Characters of fundamental groups of curve complements and orbifold pencils

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Abstract. The present work is a user's guide to the results of [7], where a description of the space of characters of a quasi-projective variety was given in terms of global quotient orbifold pencils.

Below we consider the case of plane curve complements. In particular, an infinite family of curves exhibiting characters of any torsion and depth 3 will be discussed. Also, in the context of line arrangements, it will be shown how geometric tools, such as the existence of orbifold pencils, can replace the group theoretical computations via fundamental groups when studying characters of finite order, specially order two. Finally, we revisit an Alexander-equivalent Zariski pair considered in the literature and show how the existence of such pencils distinguishes both curves.

1 Introduction

Let $\mathcal{X}$ be the complement of a reduced (possibly reducible) projective curve $\mathcal{D}$ in the complex projective plane $\mathbb{P}^2$. The space of characters of the fundamental group $\operatorname{Char}(\mathcal{X}) := \operatorname{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*)$ has an interesting stratification by subspaces, given by the cohomology of the rank one local system associated with the character:

$$\mathring{V}_k(\mathcal{X}) := \{\chi \in \operatorname{Char}(\mathcal{X}) \mid \dim H^1(\mathcal{X}, \chi) = k\}. \tag{1.1}$$

The closures $V_k(\mathcal{X})$ of these jumping loci in $\operatorname{Char}(\mathcal{X})$ were called in [23] the characteristic varieties of $\mathcal{X}$. More precisely, the characteristic varieties associated to $\mathcal{X}$ were defined in [23] as the zero sets of Fitting ideals of the $\mathbb{C}[\pi_1/\pi_1']$-module which is the complexification $\pi_1'/\pi_1'' \otimes \mathbb{C}$ of the abelianized commutator of the fundamental group $\pi_1(\mathcal{X})$ (*cf.* Section 2 for more details). In particular the characteristic varieties (un-

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like the jumping sets of the cohomology dimension greater than one) depend only on the fundamental group. Fox calculus provides an effective method for calculating the characteristic varieties if a presentation of the fundamental group by generators and relators is known.

For each character $\chi \in \mathrm{Char}(\mathcal{X})$ the *depth* was defined in [23] as

$$d(\chi) := \dim H^1(\mathcal{X}, \chi) \tag{1.2}$$

so that the strata (1.1) are the sets on which $d(\chi)$ is constant.

In [7], we describe a geometric interpretation of the depth by relating it to the pencils on $\mathcal{X}$ *i.e.* holomorphic maps $\mathcal{X} \to C$, $\dim C = 1$ having multiple fibers. In fact the discussion in [7] is in a more general context in which $\mathcal{X}$ is a smooth quasi-projective variety. [1] The viewpoint of [7] (and [8]) is that such a pencil can be considered as a map in the category of orbifolds. The orbifold structure of the curve C is matched by the structure of multiple fibers of the pencil. The main result of [7] can be stated as follows:

Theorem 1.1. *Let $\mathcal{X}$ be a quasi-projective manifold and let χ be a character of $\pi_1(\mathcal{X})$.*

(1) *Assume that there are n marked orbifold pencils* i.e. *maps $f_i : \mathcal{X} \to \mathcal{C}$ $(i = 1, \dots n)$ where $\mathcal{C}$ is a fixed orbicurve, $\rho \in \mathrm{Char}^{\mathrm{orb}}(\mathcal{C})$ and $\chi = f_i^*(\rho)$. If these pencils are strongly independent, then $d(\chi) \geq n d(\rho)$.*
(2) *If χ is a character of order two and weight two, then there are exactly $d(\chi)$ strongly independent orbifold pencils on $\mathcal{X}$ whose target is the global $\mathbb{Z}_2$-orbifold $\mathcal{C} = \mathbb{C}_{2,2}$. These pencils are marked with the character ρ of $\pi_1^{\mathrm{orb}}(\mathbb{C}_{2,2})$ characterized by the condition that ρ is nontrivial on both standard generators of the latter orbifold fundamental group.*

We refer to Section 2 for all the required definitions, and in particular, the definition of strongly independent pencils.

According to this result, the orbifold pencils on $\mathcal{X}$ whose targets have an orbifold fundamental group with characters of positive depth, induce characters in $\mathrm{Char}(\mathcal{X})$ whose depth have the lower bound given in 1.1. One can compare this statement with previous results on pencils on quasi-projective manifolds. For example, consider a character χ which belongs

[1] Much of the discussion in the first two sections below applies to general quasi-projective varieties (*cf.* [7]), but in the present paper we will stay in the hypersurface complement context. As was noted, the characteristic varieties only depend on the fundamental group, hence, by the Lefschetz-type theorems it is enough to consider the curve complement class.

to a positive dimensional component of the characteristic variety. Then the results in [2] can be applied to such a component to obtain a pencil $f : \mathcal{X} \to \mathcal{C}$ and a character $\rho \in \mathrm{Char}(\mathcal{C})$ such that $\chi = f^*(\rho)$. Here $\mathcal{C}$ is the complement in $\mathbb{P}^1$ to a finite set containing say $d > 2$ points. Moreover, the number of independent pencils in the sense of Section 2 is equal to one (*cf.* [7, Lemma 4.15]; note that the depth of $\rho \in \mathrm{Char}(\mathcal{C})$ is equal to $d-2$). Hence in this case, the inequality in Theorem 1.1 (1) is equivalent to the shown in [2, Prop.1.7] inequality $\dim H^1(\mathcal{X}, \chi) \geq \dim H^1(\mathcal{C}, \rho)$.

The orbifold structure involved in Theorem 1.1 is essential since the orbifold pencils described there and considered *without the orbifold structure*, are just rational pencils whose target might have trivial fundamental group and thus the connection with the jumping loci disappears.

The part (2) asserts a partial converse for characters of order two *i.e.* the characters of order two having positive depth are pull-backs of orbifold characters on $\mathbb{C}_{2,2}$ by orbifold pencils. Note that, as shown in [6] for characters of order 5 on an affine quintic, not all characters on complements of plane curves can be described as pull-backs of orbifold pencils.

The goal of this paper is to illustrate in detail both parts (1) and (2) of Theorem 1.1 with examples in which orbifolds are unavoidable. We start with a section reviewing mainly known results on the cohomology of local systems, characteristic varieties, orbifolds, and Zariski pairs making possible to read the rest of the paper unless one is interested in the proofs of mentioned results. Then in Section 3, a family of curves is considered for which the characteristic variety contains isolated characters having torsion of arbitrary finite order and whose depth is 3. The calculations illustrate the use of Fox calculus for finding an explicit description of the characteristic varieties. Next, in the context of line arrangements, examples of Ceva and augmented Ceva arrangements are considered in Section 4. Their characteristic varieties have been studied in the literature via computer aided calculations based on fundamental group presentations and Fox calculus. Here we present an alternative way to study such varieties independent of the fundamental group illustrating the geometric approach of Theorem 1.1. Finally, in Section 5 we discuss a Zariski pair of sextic curves whose Alexander polynomials coincide. We determine this Zariski pair by the existence of orbifold pencils.

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2 Preliminaries

In this section the necessary definitions used in Theorem 1.1 will be reviewed together with material on the characteristic varieties and Zariski pairs with the aim to keep the discussion of the upcoming sections in a reasonably self-contained manner.

2.1 Characteristic varieties

Characteristic varieties appeared first in the literature in the context of algebraic curves in [22]. They can be defined as follows.

Let $\mathcal{D} := \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r$ be the decomposition of a reduced curve $\mathcal{D}$ into irreducible components and let $d_i := \deg \mathcal{D}_i$ denote the degrees of the components $\mathcal{D}_i$. Let $\tau := \gcd(d_1, \ldots, d_r)$ and $\mathcal{X} = \mathbb{P}^2 \setminus \mathcal{D}$. Then (*cf.* [23])

$$H_1(\mathcal{X}; \mathbb{Z}) = \left\langle \bigoplus_{i=1}^{r} \gamma_i \mathbb{Z} \right\rangle / \langle d_1\gamma_1 + \cdots + d_r\gamma_r \rangle \approx \mathbb{Z}^{r-1} \oplus \mathbb{Z}/\tau\mathbb{Z}, \quad (2.1)$$

where γ_i is the homology class of a meridian of $\mathcal{D}_i$ (*i.e.* the boundary of small disk transversal to $\mathcal{D}_i$ at a smooth point).

Let $\mathbf{ab} : G := \pi_1(\mathcal{X}) \to H_1(\mathcal{X}; \mathbb{Z})$ be epimorphism of abelianization. The kernel G' of $\mathbf{ab}$, *i.e.* the commutator of G, defines the universal Abelian covering of $\mathcal{X}$, say $\mathcal{X}_{\mathbf{ab}} \xrightarrow{\pi} \mathcal{X}$, whose group of deck transformations is $H_1(\mathcal{X}; \mathbb{Z}) = G/G'$. This group of deck transformations, since it acts on $\mathcal{X}_{\mathbf{ab}}$, also acts on $H_1(\mathcal{X}_{\mathbf{ab}}; \mathbb{Z}) = G'/G''$. [2] This allows to endow $M_{\mathcal{D},\mathbf{ab}} := H_1(\mathcal{X}_{\mathbf{ab}}; \mathbb{Z}) \otimes \mathbb{C}$ (as well as $\tilde{M}_{\mathcal{D},\mathbf{ab}} := H_1(\mathcal{X}_{\mathbf{ab}}, \pi^{-1}(*); \mathbb{Z}) \otimes \mathbb{C}$) with a structure of $\Lambda_{\mathcal{D}}$-module where

$$\Lambda_{\mathcal{D}} := \mathbb{C}[G/G'] \approx \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1). \quad (2.2)$$

Note that $\operatorname{Spec} \Lambda_{\mathcal{D}}$ can be identified with the commutative affine algebraic group $\operatorname{Char} \pi_1(X)$ having τ tori $(\mathbb{C}^*)^{r-1}$ as connected components. Indeed, the elements of $\Lambda_{\mathcal{D}}$ can be viewed as the functions on the group of characters of G.

Since G is a finitely generated group, the module $M_{\mathcal{D},\mathbf{ab}}$ (respectively $\tilde{M}_{\mathcal{D},\mathbf{ab}}$) is a finitely generated $\Lambda_{\mathcal{D}}$-module: [3] in fact one can construct

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[2] This action corresponds to the action of G/G' on G'/G'' by conjugation.

