ON COMBINATORIAL INVARIANCE OF THE COHOMOLOGY OF MILNOR FIBER OF ARRANGEMENTS AND CATALAN EQUATION OVER FUNCTION FIELDS.

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Abstract. We discuss combinatorial invariance of the betti numbers of the Milnor fiber for arrangements of lines with points of multiplicity two and three and describe a link between this problem and enumeration of solutions of the Catalan equation over function field in the case when its coefficients are products of linear forms and the equation defines an elliptic curve.

1. Introduction

An interesting problem in the theory of arrangements of hyperplanes over the field of complex numbers is to determine which topological invariants of the complement depend only on the combinatorics of the arrangements i.e. the poset formed by intersections of the hyperplanes of the arrangement. A theorem, going back to Arnold, Orlik and Solomon (cf. [17]), asserts that the cohomology of the complement is a combinatorial invariant. On the other hand, it is known that the homotopy type (cf. [18]) and even the fundamental group (cf. [3]) cannot be determined from the combinatorics of the arrangement since there exist pairs of arrangements with the same combinatorics but different such invariants.

Given an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}^{n+1}$ in which the equations of hyperplanes are $L_i = 0$ where L_i are linear forms, one can consider the affine hypersurface $\Pi L_i = 1$ which is called the Milnor fiber of arrangement. It is known that for the Milnor fiber $M_{\mathcal{A}}$ the homology group $H_1(M_{\mathcal{A}}, \mathbb{Z})$ depends only on the fundamental group of the complement to $\mathcal{A}$ and more precisely it is the abelianization of the subgroup of $\pi_1(\mathbb{P}^2 - \mathcal{A})$ which the kernel of the surjection $lk : \pi_1(\mathbb{P}^2 - \mathcal{A}) \to \mathbb{Z}_{\mathrm{Card}\mathcal{A}}$ given by the total linking number with $\mathcal{A}$ cf. ([14]). One can ask about the combinatorial invariance of the cohomology of $M_{\mathcal{A}}$ and in particular $rkH^1(M_{\mathcal{A}}, \mathbb{Q})$ called (the first) Milnor number of the arrangement (for arrangements of lines, this is equivalent to the combinatorial invariance of remaining cohomology groups). To this end we show the following:

Theorem 1.1. *If two arrangements of lines $\mathcal{A}_1$ and $\mathcal{A}_2$ with the multiple points of multiplicity three and two only are combinatorially equivalent and if the monodromy on the Milnor fiber of $\mathcal{A}_1$ has an eigenvalue different from one then so does the monodromy on the Milnor fiber the arrangement $\mathcal{A}_2$.*

This theorem will be derived from the following more precise statement relating the cohomology of Milnor fiber and other data of arrangement. We say that an arrangement $\mathcal{A}$ is composed of a pencil L of plane curves (cf.[15],[8]) if $L = \{\lambda F + \mu G | degF = degG\}$ contains three curves $\lambda F + \mu G$ with equations which are products

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of linear forms such that $\mathcal{A}$ is the zero set of the forms defining these curves. A pencil is reduced if equations of the above reducible curved are reduced. (see section 2 for other definitions needed for this theorem).

Theorem 1.2. *Let $\mathcal{A}$ be an arrangement of lines with double and triple points only. If the monodromy acting on one-dimensional cohomology of the Milnor number of $\mathcal{A}$ has an eigenvalue different from one then the arrangement $\mathcal{A}$ is composed of a reduced pencil. The characteristic variety of the arrangement $\mathcal{A}$ has a component containing the identity $Char\pi_1(\mathbb{P}^2-\mathcal{A})$. Vice versa, if $\mathcal{A}$ is composed of a reduced pencil then the monodromy has eigenvalue $exp(\frac{2\pi i}{3})$.*

