Generalizations of LG/CY correspondence.

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- LG/CY correspondence: relation between invariants of isolated singularities of weighted homogeneous functions on $\mathbf{C}^n$ and invariants of Calabi Yau hypersurfaces in weighted projective spaces.
- Witten: There are generalizations of this correspondence in which CY hypersurfaces in $\mathbf{P}(w_1, ..., w_n)$ are replaced by complete intersections, hypersurfaces in products of projective spaces, Grassmanians etc. The construction is the identification of QFTs appearing to correspond to different GIT quotients of symplectic actions of Lie groups: what Witten called a hybrid models.

- **LG/CY correspondence for Euler characteristic.** $V^d_{n-2} \subset \mathbb{P}^{n-1}$

$$e(V^d_{n-2}) = \frac{1}{d}[(1-d)^n + dn - 1]$$

Milnor number of singularity $f(x_1, \cdots, x_n) = x_1^d + \cdots x_n^d = 0$ is $(d-1)^n$ and one wants to view this as "orbifoldized" Milnor number of $x_1^d + \cdots + x_n^d$ for $d = n$ i.e. CY case.

- **Orbifold Euler characteristic**: Γ -finite group acting via biholomorphic automorphisms on complex manifold $X \Rightarrow$

$$e_{orb}(X, \Gamma) = \sum_{\{\gamma\} \in \Gamma^{cong}} e(X^\gamma / C(\gamma)) = \frac{1}{|\Gamma|} \sum_{(\gamma,\gamma') | \gamma\gamma' = \gamma'\gamma} e(X^{\gamma,\gamma'})$$

- In particular: $e_{orb}(\{\mathbb{C}^d \setminus f^{-1}(t)\}, \mu_d) = \frac{1}{d}[(1-d)^d + d^2 - 1]$

- **Spectrum:** Let $f(x_1, \cdots, x_n)$ be a germ of a holomorphic function at the origin. Milnor fiber (the part if $f = t$ inside a small ball around the origin) has a limit mixed Hodge structure $(H^{n-1}(X_\infty), F, W)$ due to Steenbrink. Semi-simple part T_s of the monodromy $T = T_s T_u$ preserves this MHS. Let H^{n-1}_λ be the λ-eigenspace and

$$h^{p,q}_\lambda = dim Gr^p_F Gr^W_{p+q} H^{n-1}(X_\infty)_\lambda$$

Exponents are the rational numbers $\alpha_1 \leq \cdots \leq \alpha_\mu$ such that $\forall \lambda \in Eig(T), p \in \mathbb{Z}$ one has:

$$\sum_q h^{p,q}_\lambda = Card\{j | exp(-2\pi i \alpha_j) = \lambda, [\alpha] = n - 1 - p\}$$

$$\sum_q h^{p,q}_1 \Rightarrow Card\{j | \alpha_j = n - p\}$$

- Poincare series: $Spec(f) = \sum_\alpha Card\{j | \alpha_j = \alpha\} t^\alpha$

- $f(x_1, \cdots, x_n)$ is weighted homogeneous with weights $w_1, \cdots w_n \in \mathbb{Q}$ if $f(t^{w_1}x_1, \cdots, t^{w_n}x_n) = tf(x_1, \cdots, x_n) \Rightarrow$

$$Spec(f) = \prod \frac{t^{w_i} - t}{1 - t^{w_i}}$$

- **Spectrum of weighted homogeneous singularities and equivariant χ_y-genus.**
- X a compact complex manifold then

$$\chi_y(X) \stackrel{def}{=} \chi(\Lambda^*_{-y}\Omega^1_X) = \sum (-y)^j \chi(\Omega^j_X) =$$

$$\sum (-y)^{i+j} dim H^i(\Omega^j_X) \stackrel{R.R}{\Rightarrow} \int_X \prod_i x_i \frac{1 - ye^{-x_i}}{1 - e^{-x_i}}$$

where x_i are the Chern roots of T_X i.e. $c(T_X) = \prod_i (1 + x_i), x_i \in H^2(X)$.

