Lectures on topology of complements and fundamental groups

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Abstract

This is an introduction to the topology of the complement to plane curves and hypersurfaces in the projective space and based on the lectures given in Lumini in February and in ICTP (Trieste) in August of 2005. We discuss key problems concerning the families of singular curves, the one variable Alexander polynomials and the orders of the homotopy groups of the complements to hypersurfaces with isolated singularities. We also discuss multivariable generalizations of these invariants and the Hodge theory of infinite abelian covers used in calculations of multivariable invariants. A historical overview is included as the opening section

1 Introduction

Study of the topology of plane algebraic curves is a very old subject. In fact, its problems come up naturally after the very first definitions in a basic course on algebraic curves. And yet, the answers obtained so far, are often elusive and incomplete. If C is an algebraic curve in a complex projective plane $\mathbf{P}^2$, what is the fundamental group of $\mathbf{P}^2 - C$? Which properties of C affect complexity of this group? For which groups G there exist C such the G is the fundamental groups of the complement to C? When two curves are isotopic is appropriate sense, so that complements stay unchanged during such isotopies? What are the invariants of such isotopies? These are obvious questions and a lot is known about them but complete or even satisfactory answers still are out of reach. Below I want to describe some recent developments and I hope that this can serve as an introduction to these ideas and methods.

Perhaps the real beginning of this subject should be credited to Enriques, though some important work on construction of interesting singular curves and the numerology i.e. calculations of the number of singular points of a given type etc. started much earlier. For example in early 19 century Plücker discovered important formulas related the degree, the number of nodes and cusps of a curve to similar invariants of the dual curve, people from Newton to Puiseaux and beyond developed methods for analyzing the singular points of plane curves and already Newton classified types of singular cubics. Lefschetz ([39]) used Plücker's work to obtain first non trivial information on how many nodes and cusps a plane curve of a given degree can have (problem which still remains largely unresolved).

In the end of 19th century, undoubtedly influenced by Picard and Severi works on the topology of complex surfaces, Enriques initiated a program to extend Riemann and Hurwitz results on multivalued functions, or in a more modern terminology covering spaces of Riemann surfaces, to higher dimensions (cf. [23]). According to Riemann, a multivalued function in one variable, (e.g $w = \sqrt{z}$ or more generally a solution to the equation $w^n + a_1(z)w^{n-1} + ... + a_n(z)$ where $a_i(z)$ are single valued holomorphic functions of z) is specified by the following data. Firstly, the collection of its ramification points $B \subset \mathbf{C} \subset \mathbf{P}^1$, secondly the number d of values of the multivalued function and finally the monodromy representation $\pi_1(\mathbf{P}^1 - B) \to \Sigma_d$ of the fundamental group into the symmetric group on d letters. What makes Riemann's approach very effective for the description of multivalued functions is the fact that the fundamental group in question is always a free group since the ramification locus is just a collection of points in $\mathbf{P}^1$, and therefore the whole multivalued function is specified by ramification locus B and assignment of arbitrary permutations $\sigma_1, ..., \sigma_{\mathrm{Card}(B)}$ in the symmetric group Σ_d on d letters to the generators of $\pi_1(\mathbf{P}^1 - B)$ with the only restriction $\sigma_1 \cdots \sigma_{\mathrm{Card}(B)} = id$.

It was realized by Enriques (and others; a rather complete account of the work before mid 1930's is given by Zariski in his seminal book [84]) that similar description of multivalued functions of several variables is still valid but also that in higher dimensions such a result is much less efficient since $\sigma_1, ..., \sigma_{\mathrm{Card}(B)}$ must satisfy additional relations. For example, any algebraic curve in $\mathbf{P}^2$ can be a branching curve of a multivalued function but one cannot assign arbitrary elements of Σ_d to generators of $\pi_1(\mathbf{P}^2 - B)$ since this group is almost never free. Rather, the permutations should satisfy certain compatibility conditions (one should note that the very idea of the fundamental group did not completely crystallized around the time of the work of Enriques and therefore his statements are much less straightforward than presented here). Enriques described this conditions very explicitly and this, in modern terms, amounts to calculation of the quotient of the fundamental group by the intersection of subgroups of finite index in terms of geometric generators (those discussed in section 3; note that it is still unknown that this intersection trivial i.e. the fundamental group is residually finite; cf. section 2.2). For example, if the branching curve has degree d and is *non singular*, the fundamental group of the complement is cyclic of order d, but, at the same time, the number of geometric generators is d (cf. section 2.3). Therefore, firstly, one can assign to one geometric generator only a permutation of order d in Σ_d and, secondly, the assignment to the rest of the generators is determined by the first choice. O.Zariski (after arrival to US and visiting Princeton where Lefschetz and Alexander were working at the time) realized that the fundamental group of the complement is the central object in this theory and introduced many new at that time ideas even in context of similar problems in the knot theory. He showed how subtle the question of the fundamental group can be: not that is does depend on degree of the branching curve, as is the case for multivalued functions in one variable, but even knowing the number of nodes and cusps is not sufficient. He proved that a curve with 6 cusps can have as the fundamental group the cyclic group $\mathbf{Z}_6$ or the free product $\mathbf{Z}_2 * \mathbf{Z}_3 = PSL_2(\mathbf{Z})$. He also showed that such sextics can be distinguished by a geometric condition: in the first case the cusps must be in general position i.e. not to belong to a curve of degree 2 and in the second case they must belong to a conic. Zariski also used many technical ideas just appearing at the time in topology e.g. studying the homology of cyclic covers (which in knot theory can be traced to Alexander and Reidemeister [2], [69]). Systematic study of the branched coverings using the theory of adjoints (cf. section 5.2) allowed him to relate the homology of branched covers to the superabundances of linear systems defined by the cusps (cf [81]). He found a close relationship between the fundamental groups of the complements and braid groups by considering the dual curves for nodal rational and elliptic curves. One of the tools was his celebrated theorem on fundamental groups of hyperplane sections extending Lefschetz homological results. In the context of branched covering Zariski even obtained expressions close to Alexander polynomial (cf. [82]) as was noticed by D.Mumford (cf. [84]) and which was the basis of his questions about the role of Alexander polynomial in algebraic geometry (*).

After 1937 Zariski abruptly changed the scope of his interests and turned to ambitious project of reconstructing algebraic geometry on firm foundations of commutative algebra. Some of his students, however, continue to develop this subject, cf. [77], [40]; much later, but in a similar spirit, M.Oka ([67]) generalized Zariski calculation of the fundamental group of the complement to sextic with six cusps on conic by proving that for the curve C given by the equation $(x^p+y^p)^q+(y^q+z^q)^p = 0, \quad gcd(p,q)=1$ one has $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}_p * \mathbf{Z}_q$. The study of the topology did continue mostly in the works of O.Chisini and his students ([12]) who initiated use of braids for the study of the fundamental groups and covering spaces. Abhyankar (cf. [1]), who studied with Zariski in Harvard in the 50s, was investigating the fundamental groups, and in particular obtained important results of the fundamental groups of the complements, but the main focus was on algebraization of the fundamental groups.

One of the driving problem in the study of the fundamental groups in the 60s and 70s was the problem of commutativity for fundamental groups of the complements to curves with nodes only. Severi ([75]) outlined an argument which, as later was realized, is incomplete. It was based on an assertion that the variety of plane curves of fixed degree with a fixed number of nodes is irreducible. Zariski repeat Severi's argument in [84] but return to this issue much later (cf. [86]). Severi's statement eventually was confirmed by J.Harris ([29]). A direct algebraic proof of commutativity was found by W.Fulton (cf [28]) using Abhyankar's work and shortly

*These questions were answered later in author's papers [41] [42] and further extended in [48], [52] [56] (see references to other related works in these papers).

after that a topological argument was given by P.Deligne.(cf. [16]) . A little bit later, M.Nori (cf. [66]) clarified these results further by obtaining conditions for commutativity of the fundamental group of the complement to curves on arbitrary surfaces, in this respect continuing the work of Abhyankar (cf. [66])

In the 70's the problems about fundamental groups of the complements were mentioned infrequently. Mumford in already quoted appendix to [84] also raised the problem of investigating of the quotient G'/G'' for the fundamental groups of the complements. In introduction to the volume III of collected papers by Zariski, containing the papers on the topology of the complements, Artin and Mazur, after discussing Zariksi's study of cyclic multiple planes, note:

"Also, as far as the editors are aware, there has been no further progress in the delicate study of cyclic multiple planes for general d. There are many tantalizing questions here-there are even a number of less delicate topological issues to sort out. For example, for irreducible plane curve C with arbitrary singularities can one give some reasonable sufficient conditions for regularity of H_d in terms of zeros of "local Alexander polynomials"- that is, the Alexander polynomials of the knots associated with singularities of C?"

The answers to these questions were obtained in author's papers [41] and [42] which depend on Milnor's work on the Alexander polynomials and the infinite cyclic covers (cf. [62]). If $G = \pi_1(\mathbf{C}^2 - C)$ one has $G/G' = H_1(\mathbf{C}^2 - C) = \mathbf{Z}^r$ where r is the number of irreducible components (cf. 2.2.1). If C is irreducible one has the exact sequence:

$$0 \to G'/G'' \to G/G'' \to \mathbf{Z} \to 0$$

This sequence defines the action of $\mathbf{Z}$ on G'/G'' which after Hurewicz identification of G'/G'' with the homology of infinite cyclic cover coincides with the action induced by the action of the group $\mathbf{Z}$ of covering transformations on the universal cyclic cover. The advantage of replacing projective curve by affine is that in affine case one has infinite tower of covering spaces while in projective case the degree of the cover must divide the degree of the curve. On the other hand, if the line at infinity is transversal to a projective curve, the group of affine curve is just a central extension of the projective one (in non-transversal case the relation is much more subtle). It is shown in [41] that $G'/G'' \otimes \mathbf{Q}$, as a module over the group ring of $\mathbf{Z}$ i.e. the ring $\mathbf{Q}[t, t^{-1}]$, is a torsion module and hence the order of the latter $\Delta_C(t)$ is well defined (up to a unit of $\mathbf{Q}[t, t^{-1}]$). This is a *global* invariant of the curve in $\mathbf{C}^2$. On the other hand, with each singular point of C is associated the link i.e. the intersection of C with the boundary of a small ball about this singular point. As result one obtains a set of local Alexander polynomials Δ_P of all singularities P of the curve C (as was suggested by Artin and Masur). One need, however, another important ingredient: in [41] was introduced the Alexander polynomial at infinity Δ_∞ which is the Alexander polynomial of the link which is the intersection of C with the ball in $\mathbf{C}^2$ of a sufficiently large radius. The answer to the question of

Artin and Mazur in the above quote is given by the following divisibility theorem from [41] for these Alexander polynomials associated with the curve:

$$\Delta(C) \mid \Pi_{P\in \mathrm{Sing}(C)}\Delta_P(C)$$
$$\Delta(C) \mid \Delta_\infty(C) \tag{1}$$

and the theorem which expresses the homology of cyclic covers in terms of Alexander polynomials (cf. [41] and theorem 2.8 below). For example, for sextic curves with 6 cusps, which Zariski was considering in [80], the Alexander polynomial $\Delta_C(t)$ is equal to t^2-t+1 or 1 depending on wether six cusps are or are not on conic and both divide $(t^2-t+1)^6$ and $(t^6-1)^4(t-1)$ (which are the product of local Alexander polynomials and the Alexander polynomial at infinity respectively). The divisibility relation [41] also contains restrictions on the fundamental groups: $G'/G''\otimes \mathbf{Q}$ is trivial if Δ_∞ and $\Pi_{P\in \mathrm{Sing}(C)}\Delta_P(C)$ are relatively prime. For example if cusps are the only singularities, then $G'/G''\otimes \mathbf{Q}=0$ unless the degree is divisible by 6. The regularity condition, which were mentioned by Artin-Masur is the following: if none of the roots of local Alexander polynomial is a root of unity of degree d the degree n then the cyclic cover H_n or degree n is regular (cf. [41]).

The work [41] is topological and many of the results were extended to differential category (cf. [43]). Dependence of Alexander polynomial on position of singularities, containing generalization of several Zariski's calculation, was shown in [42]. As in [81], the irregularity of cyclic multiple plane is obtained in terms of superabundances of certain linear systems associated with the cusps but for singularities more complicated than cusps the systems are specified by more subtle geometric conditions: the local equations of the linear systems responsible for irregularity of cyclic branched covers must belong to certain ideals called in the *ideals of quasiadjunction*. Later, these ideals appeared many other contexts and often are called *multiplier ideals* (cf. [38]). Other important numerical invariants of plane singularities introduced in [42] were identified in [58] with the part of the spectrum introduced in 70's by Arnold and Steenbrink (cf [70]). The work [24] also related the irregularity of multiple planes to the position of singularities and these ideas rely on vanishing theorems which later lead to much better understanding of those (cf. [25]): a key development in algebraic geometry in 90s.

In early 80s, about the time when described above work on Alexander polynomials appeared, there was another important development in the study of plane singular curves. B.Moishezon initiated program for describing the topology of algebraic surfaces in terms of branching curve in $\mathbf{P}^2$. This is a special class of curves and these curves belong to the class of curves having nodes and cusps as singularities. If one starts with a projective surface, considers a pluricanonial embedding using a fixed multiplicity of the canonical class, and then uses a generic projection the branching curve in $\mathbf{P}^2$ becomes an invariant of the deformation type of the surface (the fact that one does not need the monodromy representation into symmetric group was conjectured by Chisini and subsequently proven in ([37]). Moishezon's first calculations deal with the branching curves of generic projections of non singular surfaces in $\mathbf{P}^3$. If the degree d of a surface is 3, one obtains as the branching curve Zariski's sextic given by the equation $f_2^3 + f_3^2 = 0$. For surfaces of arbitrary d Moishezon obtains, as the fundamental groups of the complements to the branching curves, the quotients of the Artin's braid groups by the centers (which for $d = 3$ gives $PSL_2(\mathbf{Z})$). Moishezon's important idea was that the primary invariant is not the fundamental group but rather the braid monodromy which implicitly is present in van Kampen's method of calculation of the fundamental group (Moishezon was unaware of Chisini's work [12] until he completed [63]). In this vein, the author showed that the braid monodromy defines not just the fundamental group but also the homotopy type (cf. [45], and further works by M.Teicher cf. [76]) Later Moishezon continued this work jointly with M.Teicher. Methods of braid monodromy recently found applications in symplectic geometry (cf. [4]). More recently Teicher and her students continued systematics study of the braid monodromy and the fundamental groups of the complements to the branching curves of generic projections and arrangements of lines.

In the late 80's the work started on a generalization of the the theory of complements to singular curves to higher dimensions. The case of hypersurfaces with isolated singularities it turns out remarkably similar to the case of curves. In [48], the author showed that for $n > 1$ the role of Alexander polynomial plays the order of the homotopy group $\pi_n(\mathbf{C}^{n+1} - V)$ considered as the module over $\pi_1(\mathbf{C}^{n+1} - V) = \mathbf{Z}$. The point is that this homotopy group can be canonically identified with the homology $H_n(\widetilde{\mathbf{C}^{n+1} - V}, \mathbf{Z})$ of the infinite cyclic cover of the complement. The divisibility relations (1) extends to the order of the homotopy groups and examples of hypersurfaces with non trivial homotopy appears as a natural generalizations of Zariski's sextics. For example $\pi_2(\mathbf{C}^3 - V) =\neq 0$ for V given in $\mathbf{P}^3$ by the equation: $f_{21}^2 + f_{14}^3 + f_5^7 = 0$ (f_n generic form of degree n in four variables). Analytic theory developed by the author in [42] also was extended to higher dimension in [51] by introducing the mixed Hodge structure on the homotopy group and by relating one of the Hodge components of the homotopy group to the superabundance of the linear systems defined by the singularities of hypersurface.

In the 90's were obtained the first results on a multivariable generalization of the Alexander invariants (cf. [47]). The theory of the multivariable Alexander polynomials of links, due to R.Fox, depends on a very special feature of the link groups: the first Fitting ideal of the Alexander module is "almost" principal. The fundamental groups of the complement to reducible algebraic curves in $\mathbf{C}^2$ are similar to the link groups in the sense that both have surjections on $\mathbf{Z}^r (r > 1)$. However for algebraic curves the first Fitting ideal of the Alexander module is far from principal and as result one cannot define a multivariable Alexander polynomial in a meaningful way. The puzzle of existence of multivariable invariants of algebraic curves got resolved by introducing set of zeros of the polynomials in the Fitting ideals of the Alexander modules in author's paper [47]. In the case of one variable Alexander polynomials no information get lost by replacement the Alexander polynomial by its set of zeros (at least for curves in $\mathbf{P}^2$ for which the Alexander module is semisimple) but for reducible curves zero sets provide a non trivial and very interesting invariant.