The argument for the proof of above results relies on the relation between the Alexander polynomials of plane curves and Mordell-Weil groups considered recently in [4] and study of the Catalan equation over function field. Catalan equation, i.e. $X^p - Y^q = 1$ or more generally

$$A_1X^p + A_2Y^q + A_3Z^r = 0 \tag{1}$$

has a long history and its properties of course depend on the ground field. Over function field, i.e. the case when $A_i \in \mathbb{C}(\mathcal{X})$ where $\mathcal{X}$ is a projective manifold the equation (1) was considered by J.Silverman. A finiteness results was shown in [20] in the case the equation (1) defines over the function field a curve of genus greater than 1. Elementary treatment of Fermat equation (i.e. $A_i \in \mathbb{C}, p = q = r$) is given in [10]. Finding solution in the case of genus zero is trivial. Present paper considers the case $p = q = r = 3$ and A_i are product of linear forms in 3 variables. The answer to question about enumeration of solutions depends on the geometry of the curve $A_1 \cdot A_2 \cdot A_3 = 0$ in $\mathbb{P}^2$ and we describe the cases when the number of solutions is infinite (cf. theorems 3.1 and 3.2).

The paper organized as follows. In the next section we recall terminology from the theory of characteristic varieties of arrangements and related background information. In section 3 we prove results on existence and enumeration of solutions to Catalan equation. Proof of theorems presented in the last section. The paper ends with remark on properties of Mordell Weil groups which would imply combinatorial invariance of the cohomology of Milnor fiber. The author wants to thank S.Kaliman and L.Katzarkov for hospitality during visit to University of Miami where this work was written as well as S.Yuzvinsky for a useful comments. Finally, I grateful to A.Dimca for pointing out example 4.1 which helped to correct error in original statement of theorem 1.2 and to J.I.Cogolludo-Agustin for numerous examples of Catalan equations which helped to correct theorem 3.1.

2. Preliminaries

As was already mentioned, Milnor fiber of an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}^{n+1}$ is affine hypersurface $M_{\mathcal{A}}$ in $\mathbb{C}^{n+2}$ given by equation $\Pi L_i = 1$ where L_i are linear forms defining the hyperplanes of the arrangement. The standard map $\mathbb{C}^{n+2} - \mathcal{O} \to \mathbb{P}^{n+1}$ (Ø is the origin) restricted on $M_{\mathcal{A}}$ is a cyclic cover

$$M_{\mathcal{A}} \to \mathbb{P}^{n+1} - \mathcal{A} \tag{2}$$

of degree equal to the number of hyperplanes $r = Card\mathcal{A}$ in $\mathcal{A}$. This cyclic cover corresponds to the homomorphism $\pi_1(\mathbb{P}^{n+1} - \mathcal{A}) \to \mathbb{Z}_r$ onto the cyclic group of order $r = \mathrm{Card}\mathcal{A}$ given by the total modulo r linking number with the union of hyperplanes in $\mathcal{A}$. We are interested in $dimH_1(M_{\mathcal{A}}, \mathbb{C})$ and also in the eigenspaces of the generator of the group $\mathbb{Z}_r$ of deck transformations acting on $H_1(M_{\mathcal{A}}, \mathbb{C})$. By Lefschetz theorem applied to a generic plane H in $\mathbb{P}^{n+1}$ this group is isomorphic to the corresponding group of line arrangement formed by intersections of H with the hyperplanes of $\mathcal{A}$. So from now on we shall assume that $n = 1$. Since the main results in this note deal with the case of arrangements of lines with points of multiplicity 2 and 3 from now on we shall restrict ourselves to the case of such multiplicities as well.