- Let $T = (\mathbb{C}^*)^n$ be a torus acting on X via holomorphic transformations. Then $\chi_y(X)$ is a linear combination in $Sym(Char(T))$ given by Riemann Roch formula with $x_i \in A_i^T(X)$ (equivariant Chow groups) via localization. The contribution of a fixed point $P \in X$ is

$$loc(X, T, P) = \prod_i \frac{1 - ye^{-x_i}}{1 - e^{-x_i}}, \quad x_i \in H_T^*(P)$$

- An action $t(x_1, \cdots, x_n) = (\cdots, \chi_i(t)x_i, \cdots)$ is compatible with the weights w_i if any corresponding weighted homogeneous polynomial is an eigenfunction for this action i.e. there exists $D \in \mathbb{Z}$ such that $f(t(x_1, \cdots, x_n)) = t^D f(x_1, \cdots x_n)$. If $\chi_i(t) = t^{k_i}, k_i \in \mathbb{Z}^+$, then $w_i = \frac{k_i}{D}$.
- The group μ_D leaves the polynomial invariant, i.e. f is a eigenfunction for $T' = T/\mu_D$ on $\mathbb{C}^n/\mu_D$.

Proposition

Let $T = \mathbb{C}^*$ acts on $\mathbb{C}^n$ with weights w_i and $T' = T/\mu_D$ where μ_D is the group of diagonal symmetries of f, t is generator if $CharT'$. Then the local contribution $(-1)^n loc(X, T, P)$ of the action corresponding to f for $y = t$ coincides with the spectrum of f.

- **Example.** Let $f(x_1, \cdots, x_n) = \sum_i x_i^{d_i}$. Then $t(x_1, \cdots, x_n) = (\cdots, t^{\frac{D}{d_i}} x_i, \cdots), D = lcm(d_1, \cdots, d_n)$, weights are $w_i = \frac{1}{d_i}$. Let t be the generator of $CharT' = DCharT$. The local contribution of this action is

$$\prod_i \frac{1 - yt^{-\frac{1}{d_i}}}{1 - t^{-\frac{1}{d_i}}} \overset{y=t}{=} (-1)^n \prod_i \frac{t^{\frac{1}{d_i}} - t}{1 - t^{\frac{1}{d_i}}}$$

- **Why specialization $y = t$?** One can identify the Steenbrink spectrum of weighted homogeneous polynomial with the Poincare series of associated graded of Newton filtration of exterior algebras of differential forms with differential having the degree 1. The steps are:
- 1. Use Brion-Vergne to identify the local contribution with the character of $\mathbb{C}^*$ action in deRham and log-deRham cases:

$$\sum_{i.m} mult(m, H^0(\Omega^i_{\mathbb{C}^n}))(-y)^i t^m = \prod \frac{1 - yt^{-w_i}}{1 - t^{-w_i}} = \chi_y^{\mathbb{C}^*}(\mathbb{C}^n)$$

- 2. Identify generating function for the characters with the Poincare series of associated graded of the Newton filtration on the exterior algebra (Newton filtration of monomial equals the character of this monomial):

$$P(\Omega^i_{Newton\ filtr}(logD), t) = \sum_m mult(m, H^0(\Omega^i(logD))t^m$$

where D is the divisor $x_1 \cdots x_n = 0$ in $\mathbb{C}^n$.

- 3. Use Kouchnirenko's and Saito-Varchenko-Khovanskii results that the exterior algebra with Newton filtration is the Koszul resolution of the Jacobian algebra of singularity.

$$P_{Newton\ \ filtr}(\mathbb{C}\{x_1,\cdots,x_n\}/(\cdots,x_i\frac{\partial f}{\partial x_i}\cdots)) \overset{resol.}{=}$$

$$\sum_i(-t)^i P(\Omega^i_{Newton\ \ filtr}(logD),t) \Rightarrow$$

$$\sum_i(-t)^i\chi^{\mathbb{C}^*}(\Omega^i(logD)) \overset{B-V}{=} \chi^{\mathbb{C}^*}_{y=t}(\mathbb{C}^n,log(D))$$

where D is the divisor $x_1\cdots x_n=0$ in $\mathbb{C}^n$.

- 4. This implies that the Poincare series of the associated grade of Hodge filtration on Jacobian algebra in $\mathbb{C}^n$-case is the alternating sum of the Poincare series of exterior powers of the module of differential forms Ω^i twisted by factor t^i since the differential preserves the degree with a shift by 1.