Applications followed shortly. In [32] the characteristic varieties of a group were related to the cohomology of local systems which followed the study of polynomial periodicity of Betti numbers of branched covering spaces (cf. [31]). For the curves for which all components have degree 1 i.e. arrangements of lines the components of characteristic varieties were related to the cohomology algebra of the complement (cf. [13]). The calculation of the homology of abelian covers constructed by Hirzebruch and which have universal covers biholomorphic to the ball did fall naturally in the general scheme valid for arbitrary arrangements and covers (cf. [52]). Analytic (rather than topological) theory was developed in [52] and characteristic varieties were expressed in terms of superabundances of the linear systems. Essential in this calculation were the results in [6], on the structure of the jumping loci for the cohomology of local systems. They represent an extension to quasiprojective varieties of the results of Green-Lazarsfeld, Beauville, Catanese, Simpson, Deligne and others and which asserts that the jumping loci for the cohomology of local systems are cosets of certain subgroups of the group of characters of the fundamental group.

During late 90's, the study of the topology of plane algebraic curves became much more active area of research. Many new examples of Zariski pairs due to E.Artal-Bartolo and collaborators and M.Oka showed how common is the phenomenon of curves having different equisingular isotopy type with the same local data. Many new calculations were carried out of the fundamental groups of the complements by M.Teicher's school which finally led to a general conjecture on the structure of the fundamental groups of the branching curves of generic projections (cf. [76]). Interactions with combinatorics of arrangements were important and lead to at least conjectural description of the characteristic varieties and much stronger vanishing for the cohomology of local systems than were available earlier (cf.[53],[52],[13]). Connections with symplectic topology should be noted (cf. [4]). There was further progress in the study of the complements in higher dimensions on generalizations of Zariski-van Kampen's theorem (cf. [48],[11],[26],[78]). Nevertheless, despite tremendous progress, since the first works by Enriques, Zariski and van Kampen, many problems still remains open and complete understanding of the topology of the complements to curves and hypersurfaces still out of reach.

In the text below we outlined some of the problems which resolution may clarify substantially the situation. Exposition is very elementary in the beginning describing motivation for the study in the following sections. In the later parts a reader will need more and more rely on material covered in standard courses in algebraic geometry. Moreover, some familiarity with the mixed Hodge theory is needed in the last sections. Textbook [18] also is a good reference to the background and other material discussed below. Most of the material appeared already in the literature some time ago but some results appear to be new.

I want to thank J.P. Brasselet, D.Cheniot, J.Damon, M.Oka, A.Pichon D.Trotman, N.Dutertre and C. Murolo who organized the conferences in Lumini and Trieste and the opportunity to present this beautiful area of mathematics. I am also want to thank L.Kauffman for a discussion of the history of polynomial invariants in the knot theory.

2 Fundamental groups of the complement

2.1 Problem of classification up to isotopy.

2.1.1 Stratification of the discriminant

Classically, many problems in the topology of plane curves and hypersurfaces were rooted in the study of families of curves and attempts of some kind of classification of curves and hypersurfaces (cf. [80]). We shall start by discussion in what kind classification and in which sense one may expect.

Hypersurfaces of a fixed degree d in $\mathbf{P}^n$ are parameterized by $\mathbf{P}^{\binom{n+d}{d}-1}$ by assigning to a defining equation the collection of coefficients of its monomials (in some fixed order). The discriminant $Disc(n,d)$ is the hypersurface in $\mathbf{P}^{\binom{n+d}{d}-1}$ consisting of the points corresponding to singular hypersurfaces. It has singularities in codimension one. An interesting problem is to understand the stratification of the discriminant hypersurfaces $Disc(n,d)$. By this we mean to describe the singular locus of the discriminant hypersurface having codimension one in $\mathbf{P}^{\binom{n+d}{d}-1}$, then the singular locus of singular locus (having codimension 2) and so on. More precisely, we consider the universal hypersurface of degree d i.e. $\mathcal{V} \subset \mathbf{P}^n \times \mathbf{P}^{\binom{n+d}{d}-1}$ consisting of pairs (P,V) such that $P \in V$. $Disc(n,d)$ is the image of the critical set of the projection on the second factor and its preimage in $\mathcal{V}$ is the universal singular hypersurface. The critical set of the projection on the second factor is the singular set $Sing(n,d)$ of the universal singular hypersurface. Then we consider the critical set of the restriction of the projection on the non singular part of $Sing(n,d)$. On a codimension one subset, the rank of projection on $Disc(n,d)$ drops and so on. With such a definition, Thom's isotopy theorem yields that the hypersurfaces belonging to each stratum are equisingular so the strata represent equisingular families of hypersurfaces. Note that the subset in $\mathbf{P}^{\binom{n+d}{d}-1}$ parameterizing equisingular hypersurfaces is singular in general (cf.[79]).

The case $n=1$ is already very interesting and non trivial. The discriminant consist of homogeneous polynomials $\prod_i(\alpha_i u - \beta_i v)$ in two variables u, v having multiple roots, i.e factors such that (α_i, β_i) and (α_j, β_j) satisfy $det|\begin{matrix}\alpha_i & \beta_i \\ \alpha_i & \beta_i\end{matrix}| = 0$. The strata correspond to partitions of d, i.e. the conjugacy classes of the symmetric group Σ_d. A lot is known about the geometry of these strata, for example the degrees of their closures as well as other algebro-geometric information. Cases with $n > 1$ are much more complicated. Many pieces of information are known. For example, in the case $n=2$ the degrees of the stratum corresponding to rational nodal curves have the interpretation as Gromov-Witten invariants of a projective plane and as such satisfy beautiful recurrence relations (cf. [36]). Indeed, the dimension of this stratum is $3d-1$ where d is the degree of the curves (i.e. $\frac{(d+1)(d+2)}{2} - \frac{(d-1)(d-2)}{2}$) so the degree of the corresponding stratum is the number of nodal curves of degree d passing through generic $3d-1$-points. The degrees of strata of nodal curves are subject to a conjecture of Göttsche discussed, for example, in [34].

2.1.2 Classification of quadrics and cubics. Local type.

Another class of discriminants which is well understood is case $d = 2$. Each stratum correspond to the quadrics of fixed rank. In particular each stratum is a determinantal variety.

Classification of plane cubics goes back to Newton. Codimension one stratum consists of cubic curves with one node. It has the degree equal to 12. There are two strata having codimension 2. One consists of curves with one cusp and another formed by the reducible curves having as the components a nonsingular quadric and a non tangent to it line. The rest of the strata correspond to reducible curves and each is determined by strata of curves of lower degree and the mutual position. The strata of codimension three are: unions of a non singular quadric and a tangent line (in the closure of both strata of codimension 2) and the union of three lines in general position. Note that each of these strata is described by the local type of singularities: the number of nodes, cusps, tacnodes etc. A definition of the local type is the following:

Definition 2.1 *Two reduced curves C and C' (of the same degree) have the same local type if there is a one to one correspondence between their singular points such the each pair of corresponding singular points P and P' have neighborhoods B_ϵ and $B'_{\epsilon'}$ and homeomorphisms $\phi_P : B_\epsilon \to B'_{\epsilon'}$ such that $\phi_P(C \cap B_\epsilon) = C' \cap B'_{\epsilon'}$.*
Two possibly non reduced curves have the same local type if:
(a) corresponding reduced curves have the same local type
(b) there are one to one correspondences between the components and singular points such that corresponding components have the same degrees and multiplicities and corresponding singular points belong to the corresponding components.

2.1.3 Examples with disconnected strata.

The classification of strata of curves of degree 4 provides the first example when the local type of singularities (in the sense of the first part of the definition 2.1) yields the strata with several connected components. The quartics with three nodes have two types: firstly the irreducible ones and quartics which are the unions of a non singular cubic and a generic line. The strata are distinguished by a global property.

For each degree there are finitely many irreducible families of plane curves having the same local type.

Problem 2.2 *Find discrete invariants of families of curves having the same local type.*

This problem is similar to the problem of classification of knots in S^3. Thom's isotopy theorem implies that the curves (or hypersurfaces) in a *connected* equisingular family are isotopic and hence have diffeomorphic complements. The main tool in the study of knots is the fundamental group of the complement which is one of the reasons suggesting to look at $\pi_1(\mathbf{P}^2 - C)$ or also into $\pi_k(\mathbf{P}^n - V)$ with $k > 1$ in the case of hypersurfaces of higher dimensions.

2.2 Fundamental groups of the complements

Though the classification problem of the strata of the discriminant leads to the fundamental groups of the complements as potentially an important invariant there are many other reasons for looking at the fundamental groups. One is that the fundamental groups of the complements to hypersurfaces controls the covers of projective space and any projective algebraic variety having the dimension n is a branched covering space of $\mathbf{P}^n$.

Linear representations of the fundamental groups appear as the monodromy representations of differential equations and this correspondence is a subject of the Riemann-Hilbert problem. For example monodromy representation of KZ equation yields an interesting representation of the pure braid group closely related to the discriminant $D(1, d)$.

Each of these "applications" lead to concrete problems about the fundamental groups. For example, the use for a study of the coverings suggests the following. In the above presentation of algebraic varieties as the covering spaces the degree of a cover is always finite. So the coverings are determined already by the quotient by the intersection of all subgroups of finite index. The problem is if this intersection is the identity group or in other words if the fundamental group of the complement to an algebraic hypersurface is *residually finite*. Alternatively this the the question on whether the map $\pi_1 \to \pi_1^{alg}$ into the algebraic fundamental group is injective. Note that the fundamental group of an algebraic variety does not have to be residually finite (D.Toledo). In general the problem of finding the properties of the fundamental groups of the complements or characterizing the algebraic structure of these groups is one of the central and the most difficult problems in algebraic geometry.

2.2.1 Homology of the complements.

An easily available information about the fundamental groups $\pi_1(\mathbf{P}^{n+1} - V)$ comes from calculation of the homology $H_1(\mathbf{P}^{n+1} - V)$ which, by Hurewicz theorem, is the quotient of the fundamental group by its commutator. Here is an answer:

Proposition 2.3 *Let V is the union of irreducible components $V_1, .., V_r$ having the degrees $d_1, ..., d_r$. Then $H_1(\mathbf{P}^{n+1} - V, \mathbf{Z}) = \mathbf{Z}^r/(d_1, .., d_r)$.*

For example, if $g.c.d.(d_1, ..., d_r) = 1$ then the homology group is torsion free. This is the case when one of components has the degree equal to 1 or in other words for the complements to affine hypersurfaces.

2.2.2 Examples of calculations of the fundamental groups.

In the last twenty years quite a few calculations of the fundamental groups were made. For example, as mentioned in introduction, Moishezon-Teicher calculated the fundamental groups to the branching curves of generic projections of many algebraic surfaces (cf. [76]). Oka calculated the fundamental groups of the complements to many curves having low degree, in particular to various classes of curves of degree 6 (cf. [68]). Many calculations were carried out by Artal-Carmona-Cogolludo (cf. [3]). These techniques I will discuss in a later chapter, but here I want to explain some short and elegant calculations made by Zariski 80 years back.

Proposition 2.4 *Let $\hat{C}_d$ be a curve dual to the rational nodal curve C_d having the degree d (the degree of $\hat{C}_d$ is equal to $2(d-1)$, it has $3d-6$ cusps and $2(d-2)(d-3)$ nodes). The group $\pi_1(\mathbf{P}^2 - \hat{C}_d)$ is isomorphic to the braid group of sphere on d strings. In particular the fundamental group of the complement to the quartic with 3 cusps is a non abelian group having order 12.*

Indeed, C_d is a generic projection on $\mathbf{P}^2$ of a rational normal curve C in $\mathbf{P}^d$ and dual to C_d curve is a plane section of the hypersurface $\hat{C}$ in $\mathbf{P}^d$ dual to C by a plane H. The complement to this hypersurface consists of hyperplanes in $\mathbf{P}^d$ intersecting C in d distinct points which can be chosen arbitrary. Hence the space of based loops in this complement is identified with the braids of $\mathbf{P}^1(\mathbf{C}) = S^2$. Finally the isomorphism $\pi_1(\mathbf{P}^2 - C_d) = B_d(S^2)$ follows from Lefschetz hyperplane section theorem applied to embedding of the complement in H into the complement in $\mathbf{P}^d$. In the case $d = 3$, the pure braid group of sphere can be identified with $\pi_1(PGL_1(\mathbf{C})) = \mathbf{Z}_2$ and hence one has the exact sequence: $1 \to \mathbf{Z}_2 \to B(S^2) \to S_3 \to 1$.

2.2.3 Alexander invariants of the fundamental groups.

Since the problem of characterization and understanding the fundamental group is very complicated it is reasonable to try to rather understand some invariants of the fundamental groups. An accessible and interesting invariant is the Alexander invariant of a group.

Let G be arbitrary group together with a surjective homomorphism $\phi : G \to \mathbf{Z}^r$. Let $\mathrm{Ker}\phi = K$ and let $K' = [K, K]$ be the commutator. If ϕ is the abelianization $G \to G/G'$ then $K = G', K' = G'' = [G', G']$. We have:

$$0 \to K/K' \to G/K' \to \mathbf{Z}^r \to 0 \tag{2}$$

In particular K/K' receives the action of $\mathbf{Z}^r$ and hence K/K' becomes the module over the group ring of $\mathbf{Z}^r$. This module is called the Alexander invariant of the pair (G, ϕ). In the case when ϕ is the abelianization one obtains an invariant depending on the group G only. It is denoted below as $A(G, \phi)$ or if ϕ is the abelianization as $A(G)$.

This definition can be interpreted geometrically. If X_G is a CW-complex having G as its fundamental group then the homomorphism ϕ defines a covering space $\tilde{X}_{G,\phi}$. One has $\pi_1(\tilde{X}_{G,\phi}) = K$ and $K/K' = H_1(\tilde{X}_{G,\phi}, \mathbf{Z})$. The action of $\mathbf{Z}$ corresponds to the action of the group $\mathbf{Z}$ of deck transformations on $\tilde{X}_{G,\phi}$.

For perfect groups, i.e. such that $G = G'$, this invariant is trivial (since $r = 0$ is the only possibility), but since the fundamental groups of the complements in

$\mathbf{P}^{n+1}$ are perfect only if the hypersurface is the hyperplane (cf. 2.3) for them the Alexander invariant is always interesting.

There is an algorithmic procedure for calculation of the Alexander modules due to R.Fox ("Fox calculus") (cf. [27]).