By [12], [13] (cf. also [14]) the rank of the cyclic cover (2) can be found in terms the Alexander polynomial of $\mathbb{P}^2 - \mathcal{A}$. More precisely, the Alexander polynomial of $\mathbb{P}^2 - \mathcal{A}$ is equal to

$$(t-1)^{\mathrm{Card}\mathcal{A}-2}(t^2+t+1)^s \tag{3}$$

for $s = dimH^1(\mathbb{P}^2, \mathcal{J}_{\mathcal{S}}(\frac{2\mathrm{Card}\mathcal{A}}{3} - 3))$ where $\mathcal{J}_{\mathcal{S}} \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf having stalk different from $\mathcal{O}_P$ iff P is a triple point of the arrangement and equal to maximal ideal at those P. Moreover if r is divisible by 3 then

$$\mathrm{rk}H_1(M_{\mathcal{A}})_\omega = \mathrm{rk}H_1(M_{\mathcal{A}})_{\omega^2} = s, \quad \mathrm{rk}H_1(M_{\mathcal{A}})_1 = r-2 \tag{4}$$

where $H_1(M_{\mathcal{A}})_\zeta$ is the eigenspace of the deck transformation of the covering (2) with eigenvalue ζ and $\omega = exp(\frac{2\pi\sqrt{-1}}{3})$. If r is not divisible by 3 then the dimensions of eigenspaces with eigenvalue ω, ω^2 are equal to zero. Hence from now on we shall assume that r is divisible by 3.

Together with unbranched cover $M_{\mathcal{A}}$ of degree r one can consider a non-singular model of *3-fold* branched cyclic covering $\bar{M}_{\mathcal{A}}$ of $\mathbb{P}^2$ which is a compactification of the quotient of $M_{\mathcal{A}}$ by the action of the subgroup $\mathbb{Z}_{\frac{r}{3}} \subset \mathbb{Z}_r$. The integer $\mathrm{rk}H_1(\bar{M}_{\mathcal{A}}, \mathbb{C})$ is birational invariant and hence depends only on $M_{\mathcal{A}}$. Using results of [12], one has:

$$\frac{1}{2}\mathrm{rk}H_1(\bar{M}_{\mathcal{A}}) = \mathrm{rk}H_1(\bar{M}_{\mathcal{A}})_\omega = \mathrm{rk}H_1(\bar{M}_{\mathcal{A}})_{\omega^2} = s \tag{5}$$

Recall the following definition from [4] adapting it to the case of arrangements (rather than arbitrary plane curves).

Definition 2.1. A quasi-toric relation of orbifold type $(3,3,3)$ is the identity

$$F_1f^3 + F_2g^3 + F_3h^3 = 0$$

where $F_i, i = 1,2,3$ is either a constant or a reduced curve such that $F_i = 0$ are without common components and $F_1F_2F_3$ has the arrangement $\mathcal{A}$ as its zero set.

(A'priori, of course for a given arrangement there are several possibilities for F_i's, including $F_i = 1$).

Let E be elliptic curve over $\mathbb{C}$ with j-invariant zero. As a model for E we shall use the 3-fold ramified covering of $\mathbb{C}$ branched at $0, 1$ (and infinity) which is a closure of the affine variety in $\mathbb{C}^4$ given by the equations:

$$c^3 = ab, a = s, b = 1-s \tag{6}$$

(the cyclic covering map is given by $(a,b,c,s) \to s$). As a model of $\bar{M}_{\mathcal{A}}$ for calculation of $H^1(\bar{M}_{\mathcal{A}})$ we shall use the complete intersection in $\mathbb{C}^5$ (with coordinates (x, y, A, B, C)) given by:

$$A = \frac{F_1(x,y)}{F_3(x,y)}, B = \frac{F_2(x,y)}{F_3(x,y)}, C^3 = AB \tag{7}$$

(here F_i are dehomogenized forms used in definition 2.1. The surface (7) is the threefold cyclic cover of $\mathbb{C}^2$ branched over the curve $F_1F_2F_3 = 0$ corresponding to the homomorphism of the fundamental group sending the classes of loops corresponding to the curves $F_i = 0$ to the same element of $\mathbb{Z}_3$.