[3] In most interesting examples with non-cyclic G/G' the group G'/G'' is infinitely generated.

a presentation of $M_{\mathcal{D},\mathbf{ab}}$ (respectively $\tilde{M}_{\mathcal{D},\mathbf{ab}}$) with the number of $\Lambda_{\mathcal{D}}$-generators at most $\binom{n}{2}$ (respectively n), where n is the number of generators of G. If G/G' is not cyclic (*i.e.* $r > 2$ or $r \geq 2$ and $\tau > 1$) then $\Lambda_{\mathcal{D}}$ is not a Principal Ideal Domain. One way to approach the $\Lambda_{\mathcal{D}}$-module structure of both $M_{\mathcal{D},\mathbf{ab}}$ and $\tilde{M}_{\mathcal{D},\mathbf{ab}}$ is to study their Fitting ideals (*cf.* [17]).

Let us briefly recall the relevant definitions. Let R be a commutative Noetherian ring with unity and M a finitely generated R-module. Choose a finite free presentation for M, say $\phi : R^m \to R^n$, where $M = \operatorname{coker} \phi$. The homomorphism ϕ has an associated $(n \times m)$ matrix A_ϕ with coefficients in R such that $\phi(x) = A_\phi x$ (the vectors below are represented as the column matrices).

Definition 2.1. The k-th *Fitting ideal* $F_k(M)$ of M is defined as the ideal generated by

$$\begin{cases} 0 & \text{if } k \leq \max\{0, n-m\} \\ 1 & \text{if } k > n \\ \text{minors of } A_\phi \text{ of order } (n-k+1) & \text{otherwise.} \end{cases}$$

It will be denoted F_k if no ambiguity seems likely to arise.

Definition 2.2. [22] With the above notations the *k-th characteristic variety* ($k > 0$) of $\mathcal{X} = \mathbb{P}^2 \setminus \mathcal{D}$ can be defined as the zero-set of the ideal $F_k(M_{\mathcal{D},\mathbf{ab}})$

$$V_k(\mathcal{X}) := Z(F_k(M_{\mathcal{D},\mathbf{ab}})) \subset \operatorname{Spec} \Lambda_{\mathcal{D}} = \operatorname{Char}(\mathbb{P}^2 \setminus \mathcal{D}).$$

Then $\mathring{V}_k(\mathcal{X})$ is the set of characters in $V_k(\mathcal{X})$ which do not belong to $V_j(\mathcal{X})$ for $j > k$. If a character χ belongs to $\mathring{V}_k(\mathcal{X})$ then k is called the *depth* of χ and denoted by $d(\chi)$ (*cf.* [23]).

An alternative notation for $\mathring{V}_k(\mathbb{P}^2 \setminus \mathcal{D})$ (respectively $V_k(\mathbb{P}^2 \setminus \mathcal{D})$) is $\mathring{V}_{k,\mathbb{P}}(\mathcal{D})$ (respectively $V_{k,\mathbb{P}}(\mathcal{D})$).

Remark 2.3. Essentially without loss of generality one can consider only the cases when the quotient by an ideal in the definition of the ring $\Lambda_{\mathcal{D}}$ in (2.2) is absent *i.e.* consider only the modules of the ring of Laurent polynomials. Indeed, consider a line L not contained in $\mathcal{D}$ and in general position (*i.e.* which does not contain singularities of $\mathcal{D}$ and is transversal to it). Then $\Lambda_{L\cup\mathcal{C}}$ is isomorphic to $\mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Moreover, since we assume transversality $L \pitchfork \mathcal{D}$, then the $\Lambda_{L\cup\mathcal{D}}$-module $M_{L\cup\mathcal{D},\mathbf{ab}}$ does not depend on L (see for instance [9, Proposition 1.16]). The characteristic variety $V_{k,\mathbb{P}}(L \cup \mathcal{D})$ determines $V_{k,\mathbb{P}}(\mathcal{D})$ (*cf.* [9, 23]). By abuse of language it is called the *k-th affine characteristic variety* and denoted simply by $V_k(\mathcal{D})$.

One can also use the module $\tilde{M}_{\mathcal{D},\mathbf{ab}}$ to obtain the characteristic varieties of $\mathcal{D}$. One has the following connection

$$V_k(\mathcal{X}) \setminus \overline{1} = Z(F_{k+1}(\tilde{M}_{\mathcal{D},\mathbf{ab}})) \setminus \overline{1},$$

where $\overline{1}$ denotes the trivial character.

Remark 2.4. The depth of a character appears in explicit formulas for the first Betti number of cyclic and abelian unbranched and branched covering spaces (*cf.* [20, 22, 27])

Remark 2.5. One can also define the k-th characteristic variety $V_k(G)$ of any finitely generated group G (such that the abelianization $G/G' \neq 0$ or, more generally, for a surjection $G \to A$ where A is an abelian group) as the k-th characteristic variety of the $\Lambda_G = \mathbb{C}[G/G']$-module $M_G = H_1(\mathcal{X}_{G,\mathbf{ab}})$ obtained by considering the CW-complex $\mathcal{X}_G$ associated with a presentation of G and its universal abelian covering space $\mathcal{X}_{G,\mathbf{ab}}$ (respectively considering the covering space of $\mathcal{X}_G$ associated with the kernel of the map to A). Such invariant is independent of the finite presentation of G (respectively depends only on $G \to A$). This construction will be applied below to the orbifold fundamental groups of one dimensional orbifolds.

Remark 2.6. Note that one has:

- $V_k(\mathcal{D}) = \operatorname{Supp}_{\Lambda_{\mathcal{D}}} \wedge^i (H_1(\mathcal{X}_{\mathbf{ab}}; \mathbb{C}))$,
- $\operatorname{Spec} \Lambda_{L\cup\mathcal{D}} = \mathbb{T}^r = (\mathbb{C}^*)^r$, for the affine case, and
- $\operatorname{Spec} \Lambda_{\mathcal{D}} = \mathbb{T}_{\mathcal{D}} = \{\omega^i\}_{i=0}^{\tau-1} \times (\mathbb{C}^*)^{r-1} = V(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1) \subset \mathbb{T}^r$, where ω is a τ-th primitive root of unity for the curves in projective plane.

Note also that in the case of a finitely presented group G such that $G/G' = \mathbb{Z}^r \oplus \mathbb{Z}/\tau_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\tau_s\mathbb{Z}$ one has

$$\operatorname{Spec} \Lambda_G = \mathbb{T}_G = \{(\omega_1^{i_1}, \ldots, \omega_s^{i_s}) | i_k = 0, \ldots, \tau_k - 1, k = 1, \ldots, s\} \times (\mathbb{C}^*)^r, \tag{2.3}$$

where as above $\Lambda_G = \mathbb{C}[G/G']$ and ω_i is a τ_i-th primitive root of unity.

Let $\mathcal{X}$ be a smooth quasi-projective variety such that for its smooth compactification $\overline{\mathcal{X}}$ one has $H^1(\overline{\mathcal{X}}, \mathbb{C}) = 0$. This of course includes the cases $\mathcal{X} = \mathbb{P}^2 \setminus \mathcal{D}$. The structure of $V_k(\mathcal{X})$ is given by the following fundamental result.

Theorem 2.7 ([2]). *Each $V_k(\mathcal{X})$ is a finite union of cosets of subgroups of* $\operatorname{Char}(\mathcal{X})$. *Moreover, for each irreducible component W of $V_k(\mathcal{X})$ having a positive dimension there is a pencil $f : \mathcal{X} \to C$, where C is a $\mathbb{P}^1$ with deleted points, and a torsion character $\chi \in V_k(\mathcal{X})$ such that $W = \chi f^* H^1(C, \mathbb{C}^*)$.*

2.2 Essential coordinate components

Let $\mathcal{D}' \subsetneq \mathcal{D}$ be curve whose components form a subset of the set of components of $\mathcal{D}$. There is a natural epimorphism $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{D}')$ induced by the inclusion. This surjection induces a natural inclusion Spec $\Lambda_{\mathcal{D}'} \subset$ Spec $\Lambda_{\mathcal{D}}$. With identification of the generators of $\Lambda_{\mathcal{D}}$ with components of $\mathcal{D}$ as above, this embedding is obtained by assigning 1 to the coordinates corresponding to those irreducible components of $\mathcal{D}$ which are not in $\mathcal{D}'$ (*cf.* [23]).

The embedding Spec $\Lambda_{\mathcal{D}'} \subset$ Spec $\Lambda_{\mathcal{D}}$ induces the inclusion $\mathrm{Char}(\mathcal{D}') \subset \mathrm{Char}(\mathcal{D})$ (*cf.* [23]); any irreducible component of $V_k(\mathcal{D}')$ is the intersection of an irreducible component of $V_k(\mathcal{D})$ with $\Lambda_{\mathcal{D}'}$.

Definition 2.8. Irreducible components of $V_k(\mathcal{D})$ contained in $\Lambda_{\mathcal{D}'}$ for some curve $\mathcal{D}' \subset \mathcal{D}$ are called *coordinate components* of $V_k(\mathcal{D})$. If an irreducible coordinate component V of $V_k(\mathcal{D}')$ is also an irreducible component of $V_k(\mathcal{D})$, then V is called a *non-essential coordinate component*, otherwise it is called an *essential coordinate component*.

See [4] for examples. A detailed discussion of more examples is done in Sections 3, 4, and 5.

As shown in [23, Lemma 1.4.3] (see also [15, Proposition 3.12]), essential coordinate components must be zero dimensional.

2.3 Alexander invariant

In Section 2.1 the characteristic varieties of a finitely presented group G are defined as the zeroes of the Fitting ideals of the module $M := G'/G''$ over G/G'. This module is referred to in the literature as the *Alexander invariant* of G. Note, however, that this is not the module represented by the matrix of Fox derivatives called the *Alexander module* of G.

Our purpose in this section is to briefly describe the Alexander invariant for fundamental groups of complements of plane curves and give a method to obtain a presentation of such a module from a presentation of G. In order to do so, consider $G := \pi_1(\mathbb{P}^2 \setminus \mathcal{D})$ the fundamental group of the curve $\mathcal{D}$. Without loss of generality one might assume that

$(Z1)$ $G/G'$is a free group of rank r generated by meridians $\gamma_1, \gamma_2, \ldots, \gamma_r$,

then one has the following

Lemma 2.9 ([5, Proposition 2.3]). *Any group G as above satisfying $(Z1)$ admits a presentation*

$$\langle x_1, \ldots, x_r, y_1, \ldots, y_s : R_1(\overline{x}, \overline{y}) = \ldots = R_m(\overline{x}, \overline{y}) = 1 \rangle, \tag{2.4}$$

where $\overline{x} := \{x_1, \ldots, x_r\}$ *and* $\overline{y} := \{y_1, \ldots, y_s\}$ *satisfying:*

$(Z2)$ $\mathbf{ab}(x_i)=\gamma_i$, $\mathbf{ab}(y_j)=0$, *and* R_k *can be written in terms of* $\overline{y}$ *and* $x_k[x_i, x_j]x_k^{-1}$, *where* $[x_i, x_j]$ *is the commutator of* x_i *and* x_j.