Let G be a finitely generated, finitely presented group i.e. one has a surjective map $\Phi : F_s \to G$ of the free group F_s on s generators, $x_1, ...x_s$ with the kernel being the normal closure of a finite set of elements given by the words $R_1, ...R_N$ in F_s. Consider the maps of the group rings: $\frac{\partial}{\partial x_j} : \mathbf{Z}[F_s] \to \mathbf{Z}[F_s]$ uniquely specified by the conditions:

$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} a(v) + u\frac{\partial u}{\partial x_j}; \qquad \frac{\partial x_i}{\partial x_j} = \delta_{i,j} \tag{3}$$

where $a : \mathbf{Z}[F_s] \to \mathbf{Z}$ is the augmentation surjection. Using operators $\frac{\partial}{\partial x_j}$ one can define the map of free $\mathbf{Z}[\mathbf{Z}_r]$-modules given by the Jacobi matrix:

$$(\phi_* \circ \Phi_* \frac{\partial R_i}{\partial x_j}) : \mathbf{Z}[\mathbf{Z}_r]^N \to \mathbf{Z}[\mathbf{Z}_r]^r \tag{4}$$

which entries are obtained by applying homomorphisms $\Phi_* : \mathbf{Z}[F_s] \to \mathbf{Z}[G]$ and $\phi_* : \mathbf{Z}[G] \to \mathbf{Z}[\mathbf{Z}^r]$ of group rings induced by Φ and ϕ respectively. The geometric meaning of this map is the following. With a presentation Φ one can associate the 2-complex X_G with single 0-cell, r 1-cells forming wedge $S^1 \vee ... \vee S^1$ of circles corresponding to the generators of G and N 2-cells attached so that the boundary of each is represented by the word R_i $(i = 1, ..., N)$ in $S^1 \vee ... \vee S^1$. The covering space $\tilde{X}_{G,\phi}$ corresponding to the homomorphism of the fundamental group has a canonical cell structure given by the preimages of cells in the above cell decomposition of X: each cell in X_G is replaced by cells of the same dimension corresponding to the elements of the covering group. Hence we obtain the isomorphisms $C_2(\tilde{X}_{G,\phi}) = \mathbf{Z}[\mathbf{Z}^r]^N, C_1(\tilde{X}_{G,\phi}) = \mathbf{Z}[\mathbf{Z}^r]^s$. Moreover, after this identification, the boundary operator $\partial_2 : C_2(\tilde{X}_{G,\phi}) \to C_1(\tilde{X}_{G,\phi})$ becomes identified with the operator given by (4). Since $H_0(\tilde{X}_{G,\phi}, \mathbf{Z}) = \mathbf{Z}$ and $C_0(\tilde{X}_{G,\phi}, \mathbf{Z}) = \mathbf{Z}[\mathbf{Z}]$ we have the isomorphism $\mathrm{Im}\partial_1 = \mathrm{Ker} C_0(\tilde{X}_{G,\phi}, \mathbf{Z}) \to \mathbf{Z} = I_{\mathbf{Z}[\mathbf{Z}^r]}$ where $I_{\mathbf{Z}[\mathbf{Z}^r]}$ is the augmentation ideal of the group ring. Hence, (4) determines the presentation of the module very closely related to $H_1(\tilde{X}_{G,\phi})$. More precisely, if $\hat{H}(X_{G,\phi})$ is the module having presentation (4) then we have:

$$0 \to H_1(X_{G,\phi}) \to \hat{H}(X_{G,\phi}) \to I_{\mathbf{Z}[\mathbf{Z}^r]} \to 0 \tag{5}$$

For example for (affine portions of) the curves in proposition 2.4, for the Alexander module $A(G)$ coinciding with $H_1(X_{G,\phi})$ one has (in these examples $\phi : \pi_1 \to H_1 = \mathbf{Z}$ is the canonical homomorphism):

$$A(\pi_1(\mathbf{P}^2 - C_d)) = \mathbf{Z}[t, t^{-1}]/(t^2 - t + 1) \quad (d = 4), A(\pi_1(\mathbf{P}^2 - C_d)) = 0 \quad (d \geq 5) \tag{6}$$

For the links of algebraic singularities, which all belong to the class of iterated torus link, the Alexander polynomial, *i.e. the order of* $A(G) \otimes \mathbf{Q}$ *as a* $\mathbf{Q}[t, t^{-1}]$*-module*, can be found using the data of iterations and the values of Alexander polynomial of for the torus knot: for the link of singularity $x^p = y^q \;\; g.c.d.(p,q) = 1$ one has the following.

$$\Delta(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \tag{7}$$

Another way to calculate the Alexander polynomial is to use A'Campo formula for the zeta-function of the monodromy in terms of a resolution of the singularity (cf. [22]):

$$\zeta(t) = \Pi(1-t^{m_i})^{\chi(E_i^\circ)} \tag{8}$$

where E_i are the exceptional curves of the resolution, E_i° is set of points in E_i which are non-singular points of the exceptional divisor, m_i is the order along E_i of the pullback of the equation of the singularity and χ denotes the Euler characteristic. The $\zeta(t)$ determines the Alexander polynomial of a curve singularity via: $\zeta(t) = \frac{(t-1)}{\Delta(t)}$.

2.2.4 Alexander polynomials of plane algebraic curves: divisibility theorems.

There are two types of general results concerning the Alexander invariants of the fundamental groups $\pi_1(\mathbf{C}^2 - C)$. Firstly, the Alexander polynomials of plane algebraic curves are restricted by the degree of the curve, by the local type of singularities and by position of the curve relative to the line at infinity. These restrictions sometimes yield triviality of the Alexander polynomial. On the other hand, the Alexander polynomial is completely determined by the local type and superabundances of certain linear systems given by the data depending on the singularities.

We shall start, with discussion of the first group of results. Let C be projective curve and L be the line at infinity. One has the linking number homomorphism: $\mathrm{lk} : \mathbf{C}^2 - C \to \mathbf{Z}$ associating to a loop γ in $\mathbf{C}^2 - C$ the (oriented) number of intersection points of C and an immersed disk with boundary γ. In the case when C is irreducible this homomorphism $H_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ already was used above. This defines the Alexander module and the Alexander polynomial $\Delta_C(t)$ and we shall omit mentioning the linking homomorphism used in its definition.

To each singular point $P \in C \in \mathbf{P}^2$ we associate the local Alexander polynomial which is the Alexander polynomial of the link defined as follows. In the case when $P \notin L$,the link is the intersection of C with a sufficiently small ball about P (so that the link type is independent of the radius). If this link has several components (i.e. P has several branches) the Alexander polynomial again is calculated relating to the total linking number homomorphism. In the case when $P \in L$, i.e. the curve has singularities at infinity, the local Alexander polynomial is defined as above but P considered as the singular point of $P \in C \cup L$. Note that, as follows from definitions, the local Alexander polynomials can be calculated as the characteristic polynomials of the monodromy operators (cf. [22], [60] for examples and algorithms).

On the other hand, one can define the Alexander polynomial at infinity $\Delta_{\infty,C}$ as the Alexander polynomial of the link which is the intersection of C with the boundary of a sufficiently small tubular neighborhood of L in $\mathbf{P}^2$ (or alternatively the sphere of a sufficiently large radius in $\mathbf{C}^2 = \mathbf{P}^2 - L$). For example, is C is a union of d lines passing through a point in $\mathbf{P}^2$ outside of L then the link at infinity is the Hopf link with d components and hence its Alexander polynomial is:

$$\Delta_{\infty,C} = (t^d - 1)^{d-2}(t-1) \tag{9}$$

The same equality holds for a curve which is transversal to the line at infinity since there is a deformation of such a curve to a union of d lines as above, such that transversality holds for all curves appearing during the deformation.

With these definitions we have the following:

Theorem 2.5

$$\Delta_C(t) \mid \Pi_{P \in \mathrm{Sing} C} \Delta_P(t)$$

$$\Delta_C(t) \mid \Delta_{\infty,C}(t)$$

Consider, for example and irreducible curve in $\mathbf{P}^2$ having ordinary cusps (i.e. having $x^2 = y^3$ as the local equation) and nodes (local equation: $x^2 = y^2$) as the only singularities. Then, as follows from (7), the local Alexander polynomials for each singularity is $t^2 - t + 1$ (cusp) or $t - 1$ (node). Moreover, it is not hard to show that the multiplicity of the factor $(t-1)$ is $r-1$ where r is the number of irreducible component of C (cf. [41]). Hence we obtain:

Corollary 2.6 *Let C be an irreducible curve in $\mathbf{P}^2$ having cusps and nodes as the only singularities. Then:*

$$\Delta_C(t) = (t^2 - t + 1)^s$$

for some integer $s \geq 0$.

Combining this corollary, the divisibility and the formula (9) we obtain:

Corollary 2.7 *Let C be an irreducible curve in $\mathbf{P}^2$ having cusps and nodes as the only singularities. Then $\Delta_C(t) = 1$ unless d is divisible by 6.*

We leave as an exercise for a reader to work out that $pq \not| d$ is a sufficient condition for triviality of the Alexander polynomial for an irreducible curve of degree D with singularities locally given by $x^p = y^q$.

Since the curves discussed in Proposition 2.4 (and also the branching curves of generic projections of non-singular surfaces in $\mathbf{P}^3$ cf. [63]) have the degree $d(d-1)$ it follows that the Alexander polynomial is trivial if $d \equiv 2(\mathrm{mod} 3)$ which explains with no calculation one third of equation 6 (at least after tensoring with $\mathbf{Q}$). Many additional examples of calculations of the Alexander polynomials can be found in [68]. Note, finally that it is also beneficial to consider the Alexander polynomials over finite fields $\mathbf{F}_p$, rather than over $\mathbf{Q}$ i.e. $H_1(X_{G,\phi}, \mathbf{F}_p)$ (cf. [43]).

2.2.5 Alexander polynomials of plane algebraic curves: position of singularities.

Now we shall discuss the dependence of the Alexander polynomial of the positions of singularities of the curve. To this end we shall associate the following invariants of the singularities of plane curves: rational numbers, called the constants of quasiadjunction: $\kappa_1^P, ..., \kappa_{n(P)}^P$ corresponding to each point P in the set $\mathrm{Sing}C \subset \mathbf{P}^2$ of singular points of C. Moreover, to each $\kappa \in \mathbf{Q}$, which is a constant of quasiadjunction of a point $P \in \mathrm{Sing}C$ and each $Q \in \mathrm{Sing}C$, we associate the ideal $\mathcal{J}_\kappa \subset \mathcal{O}_Q$ in the local ring of $Q \in \mathbf{P}^2$. (P and Q may be distinct). This data of constants of quasiadjunction and the ideals in the local rings of singular points determines the global Alexander polynomial $\Delta_C(t)$ completely (cf. [42] and theorem (2.11) below).

The idea of calculation is based on the relation between the Alexander polynomial and the homology of cyclic covers on one side and the classical method of adjoints to describe holomorphic 2-forms on hypersurfaces in projective space (cf. [84]).

The relationship between the Alexander polynomial and the homology of cyclic branched covering is the following:

Theorem 2.8 *Let $f(x,y)=0$ be the equation of a curve $C \in \mathbf{C}^2$. Let $\tilde{V}_n$ be a desingularization of a compactification of the surface $z^n = f(x,y)$ in $\mathbf{C}^3$. If $A(\pi_1(\mathbf{C}^2 - C)) \otimes \mathbf{Q} = \oplus \mathbf{Q}[t,t^{-1}]/(\delta_i(t))$ is the cyclic decomposition of the Alexander module of C (i.e. $\Delta_C(t) = \Pi_i \delta_i(t)$) then $\mathrm{rk} H_1(V_n, \mathbf{Q})$ is equal to the sum over i of the number of common roots of $t^n - 1$ and $\delta_i(t)$. If the line at infinity is transversal to C then the Alexander module is semisimple and the dimension of the ω_n-eigenspace of a generator of the Galois group $\mathbf{Z}_n$ acting on $H_1(V_n, \mathbf{C})$ (ω_n is a root of unity of degree n) is equal to the multiplicity of ω_n as a root of the Alexander polynomial.*

Note that the first Betti number of a non-singular projective algebraic surface is a birational invariant and hence the first Betti number of a resolution of a compactification is a well defined invariant of an affine surface $z^n = f(x,y)$. Therefore it also an invariant of affine curve C. Similar result is valid for branched covering of S^3 branched over a link: the idea of using covering spaces to derive invariants of knots goes back to Alexander and Reidemister (cf [2], [69], [82], [84]). A consequence of this theorem is that the homology of cyclic covers, in the case when line at infinity is transversal to C determine the Alexander polynomial. Another consequence, is periodicity of the homology of cyclic covers. In the abelian case the growth of the homology is polynomial periodic (cf. [31])

The calculation of the homology of cyclic covers using theory of adjoints was carried out in [81] (the case when C has cusps and nodes), [40] (the case when C has singularities of the form $x^k = y^k$ or $x^k = y^{k+1}$) and, much later, for the curves with arbitrary singularities, in [42]. The proofs for a generalization to situation including hypersurfaces having arbitrary dimension is given in [49]. In fact all these proofs yields the irregularity $q = \mathrm{dim} H^1(\tilde{V}_n, \mathcal{O}_{\tilde{V}_n}) = \mathrm{dim} H^0(\tilde{V}_n, {\Omega^1}_{\tilde{V}_n}) = \frac{1}{2} \mathrm{dim} H^1(\tilde{V}_n, \mathbf{C})$ (and in [49] the Hodge number $h^{n,0}$ for cyclic coverings of $\mathbf{P}^{n+1}$).

For details of the using this method we shall refer to [49] and section 5.2, but here we shall only remark that the adjoint ideal of a germ $(W, P) \in (\mathbf{C}^3, P)$ of isolated singularity at P consists of germs in $\mathcal{O}_P$ which restriction to W belongs to $\Phi_*(\Omega^2_{\tilde{W}})$ where $\Phi : \tilde{W} \to W$ is a resolution of singularities of W. If W is given by an equation $F = 0$, then the 2-forms on $W - P$ are residues of 3-forms $\frac{\psi(x,y,z)dx\wedge dy\wedge dz}{F(x,y,z)}$ on $\mathbf{C}^3$ having pole of order one along W i.e. restrictions of 2-forms:

$$\frac{\psi(x, y, z)dx \wedge dy}{\frac{\partial F(x,y,z)}{\partial z}} \tag{10}$$

on $W - P$.

On the other hand the 2-forms on a resolution can be described as the 2-forms on $W - P$ which can be extended over the exceptional locus of Φ. Hence a germ $\psi(x, y, z)$ is in the adjoint ideal of W if the pull back of the form (10) on resolution $\tilde{W}$ extends over the exceptional set. Such interpretation of 2-forms on resolutions allows to relate the dimensions of space 1-forms on $\tilde{V}_n$ (which is isomorphic to $H^1(\Omega^2_{\tilde{V}_n})$) to H^1 of certain sheaf of ideals on $\mathbf{P}^2$ which we are going to describe.

Let $\phi(x, y)$ be a germ of a holomorphic function. Let us consider the function $\Xi_\phi(n)$ which assigns to a n the minimal k such that $z^k\phi(x, y)$ belongs to the adjoint ideal of the singularity $z^n = f(x, y)$.

Lemma 2.9 *There exist $\kappa_\phi \in \mathbf{Q}$ (also depending on singularity $f(x, y)$) such that for $\Xi_\psi(n) = [\kappa_\phi n]$ ([..] denotes the integer part)*

The adjoint ideal of a function $F(x, y, z)$, which is generic for its Newton polytope, can be described as follows: a monomial $x^\alpha y^\beta z^\gamma$ is in the adjoint ideal of $F(x, y, z)$ if and only if the point $(\alpha+1, \beta+1, \gamma+1)$ is inside the Newton polytope of $F(x, y, z)$ (cf. [61]). Hence if $f(x, y) = x^a+y^b$ and $\phi(x, y) = x^iy^j$ then $z^kx^iy^j$ is in the adjoint ideal of $z^n = f(x, y) = x^a+y^b$ if and only if $(i+1)bn+(j+1)an+(k+1)ab > abn$ or $k + 1 > n(1 - (i + 1)\frac{1}{a} - (j + 1)\frac{1}{b})$. Therefore:

$$\Xi_{x^iy^j}(n) = \max([n(1 - (i + 1)\frac{1}{a} - (j + 1)\frac{1}{b})], 0) \tag{11}$$

This construction can be used to associate to a constant $\kappa \in \mathbf{Q}$ the following ideal in the local ring of the singular point of germ $f(x, y)$:

Definition 2.10 *Let $\kappa \in \mathbf{Q}$. The corresponding ideal of quasiadjunction is defined as follows:*

$$J_\kappa = \{\phi(x, y)|\kappa > \kappa_\phi\}$$

For example if $f(x, y) = x^2 + y^3$ then:

$$\kappa_{x^iy^j}(n) = \begin{cases} [\frac{n}{6}], & \text{(i,j)=0} \\ 0, & i + j \geq 1 \end{cases} \tag{12}$$

and hence there is only one constant of quasiadjunction $\kappa = \frac{1}{6}$. Moreover, $J_{\frac{1}{6}}$ is generated by monomials such that $i + j \geq 1$ i.e. is the maximal ideal.

Loeser and Vaquié showed that the constants of quasiadjuction are precisely the elements of Arnold-Steenbrink spectrum of singularity $f(x,y)$ which are belong to the interval $(0,1)$. In particular $exp(2\pi i\kappa_\phi)$ are the eigenvalues of the monodromy of $f(x,y) = 0$ and hence are the roots of the Alexander polynomial of the link of $f(x,y)$. After introduction of multiplier ideals it was soon realized that the ideals of quasiadjunction are closely related to multiplier ideals (cf. section 5.4 below). J.Kollar noticed the connection between the log-canonical threshold and the constants of quasiadjunction (cf. [35] and section 5.4).

Using the ideals J_κ in the local rings of points in $\mathbf{P}^2$, which are the singular points of a curve $C \in \mathbf{P}^2$, one defines the ideal sheaf

$$\mathcal{J}_\kappa = \mathrm{Ker}\mathcal{O}_{\mathbf{P}^2} \to \oplus_{P\in \mathrm{Sing}C}\mathcal{O}_P/J_{\kappa,P} \tag{13}$$

where $J_{\kappa,P}$ is the ideal corresponding to the singularity of C at P and the constant κ. Using this we can calculate the Alexander polynomial as follows:

Theorem 2.11 *Let C be a curve in $\mathbf{P}^2$ having degree d and let $\kappa_1, ..., \kappa_N$ be the collection of all constants of quasiadjunction of all singular points of C. Then the Alexander polynomial $\Delta_C(t)$ is given by:*

$$\Pi_{i,d\kappa_i\in\mathbf{Z}}[(t - exp(2\pi\sqrt{-1}\kappa_i)(t - exp(2\pi\sqrt{-1}\kappa_i)]^{\dim H^1(\mathbf{P}^2,\mathcal{J}_{\kappa_i}(d-3-d\kappa_i))}$$

Note that the exponent can be written as follows:

$$\dim H^1(\mathbf{P}^2, \mathcal{J}_{\kappa_i}(d-3-d\kappa_i)) = \dim H^0(\mathbf{P}^2, \mathcal{J}_{\kappa_i}(d-3-d\kappa_i)) - \chi(\mathcal{J}_{\kappa_i}) \tag{14}$$

(since $H^2(\mathbf{P}^2, \mathcal{J}_{\kappa_i}(d-3-d\kappa) = 0)$. In other words the exponent is the difference between the actual and "expected" dimensions of the linear system of curves of degree $d-3-d\kappa_i$ which local equations belong to the ideals of quasiadjunction corresponding to the constant κ. Therefore, (14) is what is classically called the superabundance of this linear system.