There is a correspondence between the $\mathbb{Z}_3$-equivariant maps $\phi : \bar{M}_{\mathcal{A}} \to E$ and quasi-toric relations given as follows. Note that such a map ϕ takes a fixed point of $\mathbb{Z}_3$ to one of three ramification points of $E \to \mathbb{P}^1$. Denoting the polynomials whose zero sets are the remaining components of preimages of ramification points as f, g, h respectively, one can write the morphism ϕ as a compactification of the restriction on affine portion of $\bar{M}_{\mathcal{A}}$ of $\Phi : \mathbb{C}^5 \to \mathbb{C}^4$ given by:

$$a = A(\frac{f}{h})^3, b = B(\frac{g}{h})^3, c = C\frac{fg}{h^2}, s = A(\frac{f}{h})^3 \tag{8}$$

Let us define the equivalence between quasi-toric relations 2.1 involving F_i, f, g, h and F'_i, f', g', h' saying that two such relations are equivalent if for some $\lambda, \lambda_f, \lambda_g, \lambda_h \in \mathbb{C}$, one has $F_i = \lambda F'_i$, $\frac{\lambda_f}{\lambda_h} \in \mu_3$, $\frac{\lambda_g}{\lambda_h} \in \mu_3$ and $\lambda_f \lambda_g = \lambda_h^2$.

With this we have the following:

Proposition 2.2. *There is one to one correspondence between quasi-toric relations up to above equivalence and non constant equivariant maps $\bar{M}_{\mathcal{A}} \to E$.*

Proof. Indeed it follows from formulas (6), (7) that (f, g, h) appearing in (8) satisfy the quasi-toric relation. Vice versa, if f, g, h satisfy a quasi-toric relation, the formulas (8) yield a map which when restricted on affine portion of $\bar{M}_{\mathcal{A}}$ provide a dominant map $\phi : \bar{M}_{\mathcal{A}} \to E$. □

As in [4] we shall view the maps $\bar{M}_{\mathcal{A}} \to E$ as $\mathbb{C}(\bar{M}_{\mathcal{A}})$-points of the elliptic curve E. As such they form a group with the quotient by the subgroup of constant maps (the Mordell-Weil group) being finitely generated (Mordell-Weil theorem cf. [11]). The following is a special case of result shown in [4].

Theorem 2.3. *The rank of the Mordell-Weil group $MW(\mathcal{A})$ of $\mathbb{C}(\bar{M}_{\mathcal{A}})$-points of E is equal to $2s$ where s is the degree of the factor $(t^2 + t + 1)$ is (3).*

Finally we shall need few results on characteristic varieties of arrangements. Recall (cf. [14]) that one can view them the components of affine subvariety V of the torus $Char\pi_1(\mathbb{P}^2 - \mathcal{A})$ consisting of characters χ such that

$$dimH_1(\mathbb{P}^2 - \mathcal{A}, \chi) > 0 \tag{9}$$

These affine subvarieties are finite unions of translated subgroups of $Char\pi_1(\mathbb{P}^2 - \mathcal{A})$ by the points of finite order in $\mathrm{Char}\pi_1(\mathbb{P}^2 - \mathcal{A})$ (cf. [1],[16]).

The following results is contained in [14] and [15] respectively (cf. also [5]):

Theorem 2.4. *1) The dimension of the ω-eigenspace of the monodromy acting on $H^1(M_{\mathcal{A}})$ of the Milnor number $M_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is equal to $dimH^1(\mathbb{P}^2 - \mathcal{A}, L_\omega)$ where L_ω is the local system $\pi_1(\mathbb{P}^2 - \mathcal{A}) \to \mathbb{C}^*$ sending each loop, having the linking number equal to one with a hyperplane in $\mathcal{A}$ and zero with all the others, to the a root of unity ω.*

2)The dimension of $H^1(\mathbb{P}^2 - \mathcal{A}, L_\chi)$ of the local system is not smaller than $d - 1$ where d is the dimension of the component of characteristic variety to which it belongs.

More specifically, 2) follows from semi-continuity of ranks of cohomology of local systems (as in [15]) and the fact that generic local system in a component of positive dimension is a twist of the pull back of a local system on $\mathbb{P}^1$ minus a collection of points. The latter does not change generically the rank of H^1 (cf. [1], Prop. 1.7) and relation in 2) is obvious for the local systems on complements to a finite union of point in $\mathbb{P}^1$.