A presentation satisfying $(Z2)$ is called a *Zariski presentation* of G.

From now on we will assume G admits a Zariski presentation as in (2.4). In order to describe elements of the module M it is sometimes convenient to see $\mathbb{Z}[G/G']$ as the ring of Laurent polynomials in r variables $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$, where t_i represents the action induced by γ_i on M as a multiplicative action, that is,

$$t_i g \overset{M}{=} x_i g x_i^{-1} \tag{2.5}$$

for any $g \in G'$.

Remark 2.10.

1. One of course needs to convince oneself that action (2.5) is independent, up to an element of G'', of the representative x_i as long as $\mathbf{ab}(x_i) = \gamma_i$. This is an easy exercise.
2. We denote by "$\overset{M}{=}$" equalities that are valid in M.

Example 2.11. Note that

$$[xy, z] \overset{M}{=} [x, z] + t_x[y, z], \tag{2.6}$$

where x, y, and z are elements of G and t_x denotes $\mathbf{ab}(x)$ in the multiplicative group. This is a consequence of the following

$$[xy, z] = xyzy^{-1}x^{-1}z^{-1} = x(yzy^{-1}z^{-1})x^{-1}xzx^{-1}z^{-1} \overset{M}{=} t_x[y, z] + [x, z].$$

As a useful application of (2.6) one can check that

$$[x^y, z] \overset{M}{=} [x, z] + (t_z - 1)[y, x], \tag{2.7}$$

where $x^y := yxy^{-1}$.

Note that $x_{ij} := [x_i, x_j]$, $1 \leq i < j \leq r$ and y_k, $k = 1, \ldots, s$ are elements in G', since $\mathbf{ab}(x_{ij}) = \mathbf{ab}(y_k) = 0$. Therefore

$$x_k[x_i, x_j]x_k^{-1} \overset{M}{=} t_k x_{i,j} \tag{2.8}$$

(see (2.5) and $(Z2)$). Moreover,

Proposition 2.12. *For a group* G *as above, the module* M *is generated by* $\overline{x}_{i,j} := \{x_{ij}\}_{1 \leq i < j \leq r}$ *and* $\overline{y} := \{y_k\}_{k=1,\ldots,s}$.

Example 2.13. The module M is not freely generated by the set mentioned above, for instance, note that according to $(Z2)$ and (2.8) any relation in G, say $R_i(\overline{x}, \overline{y}) = 1$ (as in (2.4)) can be written (in M) in terms of $\overline{\{x_{ij}\}}$ and $\overline{y}$ as $\mathcal{R}_i(\overline{x}_{ij}, \overline{y})$. In other words, $\mathcal{R}_i(\overline{x}_{ij}, \overline{y}) = 0$ is a relation in M.

Example 2.14. Even if G were to be the free group $\mathcal{F}_r$, M would not be freely generated by $\overline{\{x_{ij}\}}$ and $\overline{y}$. In fact,

$$J(x, y, z) := (t_x - 1)[y, z] + (t_y - 1)[z, x] + (t_z - 1)[x, y] \overset{M}{=} 0 \quad (2.9)$$

for any x, y, z in G. Using Example 2.11 repeatedly, one can check the following

$$[xy, z] = \begin{cases} \overset{M}{=} [x, z] + t_x[y, z] \\ = [y^{x^{-1}}x, z] \overset{M}{=} [y^{x^{-1}}, z] + t_y[x, z] \\ \qquad \overset{M}{=} [y, z] - (t_z - 1)[x, y] + t_y[x, z], \end{cases} \quad (2.10)$$

where $a^b = bab^{-1}$. The difference between both equalities results in $J(x, y, z) = 0$. Such relations will be referred to as *Jacobian relations* of M.

A combination of Examples 2.13 and 2.14 gives in fact a presentation of M.

Proposition 2.15 ([9, Proposition 2.39]). *The set of relations $\mathcal{R}_1,\dots,\mathcal{R}_m$ as described in Example 2.13 and $J(i, j, k) = J(x_i, x_j, x_k)$ as described in Example 2.14 is a complete system of relations for M.*

Example 2.16. Let $G = \mathcal{F}_r$ be the free group in r generators, for instance, the fundamental group of the complement to the union of $r + 1$ concurrent lines. According to Propositions 2.12 and 2.15, M has a presentation matrix A_r of size $\binom{r}{3} \times \binom{r}{2}$ whose columns correspond to the generators $x_{ij} = [x_i, x_j]$ and whose rows correspond to the coefficients of the Jacobian relations $J(i, j, k)$, $1 \leq i < j < k \leq r$. For instance, if $r = 4$

$$A_4 := \begin{bmatrix} (t_3 - 1) & -(t_2 - 1) & 0 & (t_1 - 1) & 0 & 0 \\ (t_4 - 1) & 0 & -(t_2 - 1) & 0 & (t_1 - 1) & 0 \\ 0 & (t_4 - 1) & -(t_3 - 1) & 0 & 0 & (t_1 - 1) \\ 0 & 0 & 0 & (t_4 - 1) & -(t_3 - 1) & (t_2 - 1) \end{bmatrix}.$$

Such matrices have rank $\binom{r-1}{2}$ if $t_i \neq 1$ for all $i = 1, \dots, r$, and hence the depth of a non-coordinate character is $r - 1$. On the other hand, for the trivial character $\overline{1}$, the matrix A_n has rank 0 and hence $\overline{1}$ has depth $\binom{r}{2}$ (see Definitions 2.1 and 2.2 for details on the connection between the rank of A_n and the depth of a character).

2.4 Orbicurves

As a general reference for orbifolds and orbifold fundamental groups one can use [1], see also [19, 28]. A brief description of what will be used here follows.

Definition 2.17. An *orbicurve* is a complex orbifold of dimension equal to one. An orbicurve $\mathcal{C}$ is called a *global quotient* if there exists a finite group G acting effectively on a Riemann surface C such that $\mathcal{C}$ is the quotient of C by G with the orbifold structure given by the stabilizers of the G-action on C.

We may think of $\mathcal{C}$ as a Riemann surface with a finite number of points $R := \{P_1, ..., P_s\} \subset \mathcal{C}$ labeled with positive integers $\{m(P_1), ..., m(P_s)\}$ (for global quotients those are the orders of stabilizers of action of G on C). A neighborhood of a point $P \in \mathcal{C}$ with $m(P) > 0$ is the quotient of a disk (centered at P) by an action of the cyclic group of order $m(P)$ (a rotation).

A small loop around P is considered to be trivial in $\mathcal{C}$ if its lifting in the above quotient map bounds a disk. Following this idea, orbifold fundamental groups can be defined as follows.

Definition 2.18. (*cf.* [1,19,28]) Consider an orbifold $\mathcal{C}$ as above, then the *orbifold fundamental group* of $\mathcal{C}$ is

$$\pi_1^{\text{orb}}(\mathcal{C}) := \pi_1(\mathcal{C} \setminus \{P_1, \dots, P_s\})/\langle \mu_j^{m_j} = 1\rangle$$

where μ_j is a meridian of P_j and $m_j := m(P_j)$.

According to Remark 2.5 the Definition 2.2 can be applied to the case of finitely generated groups. In particular one defines the k-th characteristic variety $V_k^{\text{orb}}(\mathcal{C})$ of an orbicurve $\mathcal{C}$ as $V_k(\pi_1^{\text{orb}}(\mathcal{C}))$. Therefore also the concepts of a character χ on $\mathcal{C}$ and its depth are well defined.

Example 2.19. Let us denote by $\mathbb{P}^1_{m_1,\dots,m_s,k\infty}$ an orbicurve for which the underlying Riemann surface is $\mathbb{P}^1$ with k points removed and s labeled points with labels $m_1, ..., m_s$. If $k \geq 1$ (respectively $k \geq 2$) we also use the notation $\mathbb{C}_{m_1,\dots,m_s,(k-1)\infty}$ (respectively $\mathbb{C}^*_{m_1,\dots,m_s,(k-2)\infty}$) for $\mathbb{P}^1_{m_1,\dots,m_s,k\infty}$. We suppress specification of actual points on $\mathbb{P}^1$. Note that

$$\pi_1^{\text{orb}}(\mathbb{P}^1_{m_1,\dots,m_s,k\infty}) = \begin{cases} \mathbb{Z}_{m_1}(\mu_1) * \dots * \mathbb{Z}_{m_s}(\mu_s) * \mathbb{Z} * \overset{k-1}{\cdots} * \mathbb{Z} & \text{if } k > 0 \\ \mathbb{Z}_{m_1}(\mu_1) * \dots * \mathbb{Z}_{m_s}(\mu_s)/\prod \mu_i & \text{if } k = 0 \end{cases}$$

(here $\mathbb{Z}_m(\mu)$ denotes a cyclic group of order m with a generator μ). Note that a global quotient orbifold of $\mathbb{P}^1 \setminus \{nk \text{ points}\}$ by the cyclic action of order n on $\mathbb{P}^1$ that fixes two points, that is, $[x : y] \mapsto [\xi_n x : y]$ (which fixes $[0 : 1]$ and $[1 : 0]$) is $\mathbb{P}^1_{n,n,k\infty}$.

Interesting examples of elliptic global quotients occur for $\mathbb{P}^1_{2,3,6,k\infty}$, $\mathbb{P}^1_{3,3,3,k\infty}$, and $\mathbb{P}^1_{2,4,4,k\infty}$, which are global orbifolds of elliptic curves $E \setminus \{6k \text{ points}\}$, $E \setminus \{3k \text{ points}\}$, and $E \setminus \{4k \text{ points}\}$ respectively, see [11] for a study of the relationship between these orbifolds ($k = 0$) and the depth of characters of fundamental groups of the complements to plane singular curves.

Definition 2.20. A *marking* on an orbicurve $\mathcal{C}$ (respectively a quasiprojective variety $\mathcal{X}$) is a non-trivial character of its orbifold fundamental group (respectively its fundamental group) of positive depth k, that is, an element of $\mathrm{Char}^{\mathrm{orb}}(\mathcal{C}) := \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathcal{C}), \mathbb{C}^*)$ (respectively $\mathrm{Char}(\mathcal{X}) := \mathrm{Hom}(\pi_1(\mathcal{X}), \mathbb{C}^*)$) which is in $V_k^{\mathrm{orb}}(\mathcal{C})$ (respectively $V_k(\mathcal{X})$).

A *marked orbicurve* is a pair $(\mathcal{C}, \rho)$, where $\mathcal{C}$ is an orbicurve and ρ is a marking on $\mathcal{C}$. Analogously, one defines a *marked quasi-projective manifold* as a pair $(\mathcal{X}, \chi)$ consisting of a quasi-projective manifold $\mathcal{X}$ and a marking on it.