As an example, let us consider the sextics with six ordinary cusps. Since only one type of singularities is present and (12) shows that there is only one constant quasiadjunction $\kappa = \frac{1}{6}$, the Alexander polynomial has the form

$$[(t - exp(\frac{2\pi i}{6}))(t - exp(-\frac{2\pi i}{6}))]^s = (t^2 - t + 1)^s$$

Now the linear system in question consists on the curves having degree $6-3-\frac{6}{6} = 2$ with local equations belonging to the maximal ideals of the singular points. Since the dimension of the space of quadrics in 6, the expected dimension of our linear system is 0 and if a quadric containg all six cusps does exist then the actual dimension is 1 (one can show that this is the maximal possible value). Hence $s = 1$ and the Alexander polynomial is $t^2 - t + 1$.

For a sextic $\hat{C}_3$ with nine cusps, which is dual to a non singular cubic, one has $\dim H^0(\mathbf{P}^2, \mathcal{J}_{\kappa_i}(d-3-d\kappa_i)) = 0$ and $\chi(\mathcal{J}_{\kappa_i}) = -3$ and hence:

$$\Delta_{\hat{C}_3}(t) = (t^2 - t + 1)^3 \tag{15}$$

For the curve from [77] given by the equation: $f_{3n}^2 + f_{2n}^3 = 0$ where f_n is a generic form of degree n, which has only ordinary cusps at $6n^2$ points forming a complete intersection of curves of degrees $2n$ and $3n$ the exponent of $t^2 - t + 1$ in theorem 2.11 is the superabundance of the curve of degree $6n - 3 - \frac{6n}{6}$ passing through this complete intersection. By a theorem of Cayley-Bacharach this superabundance is 1 and hence the Alexander polynomial is $t^2 - t + 1$.

2.3 Commutative fundamental groups

2.3.1 Commutativity in terms of local type of singularities. Nori's theorem.

Historically, much of the work on the fundamental groups of the complements, was focused on the cases when the fundamental group is abelian. In this case Prop. 2.3 yields the complete calculation of π_1. For example, as was pointed out in the introduction, F.Severi was claiming that the fundamental group of the complement to a curve having nodes as the only singularities is abelian. More precisely he claimed the irreducibility of the stratum of nodal curves (this was proven by much later by J.Harris in [29]). The irreducibility of this stratum yields that each nodal curve can be degenerated into a union of lines in general position and for such union (these days called a generic arrangement of lines) a direct calculation shows that the fundamental group of the complement is free abelian. More generally than in the case of nodal curves, one expects, speaking very vaguely, that if a curve has not too many singularities or if the singularities are sufficiently mild then the fundamental groups of the complement will be abelian. A precise result in this direction follow from a theorem of M.Nori:

Theorem 2.12 *Let D and E be a curves on a non singular surface X. Assume that D has nodes as the only singularities, that D and E intersects transversally and that for an irreducible component C of D one has $C^2 > 2r(C)$ where $r(C)$ is the number of nodes on C. Then $N = \mathrm{Ker}\pi_1(X - D - E) \to \pi_1(X - E)$ is abelian.*

For plane curves one obtains the following which extends earlier commutativity results of S.Abhyankar.

Theorem 2.13 *For a germ ϕ of a curve singularity in $\mathbf{C}^2$ let us define the invariant $e(\phi)$ as follows. Let $\Phi : S \to \mathbf{C}^2$ be a resolution of the singularity and $\Phi^*(\phi) = F + G$ where F is the proper transform of $\phi = 0$, G is the exceptional set and F and G_{red} meet transversally. Let $e(\phi) = G(G + 2F)$ and let $F(C)$, for a curve C on a non singular projective surface, be the sum of invariants $e(\phi)$ for all singularities of C. If $C^2 > F(C)$ then the extension $\pi_1(X - C) \to \pi_1(X)$ is central.*

Proof. Apply Nori's theorem the proper transform C' of a resolution of singularities of C. Then if G is the exceptional set then $C^2 = (C' + G)^2 = C'^2 + 2(C', G) + G^2 =$

$C'^2 + F(C)$. Hence the assumed inequality translates into $C'^2 > 0$. Hence Nori's theorem yields the conclusion. Note that for node we have $G = 2E$ where E is the exceptional line and $C' = L_1 + L_2$. Hence $G^2 + 2(G, C') = 4E^2 + 2 \cdot 2E(L_1 + L + 2) = 4$. For a cusp $F(\phi) = 6$. In particular on a simply-connected surface the fundamental group of the complement to a curve with δ nodese and κ cusps is abelian if $C^2 > 6\kappa + 4\delta$.

The following question is still open:

Question 2.14 *Let N be a normal subgroup of $\pi_1(X)$ generated by the images of the fundamental groups of non singular models of components. Does N has a finite index in $\pi_1(X)$*

If so, then the fundamental group of a surface, containing a rational curve with positive self-intersection, must be finite.

2.3.2 On a proof of Nori's theorem.

Let us consider a special case when $E = \emptyset$, and C is an irreducible nonsingular curve on X. Let U be a tubular neighborhood of C. Then $U - C \to C$ is a circle fibration and the fiber δ is the element of $\pi_1(U - C)$ belonging to the center of the latter group. Since in this case the assumption of the theorem is $C^2 > 0$, the theorem of Nakai and Moishezon (cf. [30]) yields that C is ample and hence a small deformation D of nC, which we may assume belongs to U, is very ample and also smooth. By usual Lefschetz theorem $\pi_1(D - C) \to \pi_1(X - C)$ is surjective and hence $\pi_1(U - C) \to \pi_1(X - C)$ is surjective as well. Therefore the image if the class of γ in $\pi_1(X - C)$ belongs to its center. On the other hand, any element in $N = \mathrm{Ker}\pi_1(X - D) \to \pi_1(X)$ is product of elements conjugated to γ. Indeed, take such element δ and consider 2-disk Δ which it bounds in X. We can assume that $\Delta \cap C$ consists of finitely many transversal intersections. Therefore $\delta = \Pi\delta_i$ where $\delta_i = \alpha_i\gamma_i\alpha^{-1}$ with γ_i being a fiber of $U - C \to C$ and α_i is a path going from the base point to a point on the boundary of U. In particular δ_i is conjugate to γ in $\pi_1(X - C)$ and hence is equal to γ. Hence δ is a power of γ i.e. N is cyclic.

Crutial in the proof of Nori's theorem in the case of nodal C is the following Nori's weak Lefschetz theorem which is very interesting by itself.

Theorem 2.15 *Let $i : H \to U$ be an embedding of connected compact complex analytic subspace (possibly non reduced) into a connected complex manifold U in which H is defined by a locally principal sheaf of ideals. Assume that $\mathcal{O}_U(H)|H$ is ample and that $dimU > 2$. Let $q : U \to X$ be holomorphic local isomorphism with the target being a smooth projective variety and $h = q \circ i$. Let R be an arbitrary Zariski closed subset and $G = \mathrm{Im}\pi_1(U - q^{-1}(R)) \to \pi_1(X - R)$. Then G is a subgroup of finite index.*

2.4 Higher homotopy groups

Another natural invariants of the homotopy type of the complement are the higher homotopy groups of the complement. However for curves, the higher homotopy groups, unlike the fundamental groups, it seems, do not have an algebro-geometric significance. Moreover, in most cases the higher homotopy groups, considered as abelian groups are infinitely generated. A more useful way to consider them is by using the action of π_1 on π_k i.e. consider π_k as a module over π_1. However unless π_1 is abelian, understanding modules over π_1 involves a subtle non commutative algebra. For curves however, as will be explained in the next section, the homotopy type of the complement is determined by another invariant of the pair $(\mathbf{P}^2, C)$ i.e. the *braid monodromy*. On the other hand for hypersurfaces in $\mathbf{P}^{n+1}$ with $n > 1$ the homotopy groups in dimensions up to n have interesting algebro-geometric meaning which we shall proceed to discuss.

2.4.1 Action of the fundamental group on higher homotopy groups

Let us start with the example which shows why the homotopy groups of simplest topological spaces are infinitely generated.

Example 2.16 *Let us consider $\pi_2(S^1 \vee S^2)$. Clearly $\pi_1(S^1 \vee S^2) = \mathbf{Z}$. On the other hand $\pi_2(S^1 \vee S^2)$ can be identified with π_2 of the universal cover of $S^1 \vee S^2$. Viewing the universal covering map of the circle as the the quotient of $\mathbf{R}$ by the subgroup of integers makes it natural to view the universal cover of $S^1 \vee S^2$ as the real line with S^2's attached at the integer points. Hence the universal cover has H_2, and by Hurewicz theorem π_2, isomorphic to $\mathbf{Z}^\infty$. On the other hand, since the deck transformation of the universal cover acts transitively on S^2's attached to $\mathbf{R}$, both H_2 and π_2 are cyclic modules over the group of deck transformations i.e $\pi_2(\widetilde{S^1 \vee S^2}) = \mathbf{Z}[t, t^{-1}]$ ($\tilde{X}$ denotes the universal cover).*

In general, the homotopy groups can be given the structure of a module over the fundamental group using the Whitehead product: $\pi_n \times \pi_m \to \pi_{n+m-1}$. In the case when $\pi_i(X) = 0$ for $2 \le i \le n-1$, if $\tilde{X}$ is the universal cover then $\pi_n(X) = \pi_n(\tilde{X}) = H_n(\tilde{X})$ and the action of $\pi_1(X)$ is just the action of the deck transformations on the homology.

Such X come up naturally:

Theorem 2.17 *Let V be a hypersurface in $\mathbf{P}^{n+1}$ having only isolated singularities. Let H be a generic hyperplane. Then $\pi_1(\mathbf{P}^{n+1} - V \cap H) = \mathbf{Z}$ and $\pi_i(\mathbf{P}^{n+1} - V \cap H) = 0$ for $2 \le i \le n-1$. Moreover, $\pi_n(\mathbf{P}^{n+1} - V \cap H) \otimes \mathbf{Q}$ is a $\mathbf{Q}[t, t^{-1}]$-torsion module.*

More generally, the Lefschetz hyperplane section theorem yields that the conclusion of the theorem holds for arbitrary hypersurfaces in $\mathbf{P}^N$ for which the singular locus has codimension $n+1$. To see this (and also the first part of theorem 2.17) recall it:

Theorem 2.18 *(***Lefschetz hyperplane section theorem***)*

(a) Let X be a projective subvariety having dimension n and let L be a codimension d linear subspace such that X is a local complete intersection outside of L. Then

$$\pi_i(X \cap L) \to \pi_i(X)$$

is isomorphism for $0 \leq i < n-d$ and surjective for $i = n-d$.

(b)Let X be a quasiprojective. The conclusion of (a) take place for generic L.

Vanishing statement in theorem 2.17 follows from this and calculation of the homotopy groups of the complement to non singular hypersurfaces.

Recently, L.Maxim ([59]) showed that the homology of infinite cyclic covers of the complement to an affine hypersurface, generic relative to hyperplane at infinity, are torsion modules in all dimensions except the top one.

2.4.2 Orders of the homotopy groups

It follows from the theorem 2.17 and the classification of modules over PIDs that

$$\pi_n(\mathbf{P}^{n+1} - V \cap H) \otimes \mathbf{Q} = \oplus \mathbf{Q}[t, t^{-1}]/\Delta_i(t)$$

for some polynomials $\Delta_i(t)$ defines up to a unit in $\mathbf{Q}[t, t^{-1}]$. We call $\Delta(t) = \Pi_i \Delta_i(t)$ the order of the group π_n. Though $\Delta(t)$ cannot be calculated in terms of a local data of singularities there is the following divisibility relation, which generalizes the divisibility relation for the Alexander polynomials:

Theorem 2.19 (**Divisibility theorem I**) *The order of $\pi_n(\mathbf{C}^{n+1} - V)$ divides the product of characteristic polynomials of the monodromy operators of singularities of V:*

$$\Delta(\mathbf{C}^{n+1} - V) | \Pi_{P_i \in Sing(V)} \Delta_{P_i}(t)$$

Note that as it stated, one should assume that V it transversal to the hyperplane at infinity. However one can define correction factors corresponding to the singularities at infinity so that, if one multiplies by these correction factors the right side in 2.19, the divisibility relation will hold.

Theorem 2.20 (**Divisibility theorem II**) *Let V be a hypersurface transversal to the hyperplane at infinity H_∞. Let S_∞ be the boundary of a small tubular neighborhood of H_∞ and let $L_\infty = V \cap S_\infty$. Then the homology of the infinite cyclic cover of $S_\infty - L_\infty$ is a torsion $\mathbf{C}[t, t^{-1}]$-module and Δ_∞ and $\Delta(\mathbf{C}^{n+1} - V) | \Delta_\infty$.*

(see [48] for a statement in the case with a weaker than transversality to H_∞ assumption)

3 Homotopy groups via pencils.

3.1 Van Kampen theorem and braid monodromy

Now let us consider how one actualy can calculate the fundamental group of a complement in the case of curves and how to calculate the first non trivial homotopy group of the complement in the case of hypersurfaces. In this section we shall deal with the curves (cf. also [26] where the case of possibly singular quasiprojective varieties is discussed).

Let C be a curve on a projective surface X for which we want to describe $\pi_1(X-C)$. Consider a line bundle $\mathcal{L}$ on X such that $\dim H^0(X,\mathcal{L})) \geq 2$ and select a 2-dimensional linear system $\mathbf{L} \subseteq H^0(X,\mathcal{L})$. Let B be the base locus of $\mathbf{L}$ (it contains at most $c_1(\mathcal{L})^2$ points). We shall assume for simplicity that $B \cap C = \emptyset$. The classical case is $X = \mathbf{P}^2$, $\mathcal{L} = \mathcal{O}(1)$ and $\mathbf{L} \subset H^0(\mathbf{P}^2, \mathcal{O}(1))$ consists of sections with the zerosets containing a fixed point. We have a regular map onto the projectivization of $\mathbf{L}$:

$$p: \ X - B \rightarrow \mathbf{P}(\mathbf{L}) = \mathbf{P}^1 \tag{16}$$

with generic fiber $L_{t_0} - L_{t_0} \cap B, t_0 \in \mathbf{P}^1$ being non singular by Bertini's (or Sard's) theorem. Though generic element of $\mathbf{L}$ may be singular at points of B, we shall make additional assumtion that L_{t_0} is non singular at any $p \in B \cap C$. The curve L_{t_0} is ample and hence $\pi_1(L_{t_0} - L_{t_0} \cap C) \rightarrow \pi_1(X-C)$ is surjective by Lefschetz theorem. We want to describe the kernel of this map. Let $Sing \subset \mathbf{P}^1$ be the (finite) subset of points $t_1, ..., t_N$ corresponding to singular members of the pencil. Each fiber of the pencil (16) is a punctured curve (which, if L_{t_0} is non singular at the points of B, has genus $g(L_{t_0}) = \frac{c_1(\mathcal{L})(K_X + c_1(\mathcal{L}))}{2} + 1$).

For each d one can define the braid group $B_d(L_{t_0} - B)$ which is the group of isotopy classes of orientation preserving diffeomorphisms of L_{t_0} which are constant in a neighborhood of B in L_{t_0}. In the case $L_{t_0} - B \cap L_{t_0} = \mathbf{C}$ one obtains the classical Artin's braid group with generators $\sigma_i, i = 1, ..., d-1$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ \ i = 1, .., d-2 \tag{17}$$

(for presentations of braid groups similar to $B_d(L_{t_0} - B)$ by generators and relations and extending this one, see [73])

We want to construct a homomorphism $\pi_1(\mathbf{P}^1 - Sing, t_0) \rightarrow B_d(L_{t_0} - B)$ called *braid monodromy* and use it to describe $\pi_1(X-C)$. We shall start by defining "good" systems of generators of $\pi_1(\mathbf{C} - Sing)$, which we shall use to give a finite presentation for this fundamental group.