3. Catalan Equation $A_1f^3 + A_2g^3 + A_3h^3 = 0$ with coefficients being products of linear forms.

The purpose of this section is to prove the following:

Theorem 3.1. *Let $A_i = \Pi_{j\le r_i} L_{i,j}, i = 1, 2, 3,$ be products of linear forms $L_{i,j} \in \mathbb{C}[x_1, x_2, x_3]$ (r_i are non negative integers). Assume that $L_{i,j}$ is not a multiple of $L_{i',j'}$ for any pair two pairs $(i, j), (i', j')$ and that lines $L_{i,j} = 0$ form an arrangement of lines with points of multiplicity 2 and 3 only.*

1) The equation

$$A_1f^3 + A_2g^3 + A_3h^3 = 0 \tag{10}$$

has solution only if $A_1A_2A_3 = A'_1A'_2A'_3$ where A'_1, A'_2, A'_3 are linear dependent over $\mathbb{C}$. In this case the solutions of (10) with linearly dependent A_i, are the pullbacks of solutions of Catalan equation over $\mathbb{C}(t)$.

2) In the case A_i are linearly dependent over $\mathbb{C}$ the set of solutions to (10) is infinite.

Proof. First we show that if the equation (10) with $A_1 = A_2 = 1$ has a solution then there exist factorization $A_3 = A'_1A'_2A'_3$ such that at least two factors are non-constant and the Catalan equation (10) with these factors as coefficients has a solution. Assume that $A_1 = A_2 = 1$, select f, g, h which form a solution with smallest possible degree of h and such that f, g, h have no common factors. Then $-A_3h^3 = \Pi(f + \omega_ig)$ ($\omega_i^3 = -1$). Let $\tilde{h}$ be an irreducible factor of h. There is only one factor $f + \omega_ig$ which is divisible by $\tilde{h}$, since if there are two, then both f and g are divisible by $\tilde{h}$ contradicting the assumption of absence of common factors for f, g, h. If each irreducible factor of h divides single factor $f + \omega_ig$ of $f^3 + g^3$, then $f + g = ah'^3A'_3$, $f + \omega_2g = b(h'')^3A'_1, f + \omega_3g = c(h''')^3A'_2$. So $3degf = (dega + degb + degc) + 3(degh' + degh'' + 3degh''') + degA_3$ i.e. $a = b = c$ are constants. Hence $A'_1h'^3, A'_2h''^3, A'_3h'''^3$ belong to a pencil i.e. h', h'', h''' are solutions to the Catalan equation with coefficients A'_1, A'_2, A'_3. If A'_1, A'_2 are constant then the degree of h''' is strictly smaller than the degree of h. Hence such descent should lead either to Catalan equation with solutions having at least two non-constant coefficients or the Catalan equation $f^3+g^3+A_3 = 0$ would have a solution. In the latter case, by unique factorization, each $f + \omega_ig$ is a product of factors of A_3 and the Catalan equation with coefficients $A''_i = f + \omega_ig$ has solution in $\mathbb{C}$ i.e. satisfies conclusion of 1) since A''_i are in the pencil generated by f, g.

Assume now that factors A_1, A_2 in (10) are not constant. Let us consider the restriction of relation (10) on one of the lines $L_{i,j}$, say $L = L_{1,j}$ which equation is a factor of A_1. Denoting the form obtained by restriction on L of a form M as ${}^LM$ we can write the resulting relation as

