ideas involved we shall outline Nori's proof of the commutativity of $\pi_1(P^2 - C)$ for nodal C and then we'll sketch the general case. First note that any loop in a connected topological space defines a conjugacy class in the fundamental group by connecting any point of this loop with the base point. Now if H is a nonsingular curve and T is a tubular neighborhood of H in a nonsingular surface, then the class of the fibre of the restriction of the normal bundle $T - H \dashrightarrow H$ is a circle whose class in $\pi_1(T - H)$ belongs to its center. Let C_i be an irreducible component of C. Because C is nodal, we can assume that C_i is the image of a generic projection $n\colon \tilde{C}_i \to C_i$ of a nonsingular curve $\tilde{C}_i$. We can assume that $\tilde{C}_i$ belongs to a surface in which we consider a sufficiently small tubular neighborhood U_i. Genericity of projection means that the projecting cone is transversal to U_i, which implies that the projection $p\colon U_i \to P^2$ is a local homeomorphism.

The main step is to show that the map $p_*\colon \pi_1(U_i - p^{-1}(\bigcup C_i)) \to \pi_1(P^2 - C)$ is surjective for any i. If so then the conjugacy class in $\pi_1(P^2 - C)$ of a fibre of the normal bundle of C_i contains a central element in $\pi_1(P^2 - C)$. But $\pi_1(P^2 - C)$ is generated by the conjugates to the fibres of normal bundles and hence the commutativity of $\pi_1(P^2 - C)$ follows.

To show the surjectivity of p_* let us consider the action $\theta\colon \mathrm{SL}(3) \times P^2 \to P^2$ of the linear group SL(3) on P^2. Let $P\colon \tilde{C}_i \times \mathrm{SL}(3) \to P^2$ be given by $P(c, g) = \theta(g)c$ ($c \in \tilde{C}_i$, $g \in \mathrm{SL}(3)$). The center of the proof is the following diagram:



Here δ is the restriction of P on $P^{-1}(P^2 - C)$, β is the restriction of the projection $p\colon U_i \to P^2$ on $p^{-1}(P^2 - \bigcup C_i)$. The curve $\tilde{C}_i'$ is a generic fibre of the map $\tilde{C}_i \times \mathrm{SL}(3) - P^{-1}(\bigcup C_i) \to \mathrm{SL}(3)$ over a point $g \in \mathrm{SL}(3)$ sufficiently close to identity and γ is the embedding as a fibre. Note that for a dominant morphism $E \to B$ of algebraic varieties with connected fibres for generic fibre one has

$$\pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1.$$

Because SL(3) is simply connected, we can assume that

$$\pi_1(C_i') \to \pi_1(\tilde{C}_i \times \mathrm{SL}(3) - P^{-1}(\bigcup C_i))$$

is surjective. Construction of α uses the genericity of $p\colon U_i \to P^2$. So $\gamma(\tilde{C}_i')$ is the part, contained in $P^2 - \bigcup C_i$, of the image of C_i under linear transformation g close to identity. This linear transformation induces a transformation of the projecting cone which produces the map of C_i onto the intersection of this cone with U_i and which makes the diagram above commutative. The map $\delta_*\colon \pi_1(\tilde{C}_i \times \mathrm{SL}(3) - P^{-1}(P^2 - \bigcup C_i)) \to \pi_1(P^2 - \bigcup C_i)$ is surjective because the fibres of δ are connected. Surjectivity of β hence follows.

In the proof of the general case Nori reconstructs a diagram somewhat similar to the one used above. For any irreducible component $C \subset D \subset X$ and its normalization $\tilde{C}$, Nori considers an open set $U \supset \tilde{C}$ with a local homeomorphism $\tilde{n}\colon U \to X$ extending the normalization map $\tilde{C} \to X$. Let $X' = X - D - E$ and $U' = \tilde{n}^{-1}(X')$. The inequality $C^2 > 2r(C)$ implies that C has positive self-intersection index on U. If C were lying on X, then by a Lefschetz-type theorem this would imply that $\pi_1(U') \to \pi_1(X')$ is surjective. In general, however, $\tilde{C} \to X$ is only a local homeomorphism but Nori shows that nevertheless the image of $\pi_1(U')$ in $\pi_1(X')$ has finite index. He calls this fact the weak Lefschetz theorem, which is very interesting for its own sake. Nori also has an estimation of the index $[\mathrm{Im}\, \pi^1(U')\colon \pi_1(X')]$. Now let us consider a covering $s\colon Y' \to X'$ corresponding to $\operatorname{Im}\pi_1(U')$. One can show that there exist a finite morphism of a normal projective variety $\varphi: Y \to X$ and a lifting $s: U \to Y$ such that the composition $\varphi \circ s$ is unbranched over D. This implies that $N = \operatorname{Ker}\pi_1(X - D - E) \to \pi_1(X - E)$ belongs to the image of $\pi_1(Y')$ in $\pi_1(X')$ because this kernel is generated by the loops which are unions of paths and the boundaries of small normal disks to D. Moreover $\pi_1(U')\pi_1(Y')$ is surjective. Hence each of these loops belongs to the center of $\pi_1(Y')$ and therefore the kernel N is abelian.

It is interesting to compare Nori's weak Lefschetz theorem with the classical one. In the latter case one assumes that C is a curve with $C^2 > 0$ and concludes that $\pi_1(C) \to \pi_1(X)$ is surjective. $\pi_1(C)$ is however much bigger then $\pi_1(\tilde{C})$ where $\tilde{C}$ is a normalization of C ($\pi_1(C)$ is a free product of $\pi_1(\tilde{C})$ and a free group). Nori assumes that $C^2 > 2r(C)$ and obtains that the fundamental group of normalization is very close to (has finite index in) $\pi_1(X)$. In fact, he poses the question of whether this inequality can be weakened; i.e., does $C^2 > 0$ imply that the image of the fundamental group of the normalization of C has finite index in $\pi_1(X)$? In [GS] a positive answer to this question is given in the case when S is an elliptic surface with at least one fibre not of the type mI_0.

As a corollary Nori obtained the strongest theorem on the commutativity of the fundamental group of the complement to a plane curve with cusps and nodes.

THEOREM 7. *Let C be an irreducible curve of degree n which has a cusps and b nodes as the only singularities. If $n > 6a + 2b$, then $\pi_1(P^2 - C)$ is abelian.*

PROOF. Blow up P several times to obtain a surface $\tilde{P}^2$ on which the proper preimage $\tilde{C}$ of C is transversal to the exceptional set E. Then $\tilde{C}^2 = C^2 - 6a$. Theorem 6 applied to $\tilde{P}^2$ implies the claim.

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