Definition 3.1 *Let $Sing = \{t_1, ..., t_N\}$. A system of generators $\gamma_i \in \pi_1(\mathbf{C} - \bigcup_i t_i, t_0)$ is called good if each of the loops $\gamma_i : S^1 \rightarrow \mathbf{C} - \bigcup_i t_i$ extends to a map of the disk $D^2 \rightarrow \mathbf{C}$ with non-intersecting images for distinct i's.*

One way to construct a good system of generators is the follwoing. Select a system of small disks Δ_i about each point t_i $i = 1, ..., N$, and choose a system of N non-intersecting paths δ_i connecting the base point t_0 with a point of $\partial\Delta_i$. Then $\gamma_i = \delta^{-1} \circ \partial\Delta_i \circ \delta_i$ is a good system of generators (with, say, the counterclockwise orientation of $\partial\Delta_i$). We shall need also good systems of generators of the fundamental groups of the complements to a finite set of N points on a *compact* Riemann surface having genus $g \geq 0$. Those are the systems of generators $\gamma'_1, ..., \gamma'_{2g}$, consisting of the images $2g$ sides of a $4g$-gon for some presentation of the surface as a $4g$-gon with identified sides and a good system of generators $\gamma_1, ..., \gamma_N$ of the complement to N points in this $4g$-gon in the above sense. We have the only relation

$$R: \ \Pi\gamma_1 \cdot ... \cdot \gamma_N = \Pi[\gamma'_i, \gamma'_{i+1}] \tag{18}$$

In the case $g = 0$ this relation becomes $\Pi\gamma_1 \cdot ... \cdot \gamma_N = 1$.

Now let us define the braid monodromy corresponding to an element $\gamma \in \pi_1(\mathbf{P}^1 -$ Sing). Let $\gamma \in \mathbf{P}^1 -$ Sing be the image of an embedding of S^1 taking the base point to t_0. We can view γ as the image of the map $\iota : I \to \mathbf{P}^1 - Sing$ (I is the unit interval) such that $\iota(0) = \iota(1) = t_0$. Then $(X - B - C) \times_{\mathbf{P}^1 - Sing} I$ is a locally trivial fibration over I and hence is a trivial fibration. This means that there is a map $\Phi : L_{t_0} - L_{t_0} \cap C \times I \to X - B$ such that $\Phi(t)|_{L_{t_0} - L_{t_0} \cap C \times t}$ is a homeomorphism onto $L_t - L_t \cap C$. Note that though Φ is not unique any two choices are isotopic via isotopies commuting with projections on I. Hence we obtain the map $\Phi(1) : L_{t_0} - L_{t_0} \cap C \to L_{t_0} - L_{t_0} \cap C$ and the isotopy class of this map is well defined. We can assume that this map keeps B fixed. One checks immediately that dependence on ι yields homotopic maps $\Phi(1)$ and a homotopy of γ extends to a homotopy of $\Phi(1)$ (but B may not be possible to preserve). Hence we obtain the braid monodromy homomorphism:

$$\pi_1(\mathbf{P}^1 - Sing) \to \pi_0(Diff(L_{t_0})) = B_d(L_{t_0}) \tag{19}$$

(where $d = (C, L_t)$ and the last group is the braid group of Riemann surface L_{t_0}.

There is a useful way to encode algebraically the homomorphism (19) using the choice of a good system of generators of $\pi_1(\mathbf{P}^1 -$ Sing). Let us fix a fiber L_{t_∞} of the pencil which we shall call the fiber at infinity. Then we can select monodromy transformations all fixing a neighborhood of B for all γ_i i.e. we obtain ordered system of braids: $\beta(\gamma_i) = \Phi_{\gamma_i}(1) \in B_d(L_{t_0} - L_{t_0} \cap C - B)$ with the order given by the order of the good systems of generators. The latter is given by the counterclockwise ordering of loops about the point t_0. Moreover, the product is a fixed word in $B_d(L_{t_0})$ independend of C. For example we obtain in the case of curves in $\mathbf{C}^2$:

$$\Pi\beta(\gamma_i) = \Delta^2 \tag{20}$$

where Δ^2 is the generators of the center of the Artin's braid group B_d (cf. [46]).

We have the following calculation in terms of the braid monodromy originated by Zariski-van Kampen:

Theorem 3.2 *Let $b \in \partial T(B) \cap L_{t_0}$ where $T(B)$ is a neighborhood of B in X and let α_j be a good system of generators of $\pi_1(L_{t_0} - L_{t_0} \cap C, b)$. Let R be the relation among α_j. Then*

$$\pi_1(X - C - L_{t_\infty}) = \pi_1(L_{t_0} - L_{t_0} - B, b)/(\beta(\gamma_i)(\alpha_i)\alpha_i^{-1})$$

(quotient by the normal subgroup generated by specified elements). The group $\pi_1(X - C)$ can be obtained by adding to the above the relation R.

In the case of plane curves we have just the homomorphism into Artin's braid group which by itself is an interesting invariant of plane curves containing more information than the fundamental group. For example the braid monodromy determines the homotopy type of the complement $\mathbf{C}^2 - C$ (cf. [45]). Many calculations are done for curves C which are the branching curves of generic projections of surfaces (cf. [76]). Recently braid monodromy found applications in symplectic geometry (cf. [4]).

3.2 Homotopy groups via pencils

Now let V be a hypersurface in $\mathbf{C}^{n+1}$ transversal to the hyperplane at infinity and having only isolated singularities. We want to describe calculation of the first non trivial homotopy group $\pi_n(\mathbf{C}^{n+1} - V)$ in terms of pencils generalizing the Zariski-van Kampen procedure described above.

We start with a high dimensional analog of the braid group and a linear representation generalizing the Burau representation of the braid group. In higher dimensions we have several candidates for such a generalization.

Let us consider a sphere S^{2n-1} in $\mathbf{C}^n$ of a sufficiently large radius. Let $\partial_\infty V = V \cap S^{2n-1}$ and let $Emb(V, \mathbf{C}^n)$ be the space of submanifolds of $\mathbf{C}^n$ with the following property: each is diffeomorphic to V and, moreover, is isotopic to the chosen embedding of V. In addition we require that for any $V' \in Emb(V, \mathbf{C}^n)$ one has $V'(V) \cup S^{2n-1} = \partial_\infty V$. We shall use the topology with the basis consisting of sets $U(V, \epsilon)$ of submanifolds $V' \subset \mathbf{C}^n$ which belong to the tubular neighborhood of V having radius ϵ and which are isotopic to V.

Let us describe a linear representation

$$\pi_1(Emb(V, \mathbf{C}^n)) \to Aut\pi_n(\mathbf{C}^n - V) \tag{21}$$

After a choice of a basis in the $\pi_1(\mathbf{C}^n - V)$-module $\pi_n(\mathbf{C}^n - V)$ this homomorphism becomes the homomorphism into $GL_r(\mathbf{Z}[t, t^{-1}])$ where r is the rank of $\tilde{H}_n(\mathbf{C}^n - V, \mathbf{Z})$ (the reduced homology of the complement). It is given in terms of the representation of another group which also is a candidate for the high-dimensional braid group.

Let $Diff(\mathbf{C}^n, S^{2n-1})$ be the group of diffeomorphisms of $\mathbf{C}^n$ acting as the identity outside S^{2n-1}. This group can be identified with the group $Diff(S^{2n}, D^{2n})$ of the diffeomorphisms of the sphere fixing a disk. Let $Diff(\mathbf{C}^n, V)$ be the subgroup of $Diff(\mathbf{C}^n, S^{2n-1})$ of the diffeomorphisms which take V into itself.

The group $Diff(\mathbf{C}^n, S^{2n-1})$ acts transitively on $Emb(V, \mathbf{C}^n)$ with the stabilizer $Diff(\mathbf{C}^n, V)$. Therefore we have the following exact sequence:

$$\pi_1(Diff(S^{2n+2}, D^{2n+2})) \to \pi_1(Emb(\mathbf{C}^n, V)) \to \pi_0(Diff(\mathbf{C}^n, V)) \tag{22}$$

$$\to \pi_0(Diff(S^{2n+2}, D^{2n+2})) \to$$

Any element in $Diff(\mathbf{C}^n, V)$ induces the self map of $\mathbf{C}^n - V$ and also the self map of the universal (in the case $n = 1$ universal cyclic) cover of this space. Hence it induces an automorphism of $H_n(\widetilde{\mathbf{C}^n - V}, \mathbf{Z}) = \pi_n(\mathbf{C}^n - V)$, $n > 1$. This gives the representation:

$$\pi_0(Diff(\mathbf{C}^n, V)) \to Aut\pi_n(\mathbf{C}^n - V) \tag{23}$$

The composition of the boundary homomorphism in (22) with the map (23) results in representation (21). The groups $\pi_1(Emb(\mathbf{C}^n, V))$ and $\pi_0(Diff(\mathbf{C}^n, V))$ are high-dimensional analogs of the braid groups and their algebraic study was not carried out so far. However some high-dimensional analogs of the mapping class groups were studied in (cf. [33]).

In the case $n = 1$, V is just a collection of points in $\mathbf{C}$, $\pi_1(Emb(\mathbf{C}, V)) = \pi_0(Diff(\mathbf{C}, V))$ is Artin's braid group, and this construction gives the homomorphism of the braid group into $AutH_1(\widetilde{\mathbf{C} - V}, \mathbf{Z})$ which, after a choice of the basis in $H_1(\widetilde{\mathbf{C} - V})$ corresponding to the choice of the generators of the braid group, gives the reduced Burau representation. In higher dimensions the isomorphism $\pi_1(Emb(V, \mathbf{C}^n)) = \pi_0(Diff(\mathbf{C}^n, V))$ fails.

Problem 3.3 *Calculate the groups $\pi_1(Emb(V, \mathbf{C}^n))$ and $\pi_0(Diff(\mathbf{C}^n, V))$*

Now we can define the relevant monodromy operator corresponding to a loop in the parameter space of a linear pencil of hyperplane sections. By our assumptions, the projective closure of V is a hypersurface in $\mathbf{P}^{n+1}$ which has only isolated singularities. Let H be the hyperplane at infinity (which is transversal to V). Let L_t, $t \in \mathbf{C}$, be a pencil of hyperplanes the projective closure of which has as the base locus a hyperplane M in H such that M also is transversal to V. Let $t_1, ..., t_N$ denote those t for which $V \cap L_t$ has a singularity. We shall assume that for any i the singularity of $V \cap L_{t_i}$ is outside of H. The pencil L_t over $\mathbf{C} - \bigcup_i t_i$ defines a locally trivial fibration τ of $\mathbf{C}^{n+1} - V$ with a non-singular hypersurface in $\mathbf{C}^n$ transversal to the hyperplane at infinity as a fiber. The restriction of this fibration on the complement to a sufficiently large ball is trivial, as follows from the assumptions on the singularities at infinity. Let $\gamma : [0, 1] \to \mathbf{C} - \bigcup_i t_i$ $(i = 1, ..., N)$ be a loop with the base point t_0. A choice of a trivialization of the pull back of the fibration τ on $[0, 1]$ using γ, defines a loop e_γ in $Emb(L_{t_0}, V \cap L_{t_0})$. Different trivializations produce homotopic loops in this space.

Definition 3.4 *The monodromy operator corresponding γ is the element in $Aut(\pi_n(L_{t_0} - L_{t_0} \cap V))$ corresponding in (21) to e_γ*

Next we will need to associate the following homomorphism with a singular fiber L_{t_i} and a loop γ with the base point t_0 in the parameter space of the pencil where γ bounds a disk Δ_{t_i} not containing other singular points of the pencil :

$$\pi_{n-1}(L_{t_i} - L_{t_i} \cap V) \to \pi_n(L_{t_0} - L_{t_0} \cap V)/Im(\Gamma - I). \tag{24}$$

Here Γ is the monodromy operator corresponding to γ.

To construct (24) let us note that the π_1-module on the right in (24) is isomorphic to the homology $H_n(\widetilde{\tau^{-1}(\partial \Delta_{t_i})}, \mathbf{Z})$ of the infinite cyclic cover of the restriction of the fibration τ on the boundary of Δ_{t_i}. This follows immediately from the Wang sequence of a fibration over a circle and the vanishing of the homotopy of $L_{t_0} - L_{t_0} \cap V$ in dimensions below n. Let B_i be a polydisk in $\mathbf{C}^{n+1}$ such that $B_i = \Delta_i \times B$ for a certain polydisk B in L_{t_0}. Then $\tau^{-1}(\widetilde{\Delta_i}) - B_i$ is a trivial fibration over Δ_i with the infinite cyclic cover $\widetilde{L_{t_i} - L_{t_i} \cap V}$ as a fiber. In particular, one obtains the map:

$$\pi_{n-1}(L_{t_0} - L_{t_0} \cap V) = H_{n-1}(\widetilde{L_{t_0} - L_{t_0} \cap V}, \mathbf{Z}) \to H_n(\tau^{-1}(\widetilde{\Delta_i}) - B_i, \mathbf{Z}) =$$

$$H_{n-1}(\widetilde{L_{t_0} - L_{t_0} \cap V}, \mathbf{Z}) \oplus H_n(\widetilde{L_{t_0} - L_{t_0} \cap V}, \mathbf{Z}) \qquad (25)$$

Definition 3.5 *The degeneration operator is the map (24) given by composition of the map (25) with the map $H_n(\tau^{-1}(\widetilde{\Delta_i}) - B_i, \mathbf{Z}) \to H_n(\tau^{-1}(\widetilde{\Delta_i}), \mathbf{Z}) = \pi_n(L_{t_0} - L_{t_0} \cap V)$ induced by inclusion.*

The following is a high-dimensional analog of the van Kampen theorem.

Theorem 3.6 *Let V be a hypersurface in $\mathbf{P}^{n+1}$ having only isolated singularities and transversal to the hyperplane H at infinity. Consider a pencil of hyperplanes in $\mathbf{P}^{n+1}$ the base locus M of which belongs to H and is transversal in H to $V \cap H$. Let $\mathbf{C}^n_t$ $(t \in \Omega)$ be the pencil of hyperplanes in $\mathbf{C}^{n+1} = \mathbf{P}^{n+1} - H$ defined by L_t (where $\Omega = \mathbf{C}$ is the set parameterizing all elements of the pencil L_t excluding H). Denote by $t_1, ..., t_N$ the collection of those t for which $V \cap L_t$ has a singularity. We shall assume that the pencil was chosen so that $L_t \cap H$ has at most one singular point outside of H. Let t_0 be different from either of t_i $(i = 1, ..., N)$. Let γ_i $(i = 1, ..., N)$ be a good collection, in the sense described in definition (3.1), of paths in Ω based in t_0 and forming a basis of $\pi_1(\Omega - \bigcup_i t_i, t_0)$ and let $\Gamma_i \in Aut(\pi_n(\mathbf{C}^n_t - V \cap \mathbf{C}^n_t))$ be the monodromy automorphism corresponding to γ_i. Let $\sigma_i : \pi_{n-1}(\mathbf{C}^n_{t_i} - V \cap \mathbf{C}^n_{t_i}) \to \pi_n(\mathbf{C}^n_{t_0} - V \cap \mathbf{C}^n_{t_0})^{\Gamma_i}$ be the degeneration operator of the homotopy group of a special element of the pencil into the corresponding quotient of covariants constructed above. Then*

$$\pi_n(\mathbf{C}^{n+1} - V \cap \mathbf{C}^{n+1}) = \pi_n(\mathbf{C}^n - V \cap \mathbf{C}^n)/(Im(\Gamma_1 - I), Im\sigma_1, ..., Im(\Gamma_N - I), \sigma_N)$$

We refer for a proof to the paper [48]. There is another way to describe this homotopy group replacing the degeneration operator by a variation operators on the homotopy groups which we shall describe now.

3.3 Variation operators

Variation operators classically defined in homology (or cohomology). The idea of defining homotopy variation operator comes from the fact that the homotopy groups on question are the homology groups of covering spaces. A description of the homotopy groups using variation operators was carried out in [11].

We shall continue to use the notations from previous section but in addition let us select $e \in M - M \cap V$ which we shall use as the base point for the homotopy groups. The homotopy variation operator is a certain homomorphism of $\mathbf{Z}[t, t^{-1}]$-modules:

$$\mathcal{V}_i\colon\ \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) \longrightarrow \pi_n(L_{t_0} - L_{t_0} \cap V, e) \qquad (26)$$

associated with each γ_i for $1 \leq i \leq N$.