$${}^LA_2({}^Lg)^3 +^L A_3({}^Lh)^3 = 0 \tag{11}$$

Let ${}^LA_i = \Pi l_{i,j}^{a_{i,j}}$. If one $a_{i,j} > 2$ then the zero of $l_{i,j}$ on L is the point with belong to more than three lines $L_{i,j} = 0$ forming the zero set of $A_1A_2A_3 = 0$ contradicting the assumption multiplicities not exceeding 3. Hence the exponent of a factor of ${}^LA_i$ is $a_{i,j} = 1$ or $a_{i,j} = 2$. Due to unique factorization in $\mathbb{C}[t]$ we obtain that none of roots of ${}^LA_2$ and ${}^LA_3$ is a root of g or h. Moreover ${}^LA_2$ and ${}^LA_3$ have the same sets of zeros on L with the same multiplicities and $\frac{{}^Lg}{{}^Lh} \in \mathbb{C}$. Hence any intersection point of L and A_i also belongs to $A_{i'}$ for $i = 2,3$ and $i' = 2,3$. This implies that the same is true for any point in $A_1 = 0$, that $degA_2 = degA_3$ and hence the conditions of Noether $AF + BG$ theorem (cf. [9]) are satisfied (the multiplicities of A_i are equal to one). Hence

$$A_1 = \lambda_2 A_2 + \lambda_3 A_3 \quad \text{where} \quad \deg\lambda_i = 0 \tag{12}$$

Next consider rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by $P \to (A_2(P), A_3(P))$. Its fibers consist of points P such that

$$\frac{A_2}{A_3} = t \in \mathbb{C} \tag{13}$$

The solution (10) for A_i is also solution to $(\lambda_2 t + \lambda_3)f^3 + tg^3 + h^3 = 0$. Since Fermat equation over the function field of the curve (13) has only constant solutions (cf. [10]) it follows that f, g, h depend only on t. The same argument shows that if $a_{i,j} = 2$ then f, g, h are pullbacks as well. Part 2) is hence proven. □

Next we shall consider solutions to equation (10) over $\mathbb{C}(t)$.

Theorem 3.2. *The solutions to (10) in case $A_i \in \mathbb{C}(t)$ form a group isomorphic to $Hom(Jac(C), E)$ where C is the 3-fold branched cover of $\mathbb{P}^1$ ramified at the set $\mathcal{S}$ which is the union of zeros of A_i $(1 \le i \le 3)$, $Jac(C)$ is its Jacobian and E is the elliptic curve with an automorphism of order 3. In particular if $\mathcal{S}$ contains only three elements then the group of solutions is isomorphic to $End(E) = \mathbb{Z}^2$.*

Proof. Non-constant solutions to (10) correspond to non-constant sections of the elliptic surface (7). This elliptic surface is iso-trivial and is trivialized over the 3-fold cyclic cover C of $\mathbb{P}^1$ containing t-line as an open set totally ramified at the union of zeros of A_i and possibly the infinity. Indeed the holonomy around each zero is trivial on such 3-fold cover. For trivial elliptic surface $C \times E$ over C non constant sections correspond to the maps $C \to E$ by Torelli theorem which correspond to the maps of the Jacobian of C to E. The last claim in the statement 2) and statement 3) follows since the Jacobian of threefold cover of $\mathbb{P}^1$ branched at three points is E. □

Example 3.3. 1) Let $\omega = exp(\frac{2\pi\sqrt{-1}}{3})$ be a primitive root of unity of degree 3. A non constant solution to Catalan equation:

$$(t-\omega)^2(t-\omega^2)^2 f^3 = (1+\omega^2)(t-1)^2(t-\omega^2)^2 g^3 + (-\omega^2)(t-\omega)^2(t-1)^2 h^3$$

is

$$f = t-1, g = t-\omega, h = t-\omega^2$$

Indeed, after substitution we get $(1+\omega^2 = -\omega)$:

$$(t^3-1)^2[(t-1)-(1+\omega^2)(t-\omega)+\omega^2(t-\omega^2)] = (t^3-1)[(t-1)+\omega(t-\omega)+\omega^2(t-\omega^2)] = 0$$