A marked orbicurve $(\mathcal{C}, \rho)$ is *a global quotient* if $\mathcal{C}$ is a global quotient of C, where C is a branched cover of $\mathcal{C}$ associated with the unbranched cover of $\mathcal{C} \setminus \{P_1, ..., P_s\}$ corresponding to the kernel of $\pi_1(\mathcal{C} \setminus \{P_1, ..., P_s\}) \to \pi_1^{\mathrm{orb}}(\mathcal{C}) \xrightarrow{\rho} \mathbb{C}^*$. In other words, the covering space in Definition 2.17 corresponds to the kernel of ρ.

2.5 Orbifold pencils on quasi-projective manifolds

Definition 2.21. Let $\mathcal{X}$ be a quasi-projective variety, C be a quasi-projective curve, and $\mathcal{C}$ an orbicurve which is a global quotient of C. A *global quotient orbifold pencil* is a map $\phi : \mathcal{X} \to \mathcal{C}$ such that there exists $\Phi : X_G \to C$ where X_G is a quasi-projective manifold endowed with an action of the group G making the following diagram commute:

$$\begin{array}{ccc} X_G & \xrightarrow{\Phi} & C \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{C} \end{array} \tag{2.11}$$

The vertical arrows in (2.11) are the quotients by the action of G.

If, in addition, $(\mathcal{X}, \chi)$ and $(\mathcal{C}, \rho)$ are marked, then the global quotient orbifold pencil $\phi : \mathcal{X} \to \mathcal{C}$ called *marked* if $\chi = \phi^*(\rho)$. We will refer to the map of pairs $\phi : (\mathcal{X}, \chi) \to (\mathcal{C}, \rho)$ as a *marked global quotient orbifold pencil* on $(\mathcal{X}, \chi)$ with target $(\mathcal{C}, \rho)$.

Definition 2.22. Global quotient orbifold pencils $\phi_i : (\mathcal{X}, \chi) \to (\mathcal{C}, \rho)$, $i = 1, ..., n$ are called *independent* if the induced maps $\Phi_i : X_G \to C$

define $\mathbb{Z}[G]$-independent morphisms of modules

$$\Phi_{i*} : H_1(X_G, \mathbb{Z}) \to H_1(C, \mathbb{Z}), \tag{2.12}$$

that is, independent elements of the $\mathbb{Z}[G]$-module $\operatorname{Hom}_{\mathbb{Z}[G]}(H_1(X_G, \mathbb{Z}), H_1(C, \mathbb{Z}))$.

In addition, if $\bigoplus \Phi_{i*} : H_1(X_G, \mathbb{Z}) \to H_1(C, \mathbb{Z})^n$ is surjective we say that the pencils ϕ_i are *strongly independent*.

Remark 2.23. Note that if either $n = 1$ or $H_1(C, \mathbb{Z}) = \mathbb{Z}[G]$, then independence is equivalent to strong independence (this is the case for Corollary 2.26(2) and Theorem 1.1(2)).

2.6 Structure of characteristic varieties (revisited)

The following are relevant improvements or additions to Theorem 2.7:

Theorem 2.24 ([8,24]). *The isolated zero-dimensional characters of $V_k(\mathcal{D})$ are torsion characters of* $\operatorname{Char}(\mathcal{D})$.

In [15, Theorem 3.9] (see also [16]) there is a description of one-dimensional components $\chi f^* H^1(C, \mathbb{C}^*) \subset V_k(\mathcal{X})$ mentioned in Theorem 2.7 and most importantly, of the order of χ in terms of multiple fibers of the rational pencil f.

In [23], an algebraic method is described to detect the irregularity of abelian covers of $\mathbb{P}^2$ ramified along $\mathcal{D}$. This method is very useful to compute *non-coordinate components* of $V_k(\mathcal{D})$ independently of a presentation of the fundamental group of the complement $\mathcal{X}$ of $\mathcal{D}$.

Theorem 1.1 (see [7]) has [15, Theorem 3.9] as a consequence, but uses the point of view of orbifold pencils. Using this result also the zero-dimensional components can be detected (in particular essential coordinate components) and in some cases characterized (see Section 4).

Another improvement of Theorem 2.7 was given in [8] were the point of view of orbifolds was first introduced as follows:

Theorem 2.25 ([8]). *Let $\mathcal{X}$ be a smooth quasi-projective variety. Let V be an irreducible component of $V_k(\mathcal{X})$. Then one of the two following statements holds:*

(1) *There exists an orbicurve $\mathcal{C}$, a surjective orbifold morphism $\rho : X \to \mathcal{C}$ and an irreducible component W of $V_k^{\text{orb}}(\mathcal{C})$ such that $V = \rho^*(W)$.*
(2) *V is an isolated torsion point not of type* (1)*.*

One has the following consequences from 1.1 (2) that allows us to characterize certain elements of $V_k(\mathcal{D})$:

Corollary 2.26. *Let $(\mathcal{X}, \chi)$ be a marked complement of $\mathcal{D}$. Then possible targets for marked orbifold pencils are $(\mathcal{C}, \rho)$ with $\mathcal{C} = \mathbb{P}^1_{m_1,\ldots,m_s,k\infty}$ (see Example* 2.19*). Assume that there are n strongly independent marked orbifold pencils with such a fixed target $(\mathcal{C}, \rho)$. Then,*

(1) *In case $\mathcal{C}$ has no orbifold points, that is $s = 0$, the character χ belongs to a positive dimensional component V of* $\operatorname{Char}(\mathcal{X})$ *containing the trivial character. In this case, $d(\chi) = \dim V - 1 = n - 2$.*
(2) *In case χ is a character of order two, there is a unique marking on $\mathcal{C} = \mathbb{C}_{2,2}$ and $d(\chi)$ is the maximal number of strongly independent orbifold pencils with target $\mathcal{C}$.*
(3) *In case χ has torsion 3,4, or 6, there is a unique marking on $\mathcal{C} = \mathbb{P}^1_{3,3,3}$, $\mathcal{C} = \mathbb{P}^1_{2,4,4}$, or $\mathcal{C} = \mathbb{P}^1_{2,3,6}$ respectively and $d(\chi)$ is the maximal number of strongly independent orbifold pencils with target $\mathcal{C}$.*

Part (1) is a direct consequence of Theorem 2.7 and part (3) had already appeared in the context of Alexander polynomials in [11].

In Section 4 we will describe in detail examples of Corollary 2.26 (2) for line arrangements.

2.7 Zariski pairs

We will give a very brief introduction to Zariski pairs. For more details we refer to [9] and the bibliography therein.

Definition 2.27 ([3]). Two plane algebraic curves $\mathcal{D}$ and $\mathcal{D}'$ form a *Zariski pair* if there are homeomorphic tubular neighborhoods of $\mathcal{D}$ and $\mathcal{D}'$, but the pairs $(\mathbb{P}^2, \mathcal{D})$ and $(\mathbb{P}^2, \mathcal{D})$ are not homeomorphic.

The first example of a Zariski pair was given by Zariski [33], who showed that the fundamental group of the complement to an irreducible sextic (a curve of degree six) with six cusps on a conic is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ whereas the fundamental group of any other sextic with six cusps is $\mathbb{Z}_6$. This paved the way for intensive research aimed to understand the connection between the topology of $(\mathbb{P}^2, \mathcal{D})$ and the position of the singularities of $\mathcal{D}$ (whether algebraically, geometrically, combinatorially...). This research has been often in the direction of a search for finer invariants of $(\mathbb{P}^2, \mathcal{D})$.

Characteristic varieties (described above) and the Alexander polynomials (*i.e.* the one variable version of the characteristic varieties), twisted polynomials [13], generalized Alexander polynomials [11, 26], dihedral covers of $\mathcal{D}$ ([32]) among many others are examples of such invariants.

Definition 2.28. If the Alexander polynomials $\Delta_{\mathcal{D}}(t)$ and $\Delta_{\mathcal{D}'}(t)$ coincide, then we say $\mathcal{D}$ and $\mathcal{D}'$ form an *Alexander-equivalent Zariski pair*.

In Section 5 we will use Theorem 1.1 to give an alternative proof that the curves in [4] Alexander-equivalent Zariski pair, without computing the fundamental group.

3 Examples of characters of depth 3: Fermat Curves

Consider the following family of plane curves:

$$\begin{aligned}\mathcal{F}_n &:= \{f_n := x_1^n + x_2^n - x_0^n = 0\},\\ \mathcal{L}_1 &:= \{\ell_1 := x_0^n - x_2^n = 0\},\\ \mathcal{L}_2 &:= \{\ell_2 := x_0^n - x_2^n = 0\}.\end{aligned}$$

We will study the characteristic varieties of the quasi-projective manifolds $\mathcal{X}_n := \mathbb{P}^2 \setminus \mathcal{D}_n$, where $\mathcal{D}_n := \mathcal{F}_n \cup \mathcal{L}_1 \cup \mathcal{L}_2$, in light of the results given in the previous sections, in particular the essential torsion characters will be considered and their depth will be exhibited as the number of strictly independent orbifold pencils.

3.1 Fundamental group

Note that $\mathcal{D}_n$ is nothing but the preimage by the Kummer cover $[x_0 : x_1 : x_2] \overset{\kappa_n}{\mapsto} [x_0^n : x_1^n : x_2^n]$ of the following arrangement of three lines in general position given by the equation

$$(x_0 - x_1)(x_0 - x_2)(x_0 - x_1 - x_2) = 0.$$

Such a map ramifies along $\mathcal{B} := \{x_0x_1x_2 = 0\}$. We will compute the fundamental group of $\mathcal{X}_n$ as a quotient of the subgroup K_n of $\pi_1(\mathbb{P}^2 \setminus \mathcal{L})$ associated with the Kummer cover, where

$$\mathcal{L} := \{x_0x_1x_2(x_0 - x_2)(x_0 - x_1)(x_0 - x_1 - x_2) = 0\} \tag{3.1}$$

is a Ceva arrangement. More precisely, the quotient is obtained as a factor of K_n by the normal subgroup generated by the meridians of the ramification locus $\kappa_n^{-1}(\mathcal{B})$ in $\mathcal{X}_n$.

The fundamental group of the complement to the Ceva arrangement $\mathcal{L}$ is given by the following presentation of G.

$$\langle e_0,\dots, e_5 : [e_1,e_2] = [e_3,e_5,e_1] = [e_3,e_4] = [e_5,e_2,e_4] = e_4e_3e_5e_2e_1e_0 = 1\rangle \tag{3.2}$$

where e_i is a meridian of the component appearing in the $(i+1)$-th place in (3.1), $[\alpha, \beta]$ denotes the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$, and $[\alpha, \beta, \delta]$ denotes the triple of commutators $[\alpha\beta\delta, \alpha]$, $[\alpha\beta\delta, \beta]$, and $[\alpha\beta\delta, \delta]$ leading to a triple of relations in (3.2).