Two Remarks:

- **Equivariant expression for spectrum plus equivariant formulas for genera of orbifolds gives "orbifoldization" of spectrum.**
- Witten obtains a LG/CY correspondence is the result of the study of $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ given by:

$$\lambda(p, x_1, ..., x_n) = (\lambda^{-n} p, \lambda x_1, ..., \lambda x_n)$$

  with LG and CY phases corresponding to two GIT quotients for this action. Two GIT quotients are $\mathbb{C}^n/\mu_n$ and $[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]$ and the problem is how to extract "the correct" invariants from the data associated with them?
- Since spectrum is the local contribution of $\mathbb{C}^*$-action on an orbifold one idea is to calculate the local contribution of $\mathbb{C}^*$-action for orbifold invariants for appropriate action. In the above case of spectrum, it is induced by the dilations $t(x_1, \cdots, x_n) \to (tx_1, \cdots, tx_n)$.

- **LG/CY, LG/Fano and mirror symmetry:** CY case

$$
\begin{array}{ccc}
A-model \quad LG & \overset{LG/CY}{\to} & A-model \quad CY \\
mirror \uparrow & & mirror \uparrow \\
symmetry \downarrow & & symmetry \downarrow \\
B-model \quad LG & \overset{LG/CY}{\to} & B-model \quad CY
\end{array}
$$

- Fano case

$$
\begin{array}{ccc}
A-model \quad LG & \overset{defect}{\leftrightarrow} & A-model \quad Fano \\
mirror \uparrow & & mirror \uparrow \\
symmetry \downarrow & & symmetry \downarrow \\
B-model \quad Fano & \overset{defect}{\leftrightarrow} & B-model \quad LG
\end{array}
$$

- For χ_y genus the exchange of type of model is the change of sign by the factor $(-1)^{dim}$ (in dimension 3 resulting from exchange $h^{1,1}(V) \leftrightarrow h^{2,1}(V'))$.

General framework for a construction of "phases" and their genera

- **GIT quotients**. Let $\mathcal{X}$ be quasi-projective, G a reductive group acting $\mathcal{X}$. G-equivariant line bundles are parametrized by $NS^G(\mathcal{X})$. $NS(\mathcal{X}) = 0 \Rightarrow NS^G(\mathcal{X}) = CharG$.
- For a line bundle $\mathcal{L}$ on $\mathcal{X}$ fix a linearization κ. $\kappa \Rightarrow \mathcal{X}^{ss} \subset \mathcal{X}$. GIT quotient is the quotient stack X^{ss}/G.
- NS^G has a partition into a union of cones such that linearizations in the same cone yield the same quotient stack.

Definition

Let $\mathcal{L}$ be the total space of a G-equivariant bundle L over a smooth quasi-projective G-manifold X. A phase corresponding to quadruple $(X, G, \mathcal{L}, \kappa)$ where κ is a linearization of G-action on $\mathcal{L}$ is the GIT quotient of the total space of linearized bundle $\mathcal{L}$ on X i.e. $\mathcal{L}^{ss}//G$.

- **Example** Consider the action of $G = \mathbb{C}^*$ on $\mathbb{C} \times \mathbb{C}^n$ with coordinates $(p, x_1, ..., x_n,)$ via:

$$\lambda(p, x_1, ..., x_n) = (\lambda^{-n} p, \lambda x_1, ..., \lambda x_n)$$

- Here $\mathcal{X} = \mathbb{C}^n$, $\mathcal{L} = \mathbb{C} \times \mathbb{C}^n$ is the total space of the trivial bundle on $\mathbb{C}^n$, additional $\mathbb{C}^*$ action on $\mathcal{L}$ by dilation of p.