2) Suppose that $F_3 = F_1 + F_2$ i.e. the arrangement is composed of a pencil. This yields the solution $f = g = h = 1$ to Catalan equation. Elliptic curve $c^3 = s^2 - s$

which is isomorphic to $a = s, b = 1 - s, c^3 = ab$ is equivalent to $y^2 = x^3 + 1$ where $s - \frac{1}{2} = y, x = 4^{\frac{1}{3}}c$. Duplication formulas from [21] p.59, are in this case:

$$x_1 := \frac{x^4 - 8x}{4x^3 + 4}, \qquad y_1 = -y - \frac{3x^2(\frac{x^4-8x}{4x^3+4} - x)}{2y}$$

(here (x_1, y_1) denotes $2(x, y)$ in the sense of addition on elliptic curve). From this we infer that if we take $F_3 = 1$ (which is always the case after replacement $F_1 \to \frac{F_1}{F_3}, F_2 \to \frac{F_2}{F_3}$) start with map $a = F_1, b = F_2$ then we get for doubling:

$$a := \frac{-F_1(F_1 - 2)^3}{(2F_1 - 1)^3} \quad b = \frac{(F_1 - 1)(F_1 + 1)^3}{(2F_1 - 1)^3}$$

i.e.

$$\frac{f}{h} = \frac{-F_2 - 1}{2F_1 - 1} \quad \frac{g}{h} = \frac{F_1 + 1}{2F_1 - 1}$$

are solutions to Catalan equation (indeed, one easily sees that $a + b = 1$).

4. On Combinatorial invariance

Proof. (Of theorems 1.1 and 1.2) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arrangements as in the statement of the theorem. Recall that the characteristic polynomial of the monodromy of Milnor fiber for each arrangement is $(t-1)^{r(\mathcal{A}_i)-2}(t^2+t+1)^{s(\mathcal{A}_i)}$ where $r(\mathcal{A}_i) = Card(\mathcal{A}_i)$ and $s(\mathcal{A}_i)$ is the superabundance of curves of degree $\frac{2d-9}{3}$ passing through the triple points of the arrangement $\mathcal{A}_i$. Hence we need to show that $s(\mathcal{A}_1) > 0$ implies $s(\mathcal{A}_2) > 0$ as well.

For the rest recall the results from [4]. Let C be a possibly reducible plane curve for which $\omega = exp(\frac{2\pi\sqrt{-1}}{3})$ is a root of the Alexander polynomial. Let $\Pi_{i \in R} F_i = 0$ where R is the set of irreducible components of C be the reduced equation of C. Then the equation of C admits a quasi-toric relation corresponding to ordinary triple point (cf. [4]) i.e. there exist an equality (cf. (2.1)):

$$\Pi_{i \in R_1} F_i f^3 + \Pi_{i \in R_2} F_i g^3 + \Pi_{i \in R_3} F_i h^3 = 0 \tag{14}$$

with distinct F_i and $R = \bigcup_{i=1,2,3} R_i, R_i \cap R_j = \emptyset, i \neq j$ is a partition of the set of components R.

Since $s(\mathcal{A}_1) > 0$, there exist non trivial quasi-toric decomposition corresponding to the curve which is a union of lines in $\mathcal{A}_1$. This means that equations of lines of arrangement can be split into 3 groups which can be used to construct Catalan equation which has solutions. By theorem 3.1, $\mathcal{A}_1$ is composed of reduced pencil.

On the other hand, to each such pencil corresponds the 2-dimensional component of characteristic variety of $\mathcal{A}_1$ containing the identity of $\mathrm{Char}(\pi_1(\mathbb{P}^2 - \mathcal{A}_1))$ (it consists of the pullbacks of local systems from the complement in $\mathbb{P}^1$ to a triple of points). Each component of characteristic variety of $\mathcal{A}_1$ containing the identity of $\mathrm{Char}(\pi_1(\mathbb{P}^2 - \mathcal{A}_1))$ yields a component of the resonance variety of $\mathcal{A}_1$ (cf. [14], [15]). Recall that the resonance variety is the collection of $a \in H^1(\mathbb{P}^2 - \mathcal{A}_1)$ such that the Aomoto complex