As in definition (3.5) of degeneration operators we shall go to the d fold cover and use the homological variation operators. Let $W \subset \mathbf{P}^{n+2}$ be the d-fold cyclic branched over V cover of $\mathbf{P}^{n+1}$, $j : V \to W$ the embedding and $\mathcal{L}_t$ be the pull back of the pencil L_t on W. By abuse of notations we shall denote by the same letter the hyperplane in $\mathbf{P}^{n+2}$ cutting the corresponding divisor on W. $\mathcal{L}_{t_0} \cap W$ is the d-fold cover of L_{t_0} branched over $V \cap L_{t_0}$. We shall consider the homological variation operators:

$$V_i\colon\ H_n(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V))) \longrightarrow H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \qquad (27)$$

associated, for $1 \leq i \leq N$, with the homotopy classes γ_i.

The definition and the properties of operators V_i are discussed in [9]. For a relative n-cycle Ξ on $\mathcal{L}_{t_0} \cap (W - j(V))$ with boundary in $\mathcal{M} \cap (W - j(V))$, one defines ($[\cdot]$ denotes the class of a cycle):

$$V_i([\Xi]_{(\mathcal{L}_{t_0} \cap (W-j(V)), \mathcal{M} \cap (W-j(V)))}) = [H_{i*}(\Xi) - \Xi]_{\mathcal{L}_{t_0} \cap (W-j(V))} \qquad (28)$$

where H_i is the geometric monodromy corresponding to γ_i. Since H_i leaves the points of $\mathcal{M} \cap (W - j(V))$ fixed the chain $H_{i*}(\Xi) - \Xi$ is actually an absolute cycle and the correspondence $\Xi \mapsto H_{i*}(\Xi) - \Xi$ induces a homomorphism V_i at the homology level ([9, Lemmas 4.6 and 4.8]). This homomorphism depends only on the homotopy class γ_i ([9, Lemma 4.8]).

Now, if $n \geq 2$ then for $1 \leq i \leq N$, using the isomorphism α_{t_0} and the homomorphism $\bar{\alpha}_{t_0}$, V_i yields the homotopical variation operator $\mathcal{V}_i$ by requiring that the following diagram will be commutative:

$$\begin{array}{ccc} H_n(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V))) & \xrightarrow{V_i} & H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \\ \uparrow \bar{\alpha}_{t_0} & & \uparrow \alpha_{t_0} \\ \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) & \xrightarrow{\mathcal{V}_i} & \pi_n(L_{t_0} - L_{t_0} \cap V, e). \end{array} \qquad (29)$$

As V_i depends only on the homotopy class γ_i so do the operators $\mathcal{V}_i$.

With these definitions we have the following (cf. [11]):

Theorem 3.7 *Let V be a hypersurface in $\mathbf{P}^{n+1}$ with $n \geq 2$ having only isolated singularities. Consider a pencil $(L_t)_{t \in \mathbf{P}^1}$ of hyperplanes in $\mathbf{P}^{n+1}$ with the base locus M transversal to V. Denote by $t_1, \ldots, t_N$ the collection of those t for which $L_t \cap V$ has singularities. Let t_0 be different from either of $t_1, \ldots, t_N$. Let γ_i be a good collection of paths in $\mathbf{P}^1$ based in t_0. Let $e \in M - M \cap V$ be a base point. Let $\mathcal{V}_i$ be the variation operator corresponding to γ_i. Then the inclusion induces an isomorphism:*

$$\pi_n(\mathbf{P}^{n+1} - V, e) \longleftarrow \pi_n(L_{t_0} - L_{t_0} \cap V, e) \Big/ \sum_{i=1}^{N} \mathcal{V}_i \qquad (30)$$

There is affine version of this theorem equivalent to this one since $\pi_n(\mathbf{P}^{n+1} - V) = \pi_n(\mathbf{C}^{n+1} - V)$ in the case when H is transversal to V.

Recently, Cheniot and Eyral proposed definition of homotopy variation operator in general showed that the map as in the above theorem is surjective (cf. [10]; see also [78] for another discussion of variation operators).

4 Local multivariable Alexander invariants: topological theory

Now we want to develop an abelian version of the cyclic theory presented so far. Though our goal at this point, as in the link theory, is to study abelian covers, what will follow deviates from the link-theoretical point of view at several points. The most important one is that the Alexander type invariants are not polynomials. The substitute for the orders of the modules over PID which were discussed before are the subvarieties of commutative algebraic groups called the characteristic varieties.

4.1 Characteristic varieties of groups.

4.1.1 Definitions.

Let us start with a classical construction of commutative algebra. Let R be a Noetherian commutative ring with a unit and let M be a finitely generated R-module. Let the homomorphism $\Phi : R^m \to R^n$ be such that $M = \mathrm{Coker}\Phi$. The k-th Fitting ideal of M is the ideal $\mathcal{F}_k(M)$ generated by $(n-k+1) \times (n-k+1)$ minors of the matrix of Φ. $\mathcal{F}_k(M)$ depends only on M rather than on Φ. The k-th characteristic variety M is the reduced sub-scheme of $\mathrm{Spec}R$ defined by $\mathcal{F}_k(M)$.

If $R = \mathbf{C}[H]$ where H is an abelian group then $\mathrm{Spec}R$ is a torus having the dimension equal to the rank of H. If H is free then after a choice of generators of H, R can be identified with the ring of Laurent polynomials and $\mathrm{Spec}R = (\mathbf{C}^*)^{\mathrm{rk}H}$ is a complex torus. In particular each k-th characteristic variety of an R-module is a subvariety $V_k(M)$ of $(\mathbf{C}^*)^{\mathrm{rk}H}$. If H has a torsion then the number of connection components of $\mathrm{Spec}\mathbf{C}[H]$ is the order of the torsion and the connected component of the unit can be identified with $\mathrm{Spec}\mathbf{C}[H/Tor(H)]$.

A more functorial description is the following (cf. [8]):

$$V_k(M) = \mathrm{Supp}_{red}(\wedge^k M) = \mathrm{Supp}_{red}(R/F_k(M)) \tag{31}$$

We shall apply this construction to the modules $A(G, \phi)$ defined in section 2.2.3. for pairs (G, ϕ) where $\phi : G \to \mathbf{Z}^r$. Prime examples which we shall consider are the following:

Example 4.1 *(i) Links in S^3. In this case $H_1(S^3 - L, \mathbf{Z}) = \mathbf{Z}^r$ where L is such a link and r is the number of its components.*
(ii) Algebraic curves in $\mathbf{C}^2$ having r irreducible components (cf. section 2.2.1)

We shall denote the corresponding characteristic varieties as $V_k(G, \phi)$ omitting ϕ when no confusion is possible.

Definition 4.2 *(cf. [52]) The depth of a component V of a characteristic variety $V_k(G)$ is the integer $i = \max\{j | V \subset V_j(G)\}$.*

In the case $r = 1$ and G is one of the groups as above, $V_1(G)$ is the zero set of the Alexander polynomial and $V_k(G)$ is determined by the zero sets of elementary divisors of the Alexander module. Vice versa, the zero sets of Fitting ideals determine the zero sets of elementary divisors of a module over PID. Since the orders of $\mathbf{Q}[t, t^{-1}]$-modules in a cyclic decomposition are determined up to a unit of the ring of Laurent polynomials the depth of each root of the Alexander polynomial, given in terms of V_k's, determines the Alexander module completely.

If $\mathrm{codim} V_1(G, \phi)$ in $\mathrm{Spec}\mathbf{C}[G/G']$ is equal to one then the information carried by V_1 is equivalent to the multivariable Alexander polynomial up to the exponent of each factor (this is the case when ϕ is the abelianization of a link group). On the other hand if the codimension is bigger than one then for the pair (G, ϕ) the Alexander polynomial is not defined (or is trivial depending on convention) but $V_1(G)$ can be very interesting.

Now, as the first example, let us calculate the characteristic varieties of a free group. If $G = F_r$ is a free group on r-generators then $G'/G'' = H_1(\widetilde{\bigvee_r S^1}, \mathbf{Z})$, where $\widetilde{\bigvee_r S^1}$ is the universal abelian cover of the wedge of r circles. It fits into the exact sequence:

$$0 \to H_1(\widetilde{\bigvee_r S^1}, \mathbf{C}) \to \mathbf{C}[\mathbf{Z}^r]^r \to I \to 0$$

with I denoting the augmentation ideal of the group ring of $\mathbf{Z}^r$: $I = \mathrm{Ker}\mathbf{C}[\mathbf{Z}^r] \to \mathbf{C}$ where the homomorphism sends each generator to $1 \in \mathbf{C}$. Indeed, as an universal abelian cover of $\widetilde{\bigvee_r S^1}$ one can take the subset of $\mathbf{R}^r$ of points having at least $r-1$ integer coordinates with the action of $\mathbf{Z}^r$ given by translations; unit vectors of the standard basis provide identification of 1-chains with $\mathbf{C}[\mathbf{Z}^r]^r$ while the module of 0-chains is identified with $\mathbf{C}[\mathbf{Z}^r]$. The boundary map sends each generator $e_i, i = 1, ..., r$ of $\mathbf{C}[\mathbf{Z}^r]^r$ to $(t_i - 1) \in \mathbf{C}[\mathbf{Z}^r]$. This is the map which also appears in the Koszul complex (cf. [74]) in which we put $R = \mathbf{C}[\mathbf{Z}]$:

$$\wedge^i R^r \longrightarrow \wedge^{i-1} R^r \longrightarrow .... \to R^r \to R \tag{32}$$

where $\partial_i(e_{j_1} \wedge ... \wedge e_{j_i}) = \sum (-1)^k (t_k - 1) e_{j_1} \wedge ... \wedge \hat{e}_{j_k} \wedge ... \wedge e_{j_i}$. Since $(t_1 - 1, ..., t_r - 1)$ is a system of parameters the complex (32) is exact. Therefore

$$H_1(\widetilde{\bigvee_r S^1}, \mathbf{C}) = \mathrm{Coker}\Lambda^{\binom{r}{3}} \mathbf{C}[\mathbf{Z}^r]^r \to \Lambda^{\binom{r}{2}} \mathbf{C}[\mathbf{Z}^r]^r \tag{33}$$

in the Koszul resolution corresponding to the $(t_1 - 1), ..., (t_r - 1)$. This implies that $V_i(F_r) = \mathbf{C}^{*r}$ for $0 < i \leq r - 1$ and $V_i(F_r) = (1, ..., 1)$ for $r \leq i \leq \binom{r}{2}$ i.e. $\mathbf{C}^{*r}$ is component having depth $r - 1$, and $1 \in \mathbf{C}^{*r}$ has depth $\binom{r}{2}$.

For arbitrary group G, as was pointed out in earlier sections, the Fox calculus provides presentation for the extension of the homology of universal abelian cover by the augmentation ideal of the group ring of the covering group. This is sufficient to determine the characteristic varieties outside of the identity character.

4.1.2 Unbranched covering

The homology of a cyclic unbranched covering X_n of a CW-complex X with $\pi_1(X) = G$ corresponding to the homomorphism $G \stackrel{\phi}{\to} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ can be found using Milnor's exact sequence (cf. [62]) i.e. the homology sequence corresponding to the exact sequence of chain complexes

$$0 \to C_*(\tilde{X}) \xrightarrow{t^n-1} C_*(\tilde{X}) \to C_*(X_n) \to 0 \tag{34}$$

The induced homology sequence:

$$\to H_1(\tilde{X}, \mathbf{C}) \xrightarrow{t^n-1} H_1(\tilde{X}, \mathbf{C}) \to H_1(X_n, \mathbf{C}) \to \mathbf{C} \to 0 \tag{35}$$

shows that $\mathrm{rk} H_1(X_n, \mathbf{C}) = \mathrm{rkCoker}(t^n - 1)|_{H_1(\tilde{X},\mathbf{C})} + 1$. In abelian case, to find the homology of the covering $X_{n_1,\ldots,n_r}$ corresponding to the homomorphism $G \stackrel{\phi}{\to} \mathbf{Z}^r \to \oplus_{i=1}^{i=r} \mathbf{Z}/n_i\mathbf{Z}$ the Milnor's sequence (35) should be replaced by the five term exact sequence corresponding to the spectral sequence of the covering group $H = \mathrm{Ker}\mathbf{Z}^r \to \oplus_i \mathbf{Z}/n_i\mathbf{Z}$ acting on the covering space $\tilde{X}$ corresponding to the homomorphism ϕ:

$$H_p(\mathbf{Z}^r, H_q(\tilde{X}, \mathbf{C})) \Rightarrow H_{p+q}(X_{n_1,\ldots,n_r}, \mathbf{C}) \tag{36}$$

This exact sequence is

$$H_2(X_{n_1,\ldots,n_r}, \mathbf{C}) \to H_2(H, \mathbf{C}) \to H_1(\tilde{X})_H \to H_1(X_{n_1,\ldots,n_r}, \mathbf{C}) \to H_1(H, \mathbf{C}) \to 0 \tag{37}$$

where for a H-module M, $M_H = M/I(H)M$ is the module of covariants ($I(H)$, as above, is the augmentation ideal of the group ring of H). This yields the following formula for the first Betti number of abelian covers:

Proposition 4.3 *Let $X_{n_1,\ldots,n_r}$ be the finite unbranched abelian cover of a CW-complex X as above which is the quotient of the infinite abelian cover corresponding to $\phi : G \to \mathbf{Z}^r$. Let $V_i(G, \phi)$ be the characteristic varieties of (G, ϕ). For $P \in \mathbf{C}^{*r}$ let $f(P, G, \phi) = \{max \quad i | P \in V_i(G, \phi)\}$. Then*

$$\mathrm{rk} H_1(X_{n_1,\ldots,n_r}) = r + \Sigma_{\omega_i^{n_i}=1, (\omega_{n_1},\ldots,\omega_{n_r}) \neq (1,\ldots,1)} f((\omega_{n_1}, .., \omega_{n_r}), (G, \phi))$$

4.1.3 Homology of local systems

Homology of rank one local systems also can be described in terms of the characteristic varieties. Such a local system is a homomorphism $\chi : G \to \mathbf{C}^*$ i.e. a character of the fundamental group. There is a natural identification: $\mathrm{Spec}\mathbf{C}[G/G']$ and $\mathrm{Char}G$. Moreover, $\mathrm{Spec}\mathbf{C}[G/\mathrm{Ker}\phi]$ can be identified with the subgroup of $\mathrm{Char}G$ of characters which can be factored through ϕ. We shall denote as $\tilde{X}_{G/G'}$ is the infinite cover corresponding to the subgroup G'. The homology $H_1(X, \chi)$ of the local system χ where $\chi \in \mathrm{Char}\pi_1(X)$ is defined as the homology of the chain complex:

$$\to C_i(\tilde{X}_{G/G'}) \otimes_{\mathbf{C}[G/G']} \mathbf{C}_\chi \to C_i(\tilde{X}_{G/G'}) \otimes_{\mathbf{C}[G/G']} \mathbf{C}_\chi \tag{38}$$

where the chains $C_i(\tilde{X}_{G/G'})$ of the universal abelian covers are given the structure of $\mathbf{C}[G/G']$-module and $\mathbf{C}_\chi$ is $\mathbf{C}$ with the module structure given by the character χ.

One has the following:

Proposition 4.4 *(cf. [32],[52]) If $\chi \neq 1$ then*

$$H_1(X, \chi) = H_1(\tilde{X}_{G/G'}, \mathbf{C}) \otimes_{\mathbf{C}[H_1(X,\mathbf{Z})]} \mathbf{C}_\chi$$

In particular, $\chi \in \mathrm{Char}G, \chi \neq 1$ belongs to $V_k(G)$ if and only if $H_1(X, \chi) \geq k$.

4.2 Links of plane curves and multivariable Alexander polynomial

For a link in S^3 with r components the characteristic varieties are just affine subvarieties of the torus. An interesting problem is the following:

Problem 4.5 *Which sequence of subvarieties can occur as $V_i(G)$ where $G = \pi_1(S^3 - L)$ for some link in S^3.*

For the multivariable Alexander polynomial one has:

$$\Delta(t_1^{-1}, ..., t_r^{-1}) = \Delta(t_1, ..., t_r) \tag{39}$$

(up to a unit of the ring $\mathbf{Z}[\mathbf{Z}^r]$ i.e. a factor $\pm t_1^{a_1} ... t_r^{a_r}$ where $a_i \in \mathbf{Z}$)

The characteristic varieties of links of algebraic singularities are very special. Let us call a translated subgroup of $\mathbf{C}^r$ a coset of a subgroup $\mathbf{C}^s$. Such a "subgroup" is said translated by an element of a finite order if this coset has finite order in $\mathbf{C}^r/\mathbf{C}^s$. Using the fact that the link of algebraic singularities are iterated torus links one can prove the following:

Proposition 4.6 *(cf. [54]) The characteristic varieties of algebraic links are unions of translated subgroups.*

For example the link of singularity $x^r - y^r = 0$ has the Alexander polynomial $t_1 \cdot ... \cdot t_r = 1$. The Alexander polynomial of $(x^2 - y^3)(x^3 - y^2) = 0$ is $(t_1^2 t_2^3 - 1)(t_1^2 t_2^3 - 1) = 0$ (cf. [54]).