- Here $NS^{\mathbb{C}^*}(\mathbb{C}\times\mathbb{C}^n)=Char\mathbb{C}^*=\mathbb{Z}$. For $\lambda\to\lambda^k, k<0$

$$\mathcal{L}^{ss}=\mathbb{C}^{n+1}\setminus(p=0)\Rightarrow\mathcal{L}^{ss}/\mathbb{C}^*=\mathbb{C}^n/\mu_d$$

- It induces on the global quotient orbifold $\mathbb{C}^n/\mu_n$ (LG-phase) the $\mathbb{C}^*$-action which can be viewed as the action induced on the μ_n-quotient as the action $\mathbb{C}^n$ given by $t(x_1,\cdots,x_n)=(t^{-1}x_1,\cdots,t^{-1}x_n)$.
- For the linearizations $\kappa=\lambda^k, k>0$ one has

$$\mathcal{L}^{ss}=\mathbb{C}\times\mathbb{C}^n\setminus(p,0,\cdots 0)\Rightarrow\mathcal{L}^{ss}/\mathbb{C}^*=[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]$$

  The action on $[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]$ (σ-model phase) is by dilations on the fibers over $\mathbb{P}^{n-1}$.
- Similar two phases are in the case of the weighted action:

$$\lambda(p,x_1,...,x_n)=(\lambda^{-\sum_1^n q_i}p,\lambda^{q_1}x_1,...,\lambda^{q_n}x_n)$$

Invariants of orbifolds

- **Orbifoldization of χ_y genus**. Let X be smooth, G finite group, acting biholomorphically with compact fixed point sets, $X^{g,h}$ fixed point set of a pair commuting elements, $\lambda \in CharG$ a character of G corresponding to an eigenbundle $T_X|_{X^{g,h}}$

$$\frac{1}{ord(G)} y^{\frac{-dim}{2}} \sum_{g} \sum_{h \in C(g)} \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_{\lambda,\lambda(h)=0} \frac{1-ye^{-x_\lambda-\lambda(g)}}{1-e^{-x_\lambda-\lambda(g)}} \times$$

$$\prod_{\lambda,\lambda(h)\neq 0} e^{2\pi i \lambda(h) z}[X^{g,h}]$$

- For $y=1$ (i.e. $z=0$) one obtains $e_{orb} = \sum_{gh=hg} e(X^{g,h})$.

- **Complex cobordisms up to K-equivalence**
- **Definition** X, Y smooth projective algebraic varieties. An basic K-equivalence between X, Y is a triple (Z, f_X, f_Y) such that exist a diagram $X \stackrel{f_X}{\leftarrow} Z \stackrel{f_Y}{\rightarrow} Y$ with properties:

  a) f_X, f_Y are birational morphisms, b) $f_X^*(K_X) = f_Y^*(K_Y)$
- **Theorem** Let $I \subset \Omega^U \otimes \mathbb{Q}$ (resp. $I^S \subset \Omega^{SU} \otimes \mathbb{Q}$) generated by classes $[X - Y]$ where X, Y are K-equivalent. Then

  $\Omega^{SU} \otimes \mathbb{Q}/I^S$ is the ring of $SL_2(\mathbb{Z})$-Jacobi forms $(\mathbb{Q}[x_2, x_3, x_4])$

  $\Omega^U \otimes \mathbb{Q}/I^S$ is the ring of $SL_2(\mathbb{Z})$-quasi-Jacobi forms $(\mathbb{Q}[x_1, x_2, x_3, x_4]$

- Explicit form of this isomorphism.
- X smooth, projective, E a bundle and $\Lambda_t(E) = \oplus t^i \Lambda^i(E), \mathcal{S}_t(E) = \oplus t^i \mathrm{Sym}^i(E)$. In $K(X)[[y^{\pm\frac{1}{2}}, q]]$

$$\mathcal{E}LL_{y,q} = y^{-\frac{d}{2}} \otimes_{n\geq 1} \left(\Lambda_{-yq^{n-1}} \Omega^1_X \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} \Omega^1_X \otimes S_{q^n} T_X \right)$$

- The elliptic genus is the holomorphic Euler characteristic of this bundle.

$$Ell(X, y, q) = \chi(\mathcal{E}LL_{y,q}) = \int_X td(X) ch \mathcal{E}LL_{y,q} =$$

$$\int_X \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)}$$

$$\theta(z, \tau) = q^{\frac{1}{8}} (2 \sin \pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l \mathrm{e}^{2\pi \mathrm{i} z})(1 - q^l \mathrm{e}^{-2\pi \mathrm{i} z})$$

$(y = e^{2\pi i z})$. This is Jacobi form if $c_1 = 0$ and $\chi_y(X) = Ell(X, y, q = 0)$.