In other to obtain (3.2) one can use the *non-generic* Zariski-Van Kampen method on Figure 3.1 (see [9, Section 1.4]). The dotted line ℓ represents a generic line where the meridians $e_0, \ldots, e_5$ are placed (note that the last relation on (3.2) is the relation in the fundamental group of $\ell \setminus (\mathcal{L} \cap \ell) \approx \mathbb{P}^1_{6\infty}$). The first two relations on (3.2) appear when moving the generic line around ℓ_1. The third and fourth relations come from moving the generic line around ℓ_4.

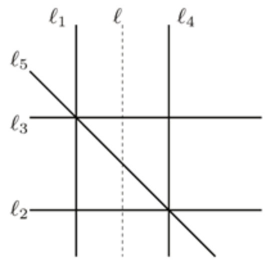

Figure 3.1. Ceva arrangement.

The fundamental group of the complement to $\mathcal{D}_n \cup \mathcal{B}$ is equal to the kernel K_n of the epimorphism

$$\begin{array}{ccc} G & \stackrel{\alpha}{\to} & \mathbb{Z}_n \times \mathbb{Z}_n \\ e_0 & \mapsto & (1,1) \\ e_1 & \mapsto & (1,0) \\ e_2 & \mapsto & (0,1) \\ e_3 & \mapsto & (0,0) \\ e_4 & \mapsto & (0,0) \\ e_5 & \mapsto & (0,0) \end{array} \tag{3.3}$$

since it is the fundamental group of the abelian cover with covering transformations $\mathbb{Z}_n \times \mathbb{Z}_n$. Therefore a presentation of the fundamental group of the complement to $\mathcal{D}_n$ can be obtained by taking a factor of K_n by the normal subgroup generated by e_0^n, e_1^n, and e_2^n (which are the meridians to the preimages of the lines x_0, x_1, and x_2 respectively). Using the Reidemeister-Schreier method (*cf.* [21]) combined with the triviality of e_0^n, e_1^n, and e_2^n one obtains the following presentation for $G_n := \pi_1(\mathcal{X}_n)$:

$$G_n = \langle\, e_{3,i,j}, e_{4,i,j}, e_{5,i,j} : \begin{array}{l} (R1)\ e_{3,i+1,j} = e_{5,i,j}^{-1} e_{3,i,j} e_{5,i,j}, \\ (R2)\ e_{4,i,j+1} = e_{5,i,j}^{-1} e_{4,i,j} e_{5,i,j}, \\ (R3)\ e_{5,i+1,j} = e_{5,i,j}^{-1} e_{3,i,j}^{-1} e_{5,i,j} e_{3,i,j} e_{5,i,j}, \\ (R4)\ e_{5,i,j+1} = e_{5,i,j}^{-1} e_{4,i,j}^{-1} e_{5,i,j} e_{4,i,j} e_{5,i,j}, \\ (R5)\ [e_{3,i,j}, e_{4,i,j}] = 1, \\ (R6)\ \prod_{k=0}^{n-1} e_{4,k,k} e_{3,k,k} e_{5,k,k} = 1 \end{array} \rangle \tag{3.4}$$

where $i, j \in \mathbb{Z}_n$ and

$$e_{k,i,j} := e_1^i e_2^j e_k e_2^{-j} e_1^{-i}, \quad k = 3, 4, 5.$$

As a brief description of the Reidemeister-Schreier method, we recall that the generators of G_n are obtained from a set-theoretical section of α in (3.3) (in our case $s : \mathbb{Z}_n \times \mathbb{Z}_n \to G$ is given by $(i, j) \mapsto e_1^i e_2^j$) as follows

$$s(i, j)\ e_k\ (\alpha(e_k)s(i, j))^{-1}.$$

Thus the set $\{e_{k,i,j}\}$ above forms a set of generators of G_n. Finally a complete set of relations can be obtained by rewriting the relations of G in (3.2) (and their conjugates by $s(i, j)$) in terms of the generators of the subgroup G_n.

Example 3.1. In order to illustrate the rewriting method we will proceed with the second relation of G in (3.2).

$$\begin{aligned} s(i, j)[e_1, e_2]s(i, j)^{-1} &= e_1^i e_2^j (e_3 e_4 e_3^{-1} e_4^{-1}) e_2^{-j} e_1^{-i} \\ &= (e_1^i e_2^j e_3 e_2^{-j} e_1^{-i})\ (e_1^i e_2^j e_4 e_2^{-j} e_1^{-i})\ (e_1^i e_2^j e_3^{-1} e_2^{-j} e_1^{-i})\ (e_1^i e_2^j e_4^{-1} e_2^{-j} e_1^{-i}) \\ &= [e_{3,i,j}, e_{4,i,j}] \end{aligned}$$

3.2 Essential coordinate characteristic varieties

Now we will discuss a presentation of G'_n/G''_n as a module over G_n/G'_n, which will be referred to as $M_{\mathcal{D}_n,\mathbf{ab}}$. For details we refer to Section 2.3. Note that G_n/G'_n is isomorphic to $\mathbb{Z}^{2n}$ and is generated by the cycles γ_5, $\gamma_{3,j}, \gamma_{4,i}$, $(i, j \in \mathbb{Z}_n)$ where $\gamma_5 = \mathbf{ab}(e_{5,i,j})$, $\gamma_{3,j} = \mathbf{ab}(e_{3,i,j})$, and $\gamma_{4,i} = \mathbf{ab}(e_{4,i,j})$ satisfying $n\gamma_5 + \sum_j \gamma_{3,j} + \sum_i \gamma_{4,i} = 0$[4]. Let t_5 (respectively $t_{3,j}, t_{4,i}$) be the generators of G_n/G'_n viewed as a multiplicative group corresponding to the additive generators γ_5 (respectively $\gamma_{3,j}, \gamma_{4,i}$). The characteristic varieties of G_n are contained in

$$(\mathbb{C}^*)^{2n} = \operatorname{Spec} \mathbb{C}[t_5^{\pm 1}, t_{3,i}^{\pm 1}, t_{4,j}^{\pm 1}] \Big/ \Big(t_5^n \prod_j t_{3,j} \prod_i t_{4,i} - 1 \Big).$$

As generators of $M_{\mathcal{D}_n,\mathbf{ab}}$ we select commutators of the generators of G_n as given in (3.2). In order to do so, note that using relations $(R1) - (R4)$ in (3.4), a presentation of G_n can be given in terms the $2n + 1$ generators $e_5 := e_{5,0,0}$, $e_{3,j} := e_{3,0,j}$, and $e_{4,i} := e_{4,i,0}$. Hence, by Proposition 2.12, $M_{\mathcal{D}_n,\mathbf{ab}}$ is generated by the $\binom{2n+1}{2}$ commutators

$$\{[e_5, e_{3,j}], [e_5, e_{4,i}], [e_{4,i}, e_{3,j}], [e_{4,i_1}, e_{4,i_2}], [e_{3,j_1}, e_{3,j_2}]\}_{i_*, j_* \in \mathbb{Z}_n}, \qquad (3.5)$$

[4] Recall that **ab** is the morphism of abelianization.

as a $\mathbb{C}[\mathbb{Z}[t_1^{\pm 1}, ..., t_4^{\pm 1}, t_5^{\pm 1}]]$-module. Also, according to Proposition 2.15, a complete set of relations of $M_{\mathcal{D}_n,\mathbf{ab}}$ is given by rewriting the following relations

$$\begin{array}{ll}
(M1)\ [\prod_{i=0}^{n-1} e_{5,i,j}, e_{3,j}] & = 0 \\
(M2)\ [\prod_{i=0}^{n-1} e_{5,i,j}, e_{4,i}] & = 0 \\
(M3)\ [e_{5,i,j+1}, e_{3,i,j+1}e_{5,i,j+1}]e_{5,i,j+1}^{-1} & =[e_{5,i+1,j}, e_{4,i+1,j}e_{5,i+1,j}]e_{5,i+1,j}^{-1} \\
(M4)\ \prod_{i=0}^{n-1} e_{4,i,i}e_{3,i,i}e_{5,i,i} & = 0
\end{array} \tag{3.6}$$

in terms of commutators (3.5) and by the Jacobian relations:

$$\begin{array}{ll}
(t_{3,j}-1)[e_5, e_{4,i}] + (t_{4,i}-1)[e_{3,j}, e_5] + (t_5-1)[e_{4,i}, e_{3,j}] & = 0, \\
(t_{3,j_1}-1)[e_5, e_{3,j_2}] + (t_{3,j_2}-1)[e_{3,j_1}, e_5] + (t_5-1)[e_{3,j_2}, e_{3,j_1}] & = 0, \\
(t_{4,i_1}-1)[e_5, e_{4,i_2}] + (t_{4,i_2}-1)[e_{4,i_1}, e_5] + (t_5-1)[e_{4,i_2}, e_{4,i_1}] & = 0, \\
\ldots &
\end{array} \tag{3.7}$$

In order to rewrite relations $(M1) - (M4)$ one needs to use (3.5) repeatedly. In what follows, we will concentrate on the characters of $\mathrm{Char}(\mathcal{D}_n)$ contained in the coordinate axes $t_{3,j} = t_{4,i} = 1$. Computations for the general case can also be performed, but are more technical and tedious.

Since we are assuming $t_{3,j} = t_{4,i} = 1$, and $t_5 \neq 1$, relations in (3.7) become $[e_{4,i}, e_{3,j}] = [e_{3,j_2}, e_{3,j_1}] = [e_{4,i_2}, e_{4,i_1}] = 0$ and hence $(R5)$ in (3.4) become redundant. A straightforward computation gives the following matrix where each line is a relation from (3.6) written in terms of the commutators $\{[e_5, e_{3,i}], [e_5, e_{4,i}]\}_{i\in\mathbb{Z}_n}$.

$$A_n := \left[\begin{array}{cccccc|cccccc}
\phi_n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \phi_n & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \ldots & 0 & \phi_n & 0 & 0 & 0 & \ldots & 0 & 0 \\
\hline\hline
0 & 0 & 0 & \ldots & 0 & 0 & \phi_n & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \phi_n & 0 & \ldots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \phi_n \\
\hline\hline
1 & -1 & 0 & \ldots & 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\
1 & -1 & 0 & \ldots & 0 & 0 & 0 & t & -t & \ldots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
1 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} \\
1 & -1 & 0 & \ldots & 0 & 0 & -t^{n-1} & 0 & 0 & \ldots & 0 & t^{n-1} \\
\hline
 & & & \vdots & & & & & & \vdots & & \\
\hline
0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} & 1 & -1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} & 0 & t & -t & \ldots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} & 0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} \\
0 & 0 & 0 & \ldots & t^{n-2} & -t^{n-2} & -t^{n-1} & 0 & 0 & \ldots & 0 & t^{n-1} \\
\hline\hline
1 & t & t^2 & \ldots & t^{n-2} & t^{n-1} & 1 & t & t^2 & \ldots & t^{n-2} & t^{n-1}
\end{array}\right]$$

More precisely, the first (respectively second) block of A_n corresponds to the n relations given in $(M1)$ (respectively $(M2)$) of (3.6), $\phi_n := \frac{t^n-1}{t-1}$, and $t = t_5$. The following n blocks of A_n (between double horizontal lines) correspond to the n^2 relations given in $(M3)$ of (3.6). Note that the last row of each of these blocks is a consequence of the remaining $n-1$ rows. The last block corresponds to the relation given in $(M4)$ of (3.6).