4.3 Links of isolated non normal crossings

Disjoint non intersecting spheres of dimension greater than one and having codimension 2 in an ambient sphere never can form a link of an algebraic singularity. There is nevertheless a *local* abelian analog of the local cyclic theory of the links of algebraic singularities. It appears when one looks at isolated non normal crossings (cf. [55], [21])

Definition 4.7 *(cf. [55]) Let $D_1, .., D_k$ be divisors of a complex manifold X and $P \in D_1 \cap ... \cap D_k$. These divisors have a normal crossing at P if there exist in a neighborhood of P in X and a system of complex analytic local coordinates $(z_1, ..., z_{\dim X})$ in it such that D_i is in this neighbourhood given by the equations $z_i = 0$. $D_1, ..., D_k$ have an isolated non normal crossing at P is there exist a ball B_ϵ in X centered at P having sufficiently small radius ϵ such that for any $Q \neq P$ in B_ϵ the divisors D_i containing Q form at Q a divisor with normal crossings.*

In particular each D_i has at most isolated singularity at P. A more general case, when the ambient space X is allowed to have a singularity at P is considered in [21]. The theory we shall describe is invariant under analytic changes of variables so we can assume that $X = \mathbf{C}^{n+1}$. The starting point is the following vanishing result:

Theorem 4.8 *(cf. [55]) Let $X = \bigcup_{i=1}^r D_i \subset \mathbf{C}^{n+1}$ be a union of r irreducible germs of hypersurfaces with normal crossings outside of the origin. If $n \geq 2$, then*

$$\pi_1((\partial B_\epsilon - \partial B_\epsilon \cap X) = \mathbf{Z}^r \quad \text{and} \quad \pi_k((\partial B_\epsilon - \partial B_\epsilon \cap X) = 0 \quad \text{for} \quad 2 \leq k < n.$$

In the case when $r = 1$ this result follows from Milnor's fibration theorem and connectivity of Milnor fibers (cf. [60]). In fact, the universal cyclic cover of the complement to a link of isolated hypersurface singularity D is homotopy equivalent to the Milnor fiber M_D. In particular $\pi_n(\partial B_\epsilon - D \cap \partial B_\epsilon) = H_n(M_D, \mathbf{Z})$. For general INNC the main invariant is $\pi_n(\partial B_\epsilon - \cup D_i \cap \partial B_\epsilon)$. This, as usual, is the module over $\mathbf{Z}[\pi_1(\partial B_\epsilon - \cup D_i \cap \partial B_\epsilon] = \mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$. We shall call it the *homotopy module of INNC*. In the case $r = 1$ this module structure is equivalent to the module structure on an abelian group with an automorphism where the abelian group is the middle homology of the Milnor fiber and the automorphism is the monodromy operator. Notice that in the case of normal crossing (i.e. when the singularity is absent), the universal abelian cover of $\partial B_\epsilon - \cup D_i \cap \partial B_\epsilon$ is contractible and all homotopy groups are trivial.

Definition 4.9 *(cf. [55]) k-th characteristic variety $V_k(X)$ of an isolated non-normal crossing $X = \cup_{i=1}^{i=r} D_i$ is the subset in* $\mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \leq i \leq r} D_i))]$ *formed by the zeros of the k-th Fitting ideal of $\pi_n(\partial B_\epsilon - \cup D_i \cap \partial B_\epsilon)$*

Let us consider an example of a non normal crossing. The simplest non trivial case is when the components are given by linear equations i.e are given in $\mathbf{C}^{n+1}$ by the equation $l_1 \cdot ... \cdot l_r = 0$, where l_i are *generic* linear forms (i.e. a cone over a generic arrangement of hyperplanes in $\mathbf{P}^n$). Since the complement to a generic arrangement of r hyperplanes in $\mathbf{P}^n$ has a homotopy type of n-skeleton of the product of $r-1$-copies of the circle S^1 (in minimal cell decomposition in which one has $\binom{r-1}{i}$ cells of dimension i) one can calculate the module structure on the π_n of such skeleton as follows (cf. [55]). The universal cover of this skeleton is obtained by removing the $\mathbf{Z}^{r-1}$ orbits of all open faces of a dimension greater than n in the unit cube in $\mathbf{R}^{r-1}$. Hence $\pi_n(\partial B_\epsilon - D) = H_n(Sk_n(\widetilde{(S^1)^{r-1}}), \mathbf{Z})$ ($Sk_n(\widetilde{(S^1)^{r-1}})$ is the universal cover of the n-skeleton). The chain complex of the universal cover of $(S^1)^{r-1}$ can be identified with the Koszul complex of the group ring of $\mathbf{Z}^{r-1} = \mathbf{Z}^r/(1,...,1)$ (so that the generators of $\mathbf{Z}^r$ correspond to the standard generators of $H_1(\partial B_\epsilon - D)$). The system of parameters of this Koszul complex is $(t_1 - 1, .., t_r - 1)$. Hence $H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z}) = Ker \Lambda^n R \to \Lambda^{n-1} R$ where $R = \mathbf{Z}[t_1, .., t_r, t_1^{-1}, ..., t_r^{-1}]/(t_1 \cdot ... \cdot t_r - 1)$. As a result, one has the following presentation:

$$\Lambda^{n+1}(\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1 ..., t_r - 1)^r) \to \Lambda^n(\mathbf{Z}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]/(t_1 ..., t_r - 1)^r) \to \tag{40}$$
$$\pi_n(\mathbf{C}^{n+1} - \bigcup D_i) \to 0$$

In particular, the support of the π_n is the subgroup $t_1 \cdot ... \cdot t_r = 1$.

The relation between the characteristic varieties, the unbranched covering spaces and the local systems described in the case of links in S^3 extends to this situation as well. We have the following:

Proposition 4.10 *(cf. [55]) (a) For each $P \in \mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$ let*

$$f(P, X) = \{\max k \ | P \in V_k(X)\}$$

Let $U_{m_1,...,m_r}$ be unbranched cover of $\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)$ corresponding to the homomorphism $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)) = \mathbf{Z}^r \to \oplus_{1 \le i \le r} \mathbf{Z}/m_i\mathbf{Z}$. Then

$$\mathrm{rk} H_p(U_{m_1,...,m_r}, \mathbf{C}) = \Lambda^p(\mathbf{Z}^r) \quad \text{for} \quad p \le n-1,$$
$$\mathrm{rk} H_n(U_{m_1,...,m_r}, \mathbf{C}) = \sum_{(...,\omega_j,...),\omega_j^{m_j}=1} f((..., \omega_j, ...), \bigcup_{1 \le i \le r} D_i)$$

(b) If $1 \ne \chi \in \mathrm{Char}\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i) = \mathrm{Spec}\mathbf{C}[\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)]$ is a character of the fundamental group then

$$H_i(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i), \chi) = 0 \quad 1 \le i \le n-1$$

$$H_n(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i), \chi) = \pi_n(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)) \otimes_{\mathbf{Z}} \mathbf{C} \otimes_{\mathbf{C}[H_1(X,\mathbf{Z})]} \mathbf{C}_\chi$$

Milnor theory [60] is applicable to INNC as to any hypersurface and one can relate relate Milnor's invariants to the characteristic varieties discussed here. We have the following:

Proposition 4.11 *(cf. [55]) The homology of the Milnor fiber F_D of the INNC singularity D is given by:*

$$H_p(F_D, \mathbf{Z}) = \Lambda^p(\mathbf{Z}^{r-1}) \quad \text{for } 1 \le p < n$$

The action of the monodromy of this homology is trivial. The multiplicity of $\omega \ne 1$ as a root of the characteristic polynomial $\Delta_n(D, t)$ of the Milnor's monodromy on $H_n(F_D, \mathbf{C})$ is equal to:

$$m_\omega = f((..., \omega, ...), D) = \max\{i | (\omega, ..., \omega) \in V_i(D) \subset \mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap D)]\}$$

In the case of INNC the unbranched covering admits a natural compactification which provides model for the abelian branched covering of the sphere S^{2n+1} with the link of INNC as the branching locus. The branching cover itself is a link of an isolated complete intersection singularity.

If the local equations of the locally irreducible components $D_1, ..., D_r$ are $f_1, ..., f_r$, then we can use as a model of abelian branched cover the link of singularity:

$$z_1^{m_1} = f_1(x_1, ..., x_{n+1}), ..., z_1^{m_1} = f_1(x_1, ..., x_{n+1}) \tag{41}$$

As a link of ICIS the link of singularity (41) is a $(n-1)$-connected manifold having the dimension equal to $2n+1$. We shall express the homology of this link in terms of the characteristic varieties of the homotopy modules associated to INNCs formed by various components of $D = \bigcup D_i$.

Proposition 4.12 *Let $V_{m_1,...,m_r}$ be the link of singularity (41) which is the branched cover of ∂B_ϵ branched over $\partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)$ with the Galois group $G = \oplus_{1 \le i \le r} \mathbf{Z}/m_i\mathbf{Z}$. For each $\chi \in \mathrm{Char} G$ let $I_\chi = \{i | 1 \le i \le r, \chi(\mathbf{Z}_{m_i}) \neq 1\}$ where $\mathbf{Z}_{m_i}$ is the i-th summand of G. Any χ can also be considered as a character of $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{i \in I_\chi} D_i))$ in which case it will be called reduced and denoted χ_{red}. Let V_χ be the branched cover of ∂B_ϵ branched over $\partial B_\epsilon \cap (\bigcup_{i \in I_\chi} D_i)$ and having $\mathrm{Im}\chi = G/\mathrm{Ker}\chi$ as its Galois group. Then*

$$\pi_p(V_{m_1,...,m_r}) = 0 \text{ for } \ 1 \le p \le n-1,$$

$$\mathrm{rk} H_n(V_{m_1,...,m_r}, \mathbf{C}) = \sum_{\chi \in \mathrm{Char}} f(\chi_{red}, \bigcup_{i \in I_\chi} D_i).$$

This proposition shows that there is a close relation between the homology of the tower of abelian covers and the characteristic varieties (at least in local case). We shall use it in the following section for the calculation of the homology of infinite abalian covers in terms algebro-geometric data such as resolution of singularities and the ideals of quasiadjunction.

5 Hodge decomposition of local Alexander invariants.

5.1 Zeros of Fitting ideals and Hodge numbers in cyclic case.

Our goal in this section is to describe the structure of the characteristic varieties in the local case of the Alexander invariants of the germs of plane curves and for germs of INNC in terms of resolutions of singularities. This will give an algebro-geometric description of these invariants. The global counterparts of the local invairants from this section will be considered in the section 6. All structures introduced in this section are essentail for describing the global case.

First let us consider the relationship between the Hodge structure of the cohomology of Milnor fiber and the Alexander invairants of the link of an isolated singularity. In the cyclic case, the calculation of the Alexander polynomial does not require Mixed Hodge theory and is a special case of A'Campo's formula for the zeta function of the monodromy of a resolution ([5]). Indeed, if D has only one component with isolated singularity, the order of $\pi_n(S^{2n+1} - D \cap S^{2n+1}) \otimes \mathbf{Q}$ is equivalent to the zeta function of the monodromy. Hence, if E_i are the components of the exceptional set of a resolution π, $\pi^*(D)$ has the multiplicity m_i along a component E_i of the exceptional locus and the euler characteristic of the set of points in E_i non-singular in the union of $\bigcup E_i$ and the proper preimage of D is $\chi(E_i^\circ)$ then (cf. [5]):

$$\Delta_n((S^{2n+1} - D \cap S^{2n+1})^{(-1)^n}(1-t) = \Pi(1 - t^{m(E_i)})^{-\chi(E_i^\circ)} \tag{42}$$

However even in cyclic case calculation of the zeros of higher Fitting ideals requires the mixed Hodge theory. We refer to [15] or [18] for the formalism of this structure.

In the rest of this section 5.1 we shall focus mainly on the case of curves i.e. $\dim D = 1$. The cohomology group H^1 of the Milnor fiber (*) of a plane curve singularity supports a mixed Hodge structure with weights 0,1 and 2, with the identification

$$N : W_2/W_1 \to W_0 \tag{43}$$

given by the logarithm of an appropriate power of the monodromy(cf. [71]). Recall that this means that one has canonically (**) defined (weight) filtration $H^1 = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0$ such that each quotient $W_n/W_{n-1} = \oplus_{p+q=n} H^{p,q}$. In fact there is a strong relation between these groups $H^{p,q}$: they all come from increasing Hodge filtration. Moreover, if T is the monodromy operator on $H^1(M, \mathbf{C})$ and $T = T_s T_u$ is the factorization into semisimple and unipotent part and if $N = log(T_u) = \sum_{i \geq 1} (-1)^{i-1} \frac{(T_u - I)^i}{i}$ then N induces the isomorphism in (43).

All Hodge groups are invariant under the action of the semisimple part T_s of the monodromy. Let $h_\zeta^{p,q}$ (cf. [71]) be the dimension of the eigenspace of this semisimple part acting on the space $H^{p,q}$. The numbers $h_\zeta^{p,q}$ determine the Jordan form of the monodromy as follows. The size of the Jordan blocks of the monodromy does not exceed 2 and the number of blocks corresponding to an eigenvalue ζ of size 1×1 (resp. 2×2) is equal to $h_\zeta^{1,0} + h_\zeta^{0,1}$ (resp. $h_\zeta^{0,0}$). As a consequence, the generators of the Fitting ideals have the form:

$$\Delta_i = \prod_{(\zeta)} (t - \zeta)^{a_{\zeta,i}}$$

where

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* we shall work with the cohomology as is more common in Hodge theory though one has the dual structures on homology. One of the differences is the presence of the negative weights in MHS in homology. See [64] where the author works with MHS on homotopy groups (also discussion below of the homotopy groups) having negative weights and where natural dual theory with positive weights is not available

** i.e. just an algebraic class of the germ at the zero of fibration $\mathbf{C}^2 \to \mathbf{C}$ given by $(x, y) \to f(x, y)$

$$a_{\zeta,i} = \begin{cases} h_\zeta^{1,0} + h_\zeta^{0,1} + 2h_\zeta^{0,0} - 2(i-1) & \text{if } 1 \le i \le h_\zeta^{0,0} \\ h_\zeta^{1,0} + h_\zeta^{0,1} - (i-1-h_\zeta^{0,0}) & \text{if } h_\zeta^{0,0} < i \le h_\zeta^{0,0} + h_\zeta^{1,0} + h_\zeta^{0,1} \\ 0 & \text{if } i > h_\zeta^{0,0} + h_\zeta^{1,0} + h_\zeta^{0,1} \end{cases}$$

In particular, Δ_i can be calculated algebraically in terms of a resolution of the singularity since all Hodge numbers $h_\zeta^{p,q}$ can be found in terms of a resolution (cf. [71]). A calculation of the Hodge numbers $h_\zeta^{p,q}$ is equivalent to the identifying the following subsets in the set of zeros of the characteristic polynomial of the monodromy operator:

$$\mathcal{H}_{p,q,k} = \{\zeta | h_\zeta^{p,q} \ge k\} \tag{44}$$

Arnold-Steenbrink spectrum [71] is also equivalent to this data. Our goal for the rest of this section 5 will be to describe the partition of (the unitary part of) the zero sets of Fitting ideal i.e. the characteristic varieties of plane curve singularities (and also INNC) into sets (44). We call this partition *the Hodge decomposition of characteristic varieties.*

In the abelian case the multivariable Alexander polynomial and hence $V_1(G)$ can be found from a resolution as well (cf. [22]).

Theorem 5.1 *Let $f_i = 0, i = 1, ...r$ be the equations of branches of a reducible curve C. Let $\pi : \tilde{\mathbf{C}}^2 \to \mathbf{C}^2$ be a resolution of singularities and $E_j, j = 1, ..., N$ be the exceptional components. Let $m_{i,j} = \mathrm{ord}_{E_j}\pi^*(f_i)$ and let E_j° be a Zariski open subset in E_j consisting of points which are non singular on the reduced total preimage of C. Then the Alexander polynomial of the link of singularity of C is:*

$$\Pi_j(1 - t_1^{m_{1,j}} \cdot ... \cdot t_r^{m_{r,j}})^{-\chi(E_i^\circ)}$$

For example for the Hopf link with r-components we obtain $(1 - t_1 \cdot ... \cdot t_r)^{r-2}$.