- Let $X, G, X^{g,h}, \lambda, x_\lambda$ be as earlier.

$$E_{orb}(X, G; z, \tau) =$$

$$\frac{1}{|G|}\sum_{gh=hg}\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}]$$

where the characteristic class Φ is the class depending on

Chern classes x_λ of bundles V_λ and characters of the action of group (g, h) on V_λ.

- Explicitly:

$$\Phi(x, g, h, z, \tau, \Gamma) = \frac{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i z\lambda(h)}[X^{g,h}]$$

- It has as $q = 0$ specialization $\chi_{y,orb}$ (and $\frac{1}{|G|}\sum_{gh=hg} e^{top}(X^{g,h})$ $(z = 0)$).

- **Remarks** $Ell_{orb}(X,G)$ can be calculated in terms of resolutions of singularities as elliptic genus of pair $Ell_{sing}(\tilde{X},E)$ (E is a $\mathbb{Q}$-divisor supported on the exceptional locus of resolution) (McKay correspondence).
- If X is smooth and T a reductive group acting on X by holomorphic transformations, $\chi(\mathcal{E}LL(X))$ can be refined to the the character $\chi^T(\mathcal{E}LL(X))$ of T, which can be expressed as characteristic class by Riemann-Roch and via localization expressed in terms of the data of the fixed point set. For abelian $G \subset T$ acting on smooth X, the global quotient X/G support action of T/G. One has $CharT/G \subset CharT$ and $Ell^T_{orb}(X,G)$ depends on the T-equivariant bundles on X, which characters are fractional characters of $CharT/G$. $Ell^T_{orb}(X,G)$ via localization is well defined for X being quasi-projective with compact T-fixed point sets.

- **The explicit form of equivariant local contribution.** Let X be quasi-projective, T a torus, and G a finite group acting on X. Assume that the actions commute $gtx = tgx, \forall x \in X, g \in G, t \in T$. Let P be an isolated fixed point of G and T. Then the orbifoldized local contribution of the action of T on X/G is given by:

$$\frac{1}{G} \sum_{gh=hg} \prod_{V_\lambda} \frac{\theta(\tau, \frac{u_\lambda}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\tau, \frac{u_\lambda}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i \lambda(h) z}$$

($\lambda \in [0,1)$ runs through the eigenspaces V_λ of T and G with the eigenvalues $exp(2\pi i \lambda(g))$ acting on the tangent space at P, since $dimP = 0$ there is no contribution from the tangent bundle from P).

- When $q \to 0$ one obtains the orbifold χ_y-genus:

$$\frac{1}{ord(G)} y^{-\frac{\dim X}{2}} (\sum_g (\sum_h \prod_{\lambda, \lambda(h)=0} \frac{1 - ye^{-x_\lambda - \lambda(g)}}{1 - e^{-x_\lambda - \lambda(g)}} \prod_{\lambda, \lambda(h) \neq 0} y^{\lambda(h)}))$$

- **Definition** Let $f(x_1, \cdots, x_n)$ be a weighted homogeneous polynomial, such that $f(t^{q_1}x_1, \cdots, t^{q_n}x_n) = t^D f(x_1, \cdots, x_n)$ and let $\mu_D \subset \mathbb{C}^*$ be the subgroup of the roots of unity of $\mathbb{C}^*$ leaving f invariant. The orbifold spectrum of f, resp. the elliptic spectrum of f:

$$\prod_{i=1}^{n} \frac{\theta(w_i u - u)}{\theta(w_i u)}$$

is defined as the $y = t$ specialization of the equivariant orbifold local contribution of $(\mathbb{C}^n/\mu_D, f)$ for the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ corresponding to f and μ_D as above, written in terms of the character of the induced $\mathbb{C}^*$ action on $\mathbb{C}^n/\mu_D$ (i.e. $t^{\frac{1}{D}}$).