Example 3.2. In order to illustrate A_n we will show how to rewrite the first relation for $n = 3$, that is,

$$[e_{5,0,j}e_{5,1,j}e_{5,2,j}, e_{3,j}] \overset{M}{=} \phi_n[e_5, e_{3,j}].$$

Using (2.6) one has

$$[e_{5,0,j}e_{5,1,j}e_{5,2,j}, e_{3,j}] \overset{M}{=} [e_{5,0,j}, e_{3,j}] + t[e_{5,1,j}, e_{3,j}] + t^2[e_{5,0,j}, e_{3,j}].$$

Therefore, it is enough to show that $[e_{5,i,j}, e_{3,j}] = [e_5, e_{3,j}]$. Note that $e_{5,i,j}$ is a conjugate of e_5 (using $(R3)$ and $(R4)$), hence, by (2.7) one obtains $[e_{5,i,j}, e_{3,j}] = [e_5, e_{3,j}]$ (since we are assuming $t_{3,j} = 1$).

Also note that, performing row operations, one can obtain the following equivalent matrix

$$B_n := \left[\begin{array}{cccccc|cccccc}
\phi_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \phi_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & 0 & \phi_n & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & 0 & \dots & 0 & 0 & \phi_n & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \phi_n & 0 & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \phi_n \\
\hline
1 & -1 & 0 & \dots & 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\
1 & -1 & 0 & \dots & 0 & 0 & 0 & t & -t & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & -t^{n-2} \\
0 & t & -t & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & -t^{n-2} \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & t^{n-2} & -t^{n-2} & 0 & 0 & 0 & \dots & t^{n-2} & -t^{n-2} \\
\hline
1 & t & t^2 & \dots & t^{n-2} & t^{n-1} & 1 & t & t^2 & \dots & t^{n-2} & t^{n-1}
\end{array}\right]$$

Finally, one can write the presentation matrix B_n in terms of the basis

$$\big\{[e_5,e_{3,i}] - [e_5,e_{3,i+1}], [e_5,e_{3,n-1}], [e_5,e_{4,i}] - [e_5,e_{4,i+1}], [e_5,e_{4,n-1}]\big\}_{i=0,\dots,n-2}$$

resulting in

$$
\left[\begin{array}{cccccc|cccccc}
\phi_n & \phi_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \phi_n & \phi_n & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & \phi_n & \phi_n & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & \phi_n & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & 0 & \dots & 0 & 0 & \phi_n & \phi_n & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \phi_n & \phi_n & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \phi_n & \phi_n \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \phi_n \\
\hline
1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & t & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & t^2 & \dots & 0 & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & 0 \\
0 & t & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & 0 \\
0 & 0 & t^2 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & 0 \\
 & & & \vdots & & & & & & \vdots & & \\
0 & 0 & 0 & \dots & t^{n-2} & 0 & 0 & 0 & 0 & \dots & t^{n-2} & 0 \\
\hline
\phi_1 & \phi_2 & \phi_3 & \dots & \phi_{n-1} & \phi_n & \phi_1 & \phi_2 & \phi_3 & \dots & \phi_{n-1} & \phi_n
\end{array}\right]
$$

One can use the units in the third block to eliminate columns, leaving the equivalent matrix

$$
\left[\begin{array}{cccccc|cccccc}
\phi_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & -\phi_n & 0 \\
0 & 0 & 0 & \dots & 0 & \phi_n & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \phi_n \\
\hline
1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & t^{n-2} & 0
\end{array}\right]
\cong
\left[\begin{array}{c|ccc}
0 & -\phi_n & 0 & 0 \\
0 & 0 & -\phi_n & 0 \\
\phi_n & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \phi_n \\
\hline
0 & -1 & t^{n-2} & 0
\end{array}\right].
$$

Finally, a last combination of row operations using the units to eliminate columns results in

$$
\left[\begin{array}{c|ccc}
0 & 0 & -\phi_n t^{n-2} & 0 \\
0 & 0 & -\phi_n & 0 \\
\phi_n & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \phi_n \\
\hline
0 & -1 & t^{n-2} & 0
\end{array}\right]
\cong
\left[\begin{array}{c|cc}
0 & -\phi_n t^{n-2} & 0 \\
0 & -\phi_n & 0 \\
\phi_n & 0 & 0 \\
\hline
0 & 0 & \phi_n
\end{array}\right]
\cong
\left[\begin{array}{c|cc}
0 & \phi_n & 0 \\
\phi_n & 0 & 0 \\
\hline
0 & 0 & \phi_n
\end{array}\right].
$$

Hence the $n-1$ non-trivial torsion characters $\chi_n^i := (\xi_n^i, 1, \dots, 1)$, $i = 1, \dots, n$ belong to $\operatorname{Char}(\mathcal{D}_n)$ and have depth 3, that is, $\chi_n^i \in V_3(\mathcal{D}_n)$.

3.3 Marked orbifold pencils

By Theorem 1.1 (1) we know there are at most three strongly independent marked orbifold pencils from the marked variety $(\mathcal{X}_n, \chi_n)$. Our purpose is to explicitly show such three strongly independent pencils. Note that

$$
\begin{aligned}
j_k : \mathbb{P}^2 \setminus (\mathcal{F}_n \cup \mathcal{L}_k) &\to \mathbb{C}_n^* = \mathbb{P}^1_{(n,[1:0]),(\infty,[0:1]),(\infty,[1:1])} \\
[x:y:z] &\mapsto [f_n : x_k^n],
\end{aligned}
\tag{3.8}
$$

for $j = 1, 2$ are two natural orbifold pencils coming from the n-ordinary points of $\mathcal{F}_n$ coming form the triple points of the Ceva arrangement $\mathcal{L}$ which are in $\mathcal{B}$. Consider the marked orbicurve $(\mathbb{C}_{n,n}, \rho_n)$, where $\rho_n = (\xi_n, 1)$, the first coordinate corresponds to the image of a meridian μ_1 around $[0:1] \in \mathbb{P}^1_{(n,[0:1]),(n,[1:0]),(\infty,[1:1])}$ and the second coordinate corresponds to the image of a meridian μ_2 around $[1:0]$ (note that $\pi_1^{\text{orb}}(\mathbb{C}_{n,n}) = \mathbb{Z}_n(\mu_1) * \mathbb{Z}_n(\mu_2)$).

In order to obtain marked orbifold pencils with target $(\mathbb{C}_{n,n}, \rho_n)$ one simply considers the following composition, where i_k and j_k are inclusions

$$\begin{aligned}\psi_k : \mathcal{X}_n \overset{i_k}{\hookrightarrow} \mathbb{P}^2 \setminus (\mathcal{F}_n \cup \mathcal{L}_k) &\overset{j_k}{\to} \mathbb{P}^1_{(n,[1:0]),(\infty,[0:1]),(\infty,[1:1])} \\ &\overset{i}{\hookrightarrow} \mathbb{P}^1_{(n,[0:1]),(n,[1:0]),(\infty,[1:1])}.\end{aligned}$$

Such pencils are clearly marked global quotient orbifold pencils from $(\mathcal{X}_n, \chi_n)$ to $(\mathbb{C}_{n,n}, \rho_n)$, where $(\mathbb{C}_{n,n}, \rho_n)$ is the marked quotient of $C_n := \mathbb{P}^1 \setminus \{[\xi_n^j : 1]\}_{j \in \mathbb{Z}_n}$ by the cyclic action $[x : y] \mapsto [\xi_n x : y]$. The resulting commutative diagrams are given by

$$\begin{array}{rcl} X_n & \overset{\Psi_k}{\to} & C_n \\ [x_0 : x_1 : x_2 : w] & \mapsto & [w : x_k] \\ \downarrow \pi & & \downarrow \\ \mathcal{X}_n & \overset{\psi_k}{\to} & \mathbb{C}_{n,n} \\ [x_0 : x_1 : x_2] & \mapsto & [f_n : x_k^n], \end{array} \tag{3.9}$$

$k = 1, 2$, where X_n is the smooth open surface given by $\{[x_0 : x_1 : x_2 : w] \in \mathbb{P}^3 \mid w^n = f_n\} \setminus \{f_n \ell_1 \ell_2 = 0\}$.

Note that there is a third quasitoric relation involving all components of $\mathcal{D}_n$, namely,

$$f_n x_0^n + \ell_1 \ell_2 = x_1^n x_2^n \tag{3.10}$$

and hence a global quotient marked orbifold map

$$\begin{array}{rcl} \psi_3 : \quad \mathcal{X}_n & \to & \mathbb{C}_{n,n} = \mathbb{P}^1_{(n,[0:1]),(n,[1:0]),(\infty,[1:1])} \\ [x : y : z] & \mapsto & [-f_n x_0^n : x_1^n x_2^n], \end{array} \tag{3.11}$$

which gives rise to the following diagram

$$\begin{array}{rcl} X_n & \overset{\Psi_3}{\to} & C_n \\ [x_0 : x_1 : x_2 : w] & \mapsto & [-w x_0 : x_1 x_2] \\ \downarrow \pi_n & & \downarrow \\ \mathcal{X}_n & \overset{\psi_k}{\to} & \mathbb{C}_{n,n} \\ [x_0 : x_1 : x_2] & \mapsto & [-f_n x_0^n : x_1^n x_2^n]. \end{array} \tag{3.12}$$

Note that, when extending π_n to a branched covering, the preimage of each line $\{\ell_{k,i} = 0\} \subset \mathcal{L}_k$ $(k = 1, 2)$ in $\mathcal{D}_n$ $(\ell_{k,i} := x_0 - \xi_n^i x_k)$ decomposes into n irreducible components $\bigcup_{j \in \mathbb{Z}_n} \ell_{k,i,j}$ and thus allows to consider $\gamma_{k,i,j}$ $(k = 1, 2, i, j \in \mathbb{Z}_n)$ meridians around each component of $\{\ell_{k,i,j} = 0\}$. Also consider a meridian γ_0 around the preimage of $\mathcal{F}_n$.