$$H^0(\mathbb{P}^2 - \mathcal{A}_1) \xrightarrow{\wedge a} H^1(\mathbb{P}^2 - \mathcal{A}_1) \xrightarrow{\wedge a} H^2(\mathbb{P}^2 - \mathcal{A}_1) \tag{15}$$

has non-zero cohomology at the middle term. Since such cohomology is determined by the cohomology algebra $H^*(\mathbb{P}^2 - \mathcal{A}_1)$ we obtain that the resonance variety of $\mathcal{A}_2$ has the same number of components as $\mathcal{A}_1$. To each component of the resonance variety of $\mathcal{A}_2$ corresponds pencil which is reduced since the corresponding component of resonance variety of $\mathcal{A}_1$ comes from reduced pencil. For a map of $\mathbb{P}^2 - \mathcal{A}_2$ onto the complement in $\mathbb{P}^1$ to a triple of points $\mathcal{P}$, the pull back of the local system on the latter assigning to a standard generator of $\pi_1(\mathbb{P}^1 - \mathcal{P})$ value $\omega_3 = exp(\frac{2\pi i}{3})$, is the local system on $\mathbb{P}^2 - \mathcal{A}_2$ which cohomology is isomorphic to the eigenspace with eigenvalue ω_3 on $H^1(M_{\mathcal{A}_2})$. Hence the claim follows.

□

Remark 4.1. The cases when $s > 1$ correspond to arrangements which are composed of several pencils. One such example is the following. Let C be an elliptic curve and $\mathcal{A}$ an arrangement of lines in plane dual to the plane containing C consisting of lines dual to nine inflection points of C. Then this arrangement has no double points, 12 triple points and the superabundance of curves of degree 3 containing these 12 points is $s = 2$ (cf. [6],[2]). Indeed, the cohomology sequence corresponding to

$$0 \to \mathcal{J}_S(3) \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_S \to 0 \tag{16}$$

is the following:

$$0 \to 0 \to \mathbb{C}^{10} \to \mathbb{C}^{12} \to H^1(\mathbb{P}^2, \mathcal{J}_S(3)) \to 0 \tag{17}$$

This arrangement is composed of 4 pencils (cf. [14] section 3.3 example 3). For example one can use as explicit equations for this arrangement union of three cubics

$$x^3 - y^3, x^3 - z^3, y^3 - z^3$$

or

$$[(y-z)(x-z)(x-y)], [(y-\omega^2 z)(z-\omega^2 y)(x-\omega z)], [(y-\omega z)(x-\omega^2 z)(x-\omega y)] \quad (\omega^3 = 1)$$

etc.

Remark 4.2. The combinatorial invariance of $s(\mathcal{A})$ would follows from the following refinement of the above argument describing the relation between $s(\mathcal{A})$ and the number $l(\mathcal{A})$ of pencils of which $\mathcal{A}$ is composed. This is the number of essential components of the resonance variety of $\mathcal{A}$. The number of Catalan equations corresponding to an arrangement $\mathcal{A}$ which have solutions by 3.1 is also $l(\mathcal{A})$. To each such pencil correspond collection the map $\bar{M}_{\mathcal{A}} \to E$ (cf. [4]) of 3-fold cyclic covers of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively defined up to automorphism of E i.e. i.e. six elements of the Mordell Weil group $MW(\mathcal{A}_1)$ differ by an automorphism of E. Recall that $MW(\mathcal{A})$ is endowed with the canonical height pairing (cf. [19]). Identification of the number of elliptic pencils on $\bar{M}_{\mathcal{A}}$ with the number of elements of minimal height in $MW(\mathcal{A}_1)$ and showing that the number of elements of minimal height in $MW(\mathcal{A}) = \mathbb{Z}[\omega_3]^{s(\mathcal{A})}$ is determined by $s(\mathcal{A})$ would imply that the number of pencils which composed $\mathcal{A}$ is determined by $s(\mathcal{A})$ and vice versa. Since, as follows from the above argument, the number of pencils composing the arrangement is a combinatorial invariant this would imply the combinatorial invariance of $s(\mathcal{A})$ and hence of the Milnor number.

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