- For χ_y case:

$$\frac{1}{D} y^{-\frac{dim}{2}} \sum_g \sum_h \prod_{\lambda,\lambda(h)=0} \frac{1 - y \cdot y^{-\lambda(g)} \omega_D^{-\lambda(g)}}{1 - y^{-\lambda} \omega_D^{-\lambda(g)}} \prod_{\lambda,\lambda(h)\neq 0} y^{\lambda(h)}$$

- **Example.** Let $f = x_1^6 + x_2^4 + x_3^4 + x_4^3$ be a defining (well-formed) polynomial of a CY hypersurface of degree 12 in $\mathbb{P}(2,3,3,4)$.
- LG phase of GIT quotient $\mathbb{C} \times \mathbb{C}^4$ by the action

$$\lambda(p, x_1, x_2, x_3, x_4) = (\lambda^{-12}p, \lambda^2 x_1, \lambda^3 x_2, \lambda^3 x_3, \lambda^4 x_4)$$

is $\mathbb{C}^4/\mu_{12}$

- The orbifold group $G = \mu_{12} = \{\omega_{12}^k, 0 \le k \le 11\}$ (resp. the group $\mathbb{C}^*$ of dilations along p-coordinate) acts on $\mathbb{C}^4$ as

$$\omega_{12}^k(x_1, \cdots x_4) = (\omega_{12}^{2k}x_1, \omega_{12}^{3k}x_2, \omega_{12}^{3k}x_3, \omega_{12}^{4k}x_4)$$

resp.

$$t(x_1, \cdots x_4) = (t^2 x_1 t^3 x_2, t^3 x_3, t^4 x_4)$$

i.e. the characters on each of 4 components are $k \to \omega_6^k, k \to \omega_4^k, k \to \omega_3^k$.

- Hence the above becomes:

$$\frac{1}{12}y^{-2}[\sum_k^{h=0} \frac{t-t^{\frac{1}{6}}\omega_6^k}{1-t^{\frac{1}{6}}\omega_6^k}(\frac{t-t^{\frac{1}{4}}\omega_4^k}{1-t^{\frac{1}{4}}\omega_4^k})^2(\frac{t-t^{\frac{1}{3}}\omega_3^k}{1-t^{\frac{1}{3}}\omega_3^k})+\sum_k^{h=1} y^{\frac{(2+3+3+4)}{12}}+$$

-

$$\sum_k^{h=2} y^{\frac{4+6+6+8}{12}}+y^{\frac{6+9+9}{12}}\sum_k^{h=3}(\frac{t-t^{\frac{1}{3}}\omega_3^k}{1-t^{\frac{1}{3}}\omega_3^k})+y^{\frac{8+4}{12}}\sum_k^{h=4}(\frac{t-t^{\frac{1}{4}}\omega_4^k}{1-t^{\frac{1}{4}}\omega_4^k})^2+$$

-

$$\sum_k^{h=5} y^{\frac{10+3+3+8}{12}}+y^{\frac{6+6}{12}}\sum_k^{h=6}(\frac{t-t^{\frac{1}{6}}\omega_6^k}{1-t^{\frac{1}{6}}\omega_6^k})(\frac{t-t^{\frac{1}{3}}\omega_3^k}{1-t^{\frac{1}{3}}\omega_3^k})+$$

-

$$\sum_k^{h=7} y^{\frac{2+9+9+4}{12}}+y^{\frac{4+8}{12}}\sum_k^{h=8}(\frac{t-t^{\frac{1}{4}}\omega_4^k}{1-t^{\frac{1}{4}}\omega_4^k})^2+$$

$$y^{\frac{6+3+3}{12}} \sum_k^{h=9} \left(\frac{t - t^{\frac{1}{3}}\omega_3^k}{1 - t^{\frac{1}{3}}\omega_3^k}\right) + \sum_k^{h=10} y^{\frac{8+6+6+4}{12}} + \sum_k^{h=11} y^{\frac{(10+9+9+8)}{12}}] =$$

$$2y^{-1} + 20 + 2y = y^{-1}\chi_y(K3)$$

LG/CY correspondence for χ_y-genus.

- For the elliptic genus, one obtains a non-trivial identity for

$$Ell(K3) = 24\wp(\tau, z)\left(\frac{\theta(\tau, z)}{\theta'_z(\tau, 0)}\right)^2$$

.