Theorem 3.3. *The marked orbifold pencils ψ_1, ψ_2, and ψ_3 described above are strongly independent and hence they form a maximal set of strongly independent pencils.*

Proof. Consider $\Psi_{\varepsilon,*} : H_1(X_n; \mathbb{Z}) \to H_1(C_n; \mathbb{Z}) = \mathbb{Z}[\xi_n]$, $\varepsilon = 1, 2, 3$ the three equivariant morphisms described above. Using the commutative diagrams (3.9) and (3.12) one can easily see that

$$\Psi_{\varepsilon,*}(\gamma_{k,i,j}) = \begin{cases} \xi_n^j & \text{if } \varepsilon = k \in \{1, 2\} \\ \xi_n^{i+j} & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

and

$$\Psi_{\varepsilon,*}(\gamma_0) = 0$$

and therefore $\Psi_{\varepsilon,*}$ are surjective $\mathbb{Z}[\xi_n]$-module morphisms. Also note that $[\gamma_{k,i,j}] = \mu_n^j[\gamma_{k,i,0}] \in H_1(X_n; \mathbb{Z})$. Consequently according to (3.13) one has

$$\left(\Psi_{1,*} \oplus \Psi_{2,*} \oplus \Psi_{3,*}\right)(\gamma_{k,i,0}) = \begin{cases} (1, 0, \xi_n^i) & \text{if } k = 1 \\ (0, 1, \xi_n^i) & \text{if } k = 2 \end{cases}$$

which implies that $\Psi_{1,*} \oplus \Psi_{2,*} \oplus \Psi_{3,*}$ is surjective. After the discussion of Section 3.2, since the depth of ξ_n^i is three, the set of strongly independent pencils is indeed maximal. □

4 Order two characters: augmented Ceva

From Theorem 1.1(2), for any order two character χ of depth k in the characteristic variety of the complement of a curve there exist k independent pencils associated with χ whose target is a global quotient orbifold of type $\mathbb{C}_{2,2}$.

Interesting examples for $k > 1$ of this scenario are the augmented Ceva arrangements CEVA$(2, s)$, $s = 1, 2, 3$ (or *erweiterte Ceva cf.* [10, Section 2.3.J, page 81]). Consider the following set of lines:

$$\begin{array}{lll} \ell_1 := x & \ell_4 := (y - z) & \ell_7 := (x - y - z) \\ \ell_2 := y & \ell_5 := (x - z) & \ell_8 := (y - z - x) \\ \ell_3 := z & \ell_6 := (x - y) & \ell_9 := (z - x - y). \end{array} \tag{4.1}$$

The curve $\mathcal{C}_6 := \left\{\prod_{i=1}^{6} \ell_i = 0\right\}$ is a realization of the Ceva arrangement CEVA(2) (a.k.a. braid arrangement or B_3-reflection arrangement). Note that this realization is different from the one used in Section 3. The curve $\mathcal{C}_7 := \left\{\prod_{i=1}^{7} \ell_i = 0\right\}$ is the augmented Ceva arrangement CEVA(2, 1) (a.k.a. a realization of the non-Fano plane). The curve $\mathcal{C}_8 := \left\{\prod_{i=1}^{8} \ell_i = 0\right\}$ is the augmented Ceva arrangement CEVA(2, 2) (a.k.a. a deleted B_3-arrangement). Finally, $\mathcal{C}_9 := \left\{\prod_{i=1}^{9} \ell_i = 0\right\}$ is the augmented Ceva arrangement CEVA(2, 3).

The characteristic varieties of such arrangements of lines are well known (*cf.* [15, 30, 31]). Such computations are done via a presentation of the fundamental group and using Fox derivatives. In most cases (except for the simplest ones) the need of computer support is basically unavoidable. In [15, Example 3.11] there is an alternative calculation of the positive dimensional components of depth 1 via pencils.

Here we will give an interpretation via orbifold pencils of the characters of depth 2, which will account for the appearance of these components of the characteristic varieties independently of computation of the fundamental group.

4.1 Ceva pencils and augmented Ceva pencils

Note that $x(y-z) - y(x-z) + z(x-y) = 0$ and hence

$$\begin{array}{cccc} f_C: & \mathbb{P}^2 & \to & \mathbb{P}^1 \\ & [x:y:z] & \mapsto & [\ell_1\ell_4 : \ell_2\ell_5] \end{array}$$

is a pencil of conics such that ($f_C^*([0:1]) = \ell_1\ell_4$, $f_C^*([1:0]) = \ell_2\ell_5$, $f_C^{-1}([1:1]) = \ell_3\ell_6$) (we will refer to it as the *Ceva pencil*). Analogously

$$x(y-z)(x-y-z)^2 - y(x-z)(y-z-x)^2 + z(x-y)(z-x-y)^2 = 0$$

and hence

$$\begin{array}{cccc} f_{SC}: & \mathbb{P}^2 & \to & \mathbb{P}^1 \\ & [x:y:z] & \mapsto & [\ell_1\ell_4\ell_7^2 : \ell_2\ell_5\ell_8^2] \end{array}$$

is a pencil of quartics such that ($f_{SC}^*([0:1]) = \ell_1\ell_4\ell_7^2$, $f_{SC}^*([1:0]) = \ell_2\ell_5\ell_8^2$, $f_{SC}^*([1:1]) = \ell_3\ell_6\ell_9^2$) (we will refer to it as the *augmented Ceva pencil*).

4.2 Characteristic varieties of $\mathcal{C}_i$, $i = 6, 7, 8, 9$

We include the structure of the characteristic varieties of these curves for the reader's convenience. As reference for such computations see [14, 18, 23, 25, 30, 31].

We will denote by $\mathcal{X}_*$ the complement of the curve $\mathcal{C}_*$ in $\mathbb{P}^2$, for $* = 6, 7, 8, 9$.

4.2.1 Arrangement $\mathcal{C}_6$. The characteristic variety $\mathrm{Char}(\mathcal{C}_6)$ consists of four non-essential coordinate components associated with the four triple points of $\mathcal{C}_6$ (see Remark 2.26 (1))[5] and one essential component of dimension 2 and depth 1 given by the Ceva pencil

$$\psi_6 := f_C|_{\mathcal{X}_6} : \mathcal{X}_6 \to \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\}.$$

4.2.2 Arrangement $\mathcal{C}_7$. The characteristic variety $\mathrm{Char}(\mathcal{C}_7)$ consists of six (respectively four) non-essential coordinate components associated with the six triple points of $\mathcal{C}_7$ (respectively four $\mathcal{C}_6$-subarrangements) of dimension 2 and depth 1. In addition, there is one extra character of order two, namely,

$$\chi_7 := (1, -1, -1, 1, -1, -1, 1)$$

of depth 2.[6] In order to check the value of the depth, one needs to find all marked orbifold pencils in $(\mathcal{X}_7, \chi_7)$ of target $(\mathbb{C}_{2,2}, \rho)$ where $\rho := (-1, -1)$ is the only possible non-trivial character of $\mathbb{C}_{2,2}$. Two such independent pencils are the following,

$$\psi_{7,1} := f_C|_{\mathcal{X}_7} : \mathcal{X}_7 \to \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\} \to \mathbb{P}^1_{(2,[1:0]),(2,[1:1]),(\infty[0:1])}$$

and

$$\psi_{7,2} := f_{SC}|_{\mathcal{X}_7} : \mathcal{X}_7 \to \mathbb{P}^1_{(2,[1:0]),(2,[1:1]),(\infty[0:1])}.$$

This is the maximal number of independent pencils by Theorem 1.1.

4.2.3 Arrangement $\mathcal{C}_8$. The characteristic variety $\mathrm{Char}(\mathcal{C}_8)$ consists of six (respectively five) non-essential coordinate components associated with the six triple points of $\mathcal{C}_8$ (respectively four $\mathcal{C}_6$-subarrangements) of dimension 2 and depth 1. In addition, there is one 3-dimensional non-essential coordinate component of depth 2 associated with its quadruple point (see Corollary 2.26(2)).

Consider the following augmented Ceva pencil

$$\psi_{8,1} := f_{SC}|_{\mathcal{X}_8} : \mathcal{X}_8 \to \mathbb{P}^1_{(2,[1:1]),(\infty[0:1]),(\infty[1:0])}.$$

5 a.k.a. local components.

6 The subscript 7 refers to the arrangement $\mathcal{C}_7$. Similar notation will be used in the examples that follow. A second subscript (when necessary) will be used to index the characters considered.

Computation of the induced map on the variety of characters shows that this map yields the only non-coordinate translated component of dimension 1 and depth 1 observed in the references above. Finally, there are two characters of order two, namely,

$$\begin{aligned} \chi_{8,1} &:= (1, -1, -1, 1, -1, -1, 1, 1) \quad \text{and} \\ \chi_{8,2} &:= (-1, 1, -1, -1, 1, -1, 1, 1) \end{aligned}$$

of depth 2. In order to check the value of the depth, one needs to find two marked orbifold pencils on $(\mathcal{X}_8, \chi_{8,1})$ with target $(\mathbb{C}_{2,2}, \rho)$, where

$$\mathbb{C}_{2,2} := \mathbb{P}^1_{(2,[1:0]),(2,[1:1]),(\infty[0:1])}$$

and $\rho := (-1, -1, 1)$ is the only non-trivial character of $\mathbb{C}_{2,2}$. Two such independent pencils can, for example, be given as follows

$$\psi_{8,2} := f_C|_{\mathcal{X}_8} : \mathcal{X}_8 \to \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\} \to \mathbb{P}^1_{(2,[1:0]),(2,[1:1]),(\infty[0:1])}$$

and

$$\psi_{8,3} := f_{SC}|_{\mathcal{X}_8} : \mathcal{X}_8 \to \mathbb{P}^1_{(2,[1:1])} \setminus \{[1:0], [0:1]\} \to \mathbb{P}^1_{(2,[1:0]),(2,[1:1]),(\infty[0:1])}.$$

4.2.4 Arrangement $\mathcal{C}_9$. The characteristic variety $\mathrm{Char}(\mathcal{C}_9)$ consists of four (respectively eleven) non-essential coordinate components associated with the four triple points of $\mathcal{C}_9$ (respectively eleven $\mathcal{C}_6$-subarrangements), which have dimension 2 and depth 1. In addition, there are three 3-dimensional non-essential coordinate components of depth 2 associated with the quadruple points of $\mathcal{C}_9$. Consider the following augmented Ceva pencil

$$\psi_{9,1} := f_{SC}|_{\mathcal{X}_9} : \mathcal{X}_9 \to \mathbb{P}^1 \setminus \{[1:0], [0:1], [1:1]\}.$$

Computations of the induced map on the variety of characters show that this pencil yields the only non-coordinate translated component of dimension 2 and depth 1 observed in the references above.