Invariants of phases and proof of LG/CY correspondence

- **Genera of Phases. Elliptic genus of $\mathcal{L}//G$ corresponding to the phase $(X, G, \mathcal{L}, \kappa)$.**
  With mild assumptions, the $\mathbb{C}^*$ action on fibers of $\mathcal{L}$ descents to the $\mathbb{C}^*$ action on the quotient $\mathcal{L}//G$ and we have the following:
  a)$\mathcal{L}//G$ has quotient singularities and admits presentation as a global quotient: $\widetilde{\mathcal{L}//G}/\Gamma = \mathcal{L}//G$ for appropriate group Γ.
  b)$\mathbb{C}^*$ action on $\mathcal{L}//G$ lifts to the $\mathbb{C}^*$ action on $\widetilde{\mathcal{L}//G}$ commuting with Γ.

- **Definition** 1. Elliptic genus of GIT quotient $\widetilde{\mathcal{L}//_{\kappa}G}$ corresponding to the component P of fixed point set of $\mathbb{C}^*$ action on $\widetilde{\mathcal{L}//_{\kappa}G}$ is

$$Ell_{orb}^{\mathbb{C}^*}(\widetilde{\mathcal{L}//_{\kappa}G}, \Gamma, P, u, z, \tau)$$

specialized to $u = z$.

- 2. Two phases are K-equivalent if there is a K-equivalence $\psi : \mathcal{L}//_{\kappa_1}G \to \mathcal{L}//_{\kappa_2}G$ that can be lifted to a $\tilde{\mathbb{C}}^*$-equivariant K-equivalence $\tilde{\psi}$:

$$\begin{array}{ccc} \widetilde{\mathcal{L}//_{\kappa_1}G} & \stackrel{\tilde{\psi}}{\to} & \widetilde{\mathcal{L}//_{\kappa_2}G} \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ \widetilde{\mathcal{L}//_{\kappa_1}G}/\Gamma = \mathcal{L}//_{\kappa_1}G & \stackrel{\psi}{\to} & \widetilde{\mathcal{L}//_{\kappa_2}G}/\Gamma = \mathcal{L}//_{\kappa_2}G \end{array}$$

for an appropriate cover $\tilde{\mathbb{C}}^*$ of the torus $\mathbb{C}^*$

- **Singular elliptic genus**
  Let X has at most log-terminal singularities i.e. X is $\mathbb{Q}$-Gorenstein and for a resolution $f : Y \to X$ and a normal crossings exceptional divisor $\bigcup E_k$ one has $K_Y = f^*K_X + \sum \alpha_k E_k$ with $\alpha_k > -1$. Then

$$Ell_{sing}(X; z, \tau) :=$$

$$\int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})}\Big) \times$$

$$\Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k + 1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k + 1)z)}\Big)$$

  Here y_l are the Chern roots of T_Y and e_i are the classes of irreducible components of the exceptional divisor
- If a resolution is crepant i.e. $\alpha_k = 0$ then the singular elliptic genus is the elliptic genus of resolution.

- **Theorem.** Let $(X, G, \mathcal{L}, \kappa_1)$ and $(X, G, \mathcal{L}, \kappa_2)$ be two K-equivalent phases. Then the local contributions of the $\mathbb{C}^*$ actions induced by dilations are the same. and hence their elliptic genera coincide.
- **Example** Two phases in the GIT quotient by Witten's action $\lambda(p, x_1, \cdots x_n) = (\lambda^{-n} p, \lambda x_1, \cdots x_n)$ are

$$[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)] \to \mathbb{C}^n/\mu_n$$

  which is a K-equivalence.
- The induced $\mathbb{C}^*$-action from $\mathbb{C} \times \mathbb{C}^n$ is the action by dilations. Equivariant Chern roots of restriction $T([\mathcal{O}_{\mathbb{P}^{n-1}(-n)}])|_{\mathbb{P}^{n-1}}$ are $x, ..., x$ (n-times) and $-nx + u$.

Hence **specializing** the equivariant contribution one obtains:

$$(\frac{x\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})})^n \frac{\theta(-\frac{nx}{2\pi\mathrm{i}}+u-z)}{\theta(-\frac{nx}{2\pi\mathrm{i}}+u,\tau)}[\mathbb{P}^{n-1}] \stackrel{u=z}{=}$$

$$(\frac{x\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})})^n \frac{\theta(\frac{nx}{2\pi\mathrm{i}})}{\theta(\frac{nx}{2\pi\mathrm{i}}-z,\tau)}[\mathbb{P}^{n-1}]$$

Elliptic genus of hypersurface of degree n in $\mathbb{P}^{n-1}$ is

$$(\frac{x\theta(\frac{x}{2\pi\mathrm{i}}-z)}{\theta(\frac{x}{2\pi\mathrm{i}})})^n \frac{\theta(\frac{nx}{2\pi\mathrm{i}})}{nx\theta(\frac{nx}{2\pi\mathrm{i}}-z,\tau)}[V_n^{n-2}]$$

Two expressions are the same since $[V_n^{n-2}] = nx \cap [\mathbb{P}^{n-1}]$.

- **LG and hypersurfaces in weighted projective spaces**. Two phases in the GIT quotient by Witten's action $\lambda(p, x_1, \cdots x_n) = (\lambda^{-\sum_1^n q_i} p, \lambda^{q_1} x_1, \cdots \lambda^{q_n} x_n)$ are coarse moduli space of the total space of the orbibundle:

$$[\mathcal{O}_{\mathbb{P}(q_1, \cdots q_n)}(-\sum q_i)] \to \mathbb{C}^n / \mu_{\sum q_i}$$

  which is a K-equivalence. The induced $\mathbb{C}^*$-action from $\mathbb{C} \times \mathbb{C}^n$ is the action by dilations.
- **Theorem** McKay's correspondence identifies the local contribution on the LG phase with the **orbifold** elliptic genus of the diagonal hypersurface in $\mathbb{P}(q_1, \cdots, q_n)$.

- **Non CY case.**
  For the action $\lambda(p, x_1, \cdots x_n) = (\lambda^{-k} p, \lambda x_1, \cdots \lambda x_n), k \neq n$ there is no equality of genera, but the difference between the Euler characteristic of $V_k \in \mathbb{P}^{n-1}$ and the orbifoldized Milnor number (value of spectrum at $t = 1$) is $k - n$.
- The local contribution into χ_y-genus of the LG phase for $y = t$ is the spectrum (i.e. the term corresponding to $(1, 1) \in \mu_k \times \mu_k$ in the orbifoldized contribution).
- **Definition** The defect is the difference between the local contribution of the action by dilation on $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$ and the orbifold elliptic genus local contribution of the action induced by dilations on $\mathbb{C}^n/\mu_k$.

- **Theorem.**
- The defect of the correspondence between the phases $[\mathcal{O}_{\mathbb{P}^{n-1}}(k)]$ and $\mathbb{C}^n/\mu_k$ is given by:

$$coef.\ of\ x^{n-1}\ in\ (\frac{x\theta(x-z)}{\theta(x)})^n \frac{\theta'(0)}{\theta(-z)} \frac{\theta(-kx-z+t)}{\theta(-kx+t)} -$$

$$(\frac{x\theta(x-z)}{\theta(x)})^n \frac{\theta'(0)}{\theta(-z)} \frac{\theta(-kx-(disc)z-z+t)\theta(-z)}{\theta(-kx+t)\theta(-(disc)z-z)}$$

- Specialization $t = z$ gives the difference between the elliptic (resp. χ_y) genus and the orbifold (resp. χ_y) elliptic genus of LG model. The latter is the orbifoldized spectrum of $\sum_1^n x_i^k$.
- Specialization $q = 0$ is rigid (i.e. independent of t).

- **Example**. χ_y genus of cubic surface in $\mathbb{P}^3$.
- Defect is

$$y^{1-\frac{n}{2}} + \cdots y^{\frac{n}{2}-1} - y^{\frac{n}{k}-\frac{n}{2}} - \cdots y^{\frac{(k-1)n}{k}-\frac{n}{2}}$$

$$\stackrel{n=4,k=3}{=} y^{-1} + 1 + y - y^{\frac{-2}{3}} - y^{\frac{4}{3}}$$

- LG/Fano relation:

$$(Spectrum_{orb} = y^{\frac{2}{3}} + 6 + y^{\frac{2}{3}}) + defect = y^{-1} + 7 + y$$