Finally, there are also three characters of order two

$$\begin{aligned} \chi_{9,1} &:= (-1, -1, 1, -1, -1, 1, 1, 1, 1), \\ \chi_{9,2} &:= (-1, 1, -1, -1, 1, -1, 1, 1, 1), \quad \text{and} \\ \chi_{9,3} &:= (1, -1, -1, 1, -1, -1, 1, 1, 1) \end{aligned}$$

of depth 2. In order to check the value of the depth, one needs to find two independent marked orbifold pencils on $(\mathcal{X}_9, \chi_{9,1})$ with target $(\mathbb{C}_{2,2}, \rho)$ where $\mathbb{C}_{2,2} := \mathbb{P}^1_{(2,[0:1]),(2,[1:0]),(\infty[1:1])}$ and $\rho := (-1, -1, 1)$ is the only non-trivial character on $\mathbb{C}_{2,2}$. Two such independent pencils can be given, for example, as follows

$$\psi_{9,2} := f_C|_{\mathcal{X}_9} : \mathcal{X}_9 \to \mathbb{P}^1 \setminus \{[0:1],[1:0],[1:1]\} \to \mathbb{P}^1_{(2,[0:1]),(2,[1:0]),(\infty[1:1])}$$

and

$$\psi_{9,3} := f_{SC}|_{\mathcal{X}_9} : \mathcal{X}_9 \to \mathbb{P}^1 \setminus \{[0:1],[1:0],[1:1]\} \to \mathbb{P}^1_{(2,[0:1]),(2,[1:0]),(\infty[1:1])}.$$

Remark 4.1. Note that the depth 2 characters in $\mathrm{Char}(\mathcal{C}_8)$ and $\mathrm{Char}(\mathcal{C}_9)$ lie in the intersection of positive dimensional components and this fact forces them to have depth greater than 1, see [8, Proposition 5.9].

4.3 Comments on independence of pencils

- **Depth conditions on the target:** First of all note that the condition on the target $(\mathcal{C}, \rho)$ to have $d(\rho) > 0$ is essential in the discussion above, *i.e.* pencils with target satisfying $d(\rho) = 0$ may not contribute to the characteristic varieties. For instance, the space $\mathcal{X}_6$ also admits several global quotient pencils coming from the augmented Ceva pencil, namely

  $$\psi_6' := f_{SC}|_{\mathcal{X}_6} : \mathcal{X}_6 \to \mathbb{P}^1_{(2,[0:1]),(2,[1:0]),(2,[1:1])} \to \mathbb{P}^1_{(2,[0:1]),(2,[1:0])}.$$

  However, the orbifold $\mathbb{P}_{2,2}$ is a global quotient orbifold whose orbifold fundamental group is abelian, so no non-trivial characters belong to its characteristic variety.
- **Independence of pencils.** Here is an explicit argument for independence of pencils for one of the cases discussed in last section. Consider the pencils $\psi_{9,2}$ and $\psi_{9,3}$ described above as marked pencils from $(\mathcal{X}_9, \chi_{9,1})$ having $(\mathbb{C}_{2,2}, \rho)$ as target. The marking produces the following commutative diagrams:

  $$\begin{array}{rcl} X_{9,2} & \overset{\Psi_{9,2}}{\to} & C_2 \\ [x:y:z:w] & \mapsto & [\ell_1\ell_4 : w] \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathcal{X}_9 & \overset{\psi_{9,2}}{\to} & \mathbb{C}_{2,2} \\ [x:y:z] & \mapsto & [\ell_1\ell_4 : \ell_2\ell_5], \end{array}$$

  and

  $$\begin{array}{rcl} X_{9,2} & \overset{\Psi_{9,3}}{\to} & C_2 \\ [x:y:z:w] & \mapsto & [\ell_1\ell_4\ell_7 : w\ell_8] \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathcal{X}_9 & \overset{\psi_{9,3}}{\to} & \mathbb{C}_{2,2} \\ [x:y:z] & \mapsto & [\ell_1\ell_4\ell_7^2 : \ell_2\ell_5\ell_8^2], \end{array}$$

where $X_{9,2}$ is contained in $\{[x : y : z : w] \mid w^2 = \ell_1\ell_4\ell_2\ell_5\}$, $C_2 := \mathbb{P}^1 \setminus \{[1:1],[1:-1]\}$ and $\tilde{\pi}$ is given by $[u : v] \mapsto [u^2 : v^2]$.
Consider $\gamma_{i,k}$, $i = 3, 6, 7, 8, 9$, $k = 1, 2$ the lifting of meridians around ℓ_i in $X_{9,2}$. Also denote by $\mathbb{Z}[\mathbb{Z}_2]$ the ring of deck transformations of $\tilde{\pi}$ as before, where $\mathbb{Z}_2$ acts by multiplication by $\xi_2 = (-1)$. Note that, as before $\Psi_{9,2}(\gamma_{3,k}) = \Psi_{9,2}(\gamma_{3,k}) = (-1)^k$ and $\Psi_{9,3}(\gamma_{4,k}) = \Psi_{9,3}(\gamma_{4,k}) = (-1)^{k+1}$. However, $\Psi_{9,2}(\gamma_{9,k}) = 0$ and $\Psi_{9,3}(\gamma_{9,k}) = (-1)^k$. Therefore $\psi_{9,2}$ and $\psi_{9,3}$ are independent pencils of $(\mathcal{X}_9, \chi_{9,1})$ with target $(\mathbb{C}_{2,2}, \rho)$.

5 Curve arrangements

Consider the space $\mathcal{M}$ of sextics with the following combinatorics:

1. $\mathcal{C}$ is a union of a smooth conic $\mathcal{C}_2$ and a quartic $\mathcal{C}_4$;
2. $\operatorname{Sing}(\mathcal{C}_4) = \{P, S\}$ where S is a cusp of type $\mathbb{A}_4$ and P is a node of type $\mathbb{A}_1$;
3. $\mathcal{C}_2 \cap \mathcal{C}_4 = \{S, R\}$ where S is a $\mathbb{D}_7$ on $\mathcal{C}$ and R is a $\mathbb{A}_{11}$ on $\mathcal{C}$.

In [4] it is shown that $\mathcal{M}$ has two connected components, say $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$. The following are equations for curves in each connected component:

$$f_6^{(1)} = f_2^{(1)} f_4^{(1)} := \left((y+3x)\,z + \tfrac{3y^2}{2}\right)$$
$$\left(x^2z^2 - \left(xy^2 + \tfrac{15}{2}\,x^2y + \tfrac{9}{2}x^3\right)z - 3x\,y^3 - \tfrac{9x^2y^2}{4} + \tfrac{y^4}{4}\right)$$

for $\mathcal{C}_6^{(1)} \in \mathcal{M}^{(1)}$ and

$$f_6^{(2)} = f_2^{(2)} f_4^{(2)} := \left(\left(y + \tfrac{x}{3}\right)z - \tfrac{y^2}{6}\right)$$
$$\left(xz^2 - \left(xy^2 + \tfrac{9x^2y}{2} + \tfrac{3x^3}{2}\right)z + \tfrac{y^4}{4} + \tfrac{3x^2y^2}{4}\right)$$

for $\mathcal{C}_6^{(2)} \in \mathcal{M}^{(2)}$.

The curves $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_6^{(2)}$ form a Zariski pair since their fundamental groups are not isomorphic. This cannot be detected by Alexander polynomials since both are trivial. In [4] the existence of an essential coordinate character of order two in the characteristic variety of $\mathcal{C}_6^{(2)}$ was shown enough to distinguish both fundamental groups, since the characteristic variety of $\mathcal{C}_6^{(1)}$ is trivial.

By Theorem 1.1 (2) this fact can also be obtained by looking at possible orbifold pencils. Note that there exists a conic $\mathcal{Q} := \{q = 0\}$ passing through S and R such that $(\mathcal{Q}, \mathcal{C}_4^{(1)})_S = 4$, $(\mathcal{Q}, \mathcal{C}_4^{(2)})_S = 5$, and

$(\mathcal{Q}, \mathcal{C}_2^{(2)})_R = 3$, $(\mathcal{Q}, \mathcal{C}_2^{(2)})_R = 3$. Consider $L := \{\ell = 0\}$ the tangent line to $\mathcal{Q}$ at S. One has the following list of multiplicities of intersection:

$$\begin{array}{ll}
(\mathcal{Q}, \mathcal{C}_2^{(2)} + 2L)_S = (\mathcal{Q}, \mathcal{C}_4^{(2)})_S = 5 & (\mathcal{Q}, \mathcal{C}_2^{(2)} + 2L)_R = (\mathcal{Q}, \mathcal{C}_4^{(2)})_R = 3 \\
(\mathcal{C}_4^{(2)}, 2\mathcal{Q})_S = (\mathcal{C}_4^{(2)}, \mathcal{C}_2^{(2)} + 2L)_S = 10 & (\mathcal{C}_4^{(2)}, 2\mathcal{Q})_R = (\mathcal{C}_4^{(2)}, \mathcal{C}_2^{(2)} + 2L)_R = 6 \\
(\mathcal{C}_2^{(2)}, \mathcal{C}_4^{(2)})_S = (\mathcal{C}_2^{(2)}, 2\mathcal{Q})_S = 2 & (\mathcal{C}_2^{(2)}, \mathcal{C}_4^{(2)})_R = (\mathcal{C}_2^{(2)}, 2\mathcal{Q})_R = 6 \\
(L, \mathcal{C}_4^{(2)})_S = (L, 2\mathcal{Q})_S = 4 & (L, \mathcal{C}_4^{(2)})_R = (L, 2\mathcal{Q})_R = 0.
\end{array}$$

By [12], this implies that $(\mathcal{C}_2^{(2)} + 2L, \mathcal{C}_2^{(2)}, 2\mathcal{Q})$ are members of a pencil of quartics. In other words, there is a marked orbifold pencil from $\mathcal{C} := \mathbb{P}^2 \setminus \mathcal{C}_6^{(2)}$ marked with $\chi := (-1, 1)$ to $\mathbb{P}^1_{(2,[0,1]),(2,[1:0]),(\infty[1:1])}$ given by $[x : y : z] \mapsto [f_2^{(2)}\ell^2 : q^2]$ whose target mark is the character $\rho := (-1, -1, 1)$.

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