We shall use the description of the cohomology of branched coverings from the last lecture to calculate higher $V_i(G)$. The information about the characteristic varieties is closely related to the information about the cohomology of branched abelian covers (cf. Prop. 4.12). Those are the links of singularities of complete intersection and hence have the mixed hodge structure. We shall calculated the eigenspaces of the deck transformations acting on the Hodge spaces of the cohomology of branched coverings in terms of the algebraic data of the singularity i.e. some associated ideals generalizing the ideals of quasiadjunction defined earlier in this paper. This will give us the calculation of the sets (44) and hence the characteristic varieties. The MHS on the link can be described in terms of resolution of the singularity so we shall need a description of the forms of resolution of singularities which we shall describe in the next section. We shall start it by description of a global precursor: adjoint hypersurfaces.

5.2 Theory of adjoints

The classical theory of adjoints (which already was briefly mentioned in section 2.2.5 and which was used there to calculated one variable Alexander polynomial) gives a presentation of the geometric genus of a resolution of singularities of a hypersurface or a complete intersections in $\mathbf{P}^N$ in terms of the degree of hypersurface (or the multidegree in the case of a CI) and local data about the singularities. A starting point maybe an observation that the genus of a plane curve C of degree d (i.e. $h^{1,0} = \dim H^0(\Omega_C))$ is equal to $\frac{(d-1)(d-2)}{2}$ which also is equal to the dimension of plane curves of degree $d-3$. If C is not smooth but have δ nodes then the genus of desingularization $\frac{(d-1)(d-2)}{2} - \delta$ is the the dimension of space of plane curves of degree $d-3$ passing through the nodes. In the case when C has singularities more complicated than nodes one can associate with each singular point P the ideal in the local ring $\mathcal{O}_P$ (adjoint ideal) such that the genus of desingularization is the dimension of the space of curves of degree $d-3$ passing through the singularities of C and which local equations at P belong to the adjoint ideal in $\mathcal{O}_P$.

Explicitly, these ideals can be described as follows. Let $X \subset \mathbf{P}^{n+1}$ be a hypersurface and let $f_* : \tilde{X} \to X$ be a resolution of singularities. Let $\mathcal{A} = f_*(\Omega^n_{\tilde{X}})(-d+n+2)$ and $\mathcal{A}' = \pi^{-1}(\mathcal{A})$ where $\pi : \mathcal{O}_{\mathbf{P}^{n+1}} \to \mathcal{O}_X$ is the restriction map. Then $\mathcal{A}'$ is called the sheaf of adjoint ideals and for example $H^0(\mathcal{A}'(d-n-2)) = h^{n,0}(\tilde{X})$.

The sheaf $f_*(\Omega^n_{\tilde{X}})$ is a subsheaf of $i_*(\Omega^n_{X-\mathrm{Sing}X})$ where $i : X - \mathrm{Sing}X \to X$ is the embedding. One has the residue map which fit into the exact sequence:

$$0 \to \Omega^{n+1}_{\mathbf{P}^{n+1}} \to \Omega^{n+1}_{\mathbf{P}^{n+1}}(X) \xrightarrow{\mathrm{Res}} i_*(\Omega^n_{X-\mathrm{Sing}X}) \to 0 \tag{45}$$

The residue map sends a form $\omega = \frac{f(z_1,...,z_n)dz_1 \wedge ... \wedge dz_n}{F}$, defined in a chart with coordinates $z_1, ..., z_n$ and having the pole of order one along X (given by in this chart by the equation $F = 0$) to $(-1)^{j-1} \frac{f dz_1 \wedge ... \hat{dz_j} \wedge dz_n}{F_{z_j}}|_{X-\mathrm{Sing}X}$ (the restriction is independent of $j, 1 \le j \le n$). From this point of view the stalk of the sheaf $\mathcal{A}'$ at $P \in \mathbf{P}^{n+1}$ consists of $f \in \mathcal{O}_P$ such that $Res(\omega)$ extends to a holomorphic form on some resolution $\tilde{X}$.

If the case when a subvariety $X \subset \mathbf{P}^{n+r}$ is a complete intersection of hypersurfaces $F_1 = ... = F_r$ the stalk of the sheaf of adjoint ideals at a singular point can be described using the fact that a holomorphic n-form can be obtained as a multiple of the residue of $(n+r)$-form having poles of order one along each hypersurface $F_i = 0$ and that

$$Res \frac{dw_1 \wedge .... \wedge dw_{n+r}}{F_1 \cdot .... \cdot F_r} = \frac{... \wedge \hat{dw}_{i_1} ... \wedge \hat{dw}_{i_r} ...}{\frac{\partial(F_1,...,F_r)}{\partial(w_{i_1},...,w_{i_r})}}|_X \tag{46}$$

where $\frac{\partial(F_1,...,F_r)}{\partial(w_{i_1},...,w_{i_r})}$ is the Jacobian of partial derivatives of the system $(F_1, ..., F_r)$ relative to variables $(w_{i_1}, ..., w_{i_r})$ (it is easy to check that the restriction up to sign is independent of collection of variables $(w_{i_1}, ..., w_{i_r})$).

This construction describes the differential forms on a resolution in terms of the linear systems of hypersurfaces in $\mathbf{P}^{n+r}$ given by ideal sheaves on the latter. The cohomology and the Hodge structure on the link of a complete intersection singularity can be described in terms of differential forms (cf [71]) and this description can be used to calculate the cohomology of the link in terms of certain ideals in the local ring of singular point. We shall review the construction of MHS on various cohomology associated with singular points of complex spaces and in the next section we shall discuss the connections with the ideals of quasiadjunction.

The cohomology of the link L of an isolated singularity x of a complex space X ($\dim X = n$) can be given a Mixed Hodge structure, for example using canonical identification $H^k(L) = H^*_{\{x\}}(X)$ with the local cohomology (*) . The mixed Hodge structure on the latter was described in [70]. The Hodge numbers: $h^{kpq}(L) = \dim Gr^p_F Gr^W_{p+q} H^k(L)$ have the following symmetry properties:

$$h^{kpq} = h^{2n-k-1,n-p,n-q} \tag{47}$$

If E is the exceptional divisor for a resolution, then for $k < n$ one has

$$h^{kpq}(L) = h^{kpq}(E) \quad if\ p+q < k$$

$$h^{kpq}(L) = h^{kpq}(E) - h^{2n-k,n-p,n-q}(E) \quad if\ p+q = k \tag{48}$$

$$h^{kpq}(L) = 0 \quad if\ p+q > k$$

The local cohomology $H^*_E(\tilde{X})$ ([70]) where $\tilde{X}$ is a resolution of X support the canonical mixed Hodge structure. Let us consider it in more detail in the case $dim_{\mathbf{C}} \tilde{X} = 2$ which we shall need to describe the characteristic varieties in the case of germs of plane curves. We have

$$H^*_E(\tilde{X}) = Hom(H^{4-*}(E), \mathbf{Q}(-2)) \tag{49}$$

where $\mathbf{Q}(-2)$ is Tate Hodge of type $(2,2)$. Since the Hodge and weight filtrations on $H^1(E)$ have the form:

$$H^1(E) = W_1 \supset W_0 \supset 0, \quad H^1(E) = F^0 \supset F^1 \supset F^2 = 0$$

on $H^3_E(\tilde{X})$ we have:

$$H^3_E(\tilde{X}) = W_4 \supset W_3 \supset W_2 = 0, H^3_E(\tilde{X}) = F^1 \supset F^2 \supset F^3 = 0$$

Moreover

$$F^1H^1(L) = F^1H^1(E) = F^2H^3_E(\tilde{X}) \tag{50}$$

In order to describe this mixed Hodge structure one can use the following complex:

$$0 \to A^2_E(\tilde{X}) \to A^3_E(\tilde{X}) \to 0 \tag{51}$$

* Recall that if Y is a subset in a topological space X and $\mathcal{F}$ is a sheaf on X then $H^i_Y(X)$ is the right derived functor of the functor $\Gamma_Y(X, \mathcal{F})$ of sections of $\mathcal{F}$ supported on Y. It fits into long exact sequence: $... \to H^i_Y(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(X - Y, \mathcal{F}) \to H^{i+1}_Y(X, \mathcal{F}) \to ...$ (cf. [30])

where

$$A_E^2(\tilde{X}) = \Omega^1_{\tilde{X}}(log\ E)/\Omega^1_{\tilde{X}}, A_E^3(\tilde{X}) = \Omega^2_{\tilde{X}}(log\ E)/\Omega^2_{\tilde{X}}$$

with filtrations given by

$$F^2 A_E^p(\tilde{X}) = 0\ for\ p < 3, F^2 A_E^p(\tilde{X}) = A_E^p(\tilde{X})\ for\ p \geq 3$$

$$W_3 A_E^3(\tilde{X}) = W_1 \Omega^2_{\tilde{X}}(log\ E)/\Omega^2_{\tilde{X}}$$

Since $H^3(E) = 0$, the relations (48) and (49) yield that the complex (51) completely determines h^{1pq} (and hence all Hodge numbers h^{kpq} by (47)).

Putting all this together we obtain the following isomorphism:

$$H^0(\Omega^2_{\tilde{X}}(log\ E))/H^0(\Omega^2_{\tilde{X}}) = F^1 H^1(L) \tag{52}$$

Our next goal will be to apply this to the link of complete intersection singularity which is the abelian cover branched over the a link plane curve singularity i.e. to the link (41) which in case of curves with r components is the following link in $\mathbf{C}^{r+2}$:

$$z_1^{m_1} = f_1(x, y), \quad ... \quad z_1^{m_1} = f_r(x, y) \tag{53}$$

We want to calculate the eigenspaces corresponding to the characters of the Galois group acting on the Hodge spaces $H^{p,q,k}$ of singularity (53) by interpreting the left hand side of (52) in terms of ideals in the local ring of the singularity $f_1 \cdots f_r = 0$ in $\mathbf{C}^2$.

5.3 Ideal of quasiadjunction and log-quasiadjunction

We shall start with the following multivariable generalization of the ideals of quasiadjunction introduced in section 2.2.5.

Definition 5.2 *(cf. [50] [52]) An ideal of quasiadjunction of type $(j_1, .., j_r | m_1, .., m_r)$ is the ideal in the local ring of the singularity of C (i.e. $O \in \mathbf{C}^2$) consisting of germs ϕ such that the 2-form:*

$$\omega_\phi = \frac{\phi(x, y) z_1^{j_1} \cdots z_r^{j_r} dx \wedge dy}{z_1^{m_1 - 1} \cdots z_r^{m_r - 1}},$$

extends to a holomorphic form on a resolution of the singularity of the abelian cover of a ball B of type $(m_1, ..., m_r)$, i.e. a resolution of (53) (we suppress dependence of ω_ϕ on $j_1, ..j_r, m_1, ...m_r$). In other words, $\phi z_1^{j_1} \cdots z_r^{j_r}$ belongs to the adjoint ideal of the singularity (53). In particular the condition on ϕ is independent of resolution.

Note that ω_ϕ in definition (5.2) is the residue of the form $\frac{\phi z_1^{j_1} \cdots z_r^{j_r} dx \wedge dy}{(z_1^{m_1} - f_1(x,y)) \cdots (z_r^{m_r} - f_r(x,y))}$ (cf. (46)) We always shall assume that $0 \leq j_1 < m_1, .., 0 \leq j_r < m_r$. Also, notice that forms ω_ϕ in definition (5.2) are exactly the forms on the abelian cover (53) which are the eigenforms corresponding to the character of the Galois group taking value

$exp(\frac{2\pi i(j_i-m_i+1)}{m_i})$ on the automorphism of the surface (53) induced by multiplication of the i-th coordinate by $exp(\frac{2\pi i}{m_i})$.

An ideal of log-quasiadjunction (resp. *an ideal of weight one log-quasiadjunction*) of type $(j_1,..,j_r|m_1,..,m_r)$ is the ideal in the same local ring consisting of germs ϕ such that ω_ϕ extends to a log-form (resp. weight one log-form) on a resolution of the singularity of the same abelian cover. Recall (cf. [15]) that a holomorphic 2-form is weight one log-form if it is a combination of forms having poles of order at most one on each component of the exceptional divisor and not having poles of order one on a pair of intersecting components. These ideals are also independent of a resolution (cf. [54]).

One can show (cf. [52]) that an ideal of quasiadjunction $\mathcal{A}(j_1,..,j_r|m_1,...,m_r)$ is determined by the vector (i.e. depends only on the collection of ratios):

$$(\frac{j_1+1}{m_1},....,\frac{j_r+1}{m_r}). \tag{54}$$

This is also the case for the ideals of log-quasiadjunction and weight one log-quasiadjunction. Indeed, these ideals can be described in terms of resolutions as follows. For a given embedded resolution $\pi : V \to \mathbf{C}^2$ of the germ $f_1 \cdots f_r = 0$ with the exceptional curves $E_1,..,E_k,...,E_s$ let $a_{k,i}$ (resp. c_k, resp. $e_k(\phi)$) be the multiplicity of the pull back on V of f_i $(i=1,..,r)$ (resp. $dx \wedge dy$, resp. ϕ) along E_k. Then ϕ belongs to the ideal of quasiadjunction of type $(j_1,..,j_r|m_1,..,m_r)$ if and only if for any k

$$a_{k,1}\frac{j_1+1}{m_1}+....+a_{k,r}\frac{j_r+1}{m_r} > a_{k,1}+...+a_{k,r}-e_k(\phi)-c_k-1 \tag{55}$$

(cf. [52]). Similar calculation shows that a germ ϕ belongs to the ideal of log-quasiadjunction corresponding to $(j_1,..,j_r|m_1,..,m_r)$ if and only if the inequality

$$a_{k,1}\frac{j_1+1}{m_1}+....+a_{k,r}\frac{j_r+1}{m_r} \geq a_{k,1}+...+a_{k,r}-e_k(\phi)-c_k-1 \tag{56}$$

is satisfied for any k. In addition, a germ ϕ belongs to the ideal of weight one log-quasiadjunction if and only if this germ is a linear combination of germs ϕ satisfying inequality (56) for any collection of k's such that corresponding components do not intersect and satisfying the inequality (55) for k outside of this collection. We shall denote the ideal of quasiadjunction (resp. log-quasiadjunction, resp. weight one log-quasiadjunction) corresponding to $(j_1,..,j_r|m_1,..,m_r)$ as $\mathcal{A}(j_1,..,j_r|m_1,..,m_r)$ (resp. $\mathcal{A}''(j_1,..,j_r|m_1,..,m_r)$, resp. $\mathcal{A}'(j_1,..,j_r|m_1,..,m_r)$). Note the inclusions:

$$\mathcal{A}(j_1,..,j_r|m_1,..,m_r) \subseteq \mathcal{A}'(j_1,..,j_r|m_1,..,m_r) \subseteq \mathcal{A}''(j_1,..,j_r|m_1,..,m_r)$$

Both (55) and (56) follow from the following calculation (cf. [52] section 2 for complete details). One can use the normalization of the fiber product $\widetilde{V}_{m_1,...,m_r} = V \times_{\mathbf{C}^2} V_{m_1,..,m_r}$ as a resolution of singularity (53) in the category of manifolds with quotient singularities (cf. [57]). We have:

$$\begin{array}{ccc} \tilde{V}_{m_1,..,m_r} & \stackrel{\tilde{p}}{\to} & V \\ \tilde{\pi} \downarrow & & \pi \downarrow \\ V_{m_1,..,m_r} & \stackrel{p}{\to} & \mathbf{C}^2 \end{array} \tag{57}$$

The preimage of the exceptional divisor of $V \to \mathbf{C}^2$ in $\widetilde{V}_{m_1,\ldots,m_r}$ forms a divisor with normal crossings (cf. [71]), though the preimage of each component is reducible in general. In this case the irreducible components above each exceptional curve do not intersect. If the Galois group G of $\tilde{p}$ is abelian (as we always assume here) and, in particular, is the quotient of $H_1(B - C \cap B, \mathbf{Z})$, then the Galois group of $\tilde{p}^{-1}(E_i) \to E_i$ is $G/(\gamma_i)$ where for an exceptional curve E_k, γ_k is the image in the Galois group of the homology class of the boundary of a small disk transversal to E_k in V. The components of $\tilde{p}^{-1}(E_i)$ correspond to the elements of $G/(\gamma_i, ...\gamma_l..)$ where l runs through indices of all exceptional curves intersecting E_i, while $\tilde{p}_i$ restricted on each component has $(\gamma_i, ...\gamma_l..)/(\gamma_i)$ as the Galois group. The points $\tilde{p}^{-1}(E_i \cap E_j)$ correspond to the elements of $G/(\gamma_i, \gamma_j)$ and the points of $\tilde{p}^{-1}(E_i \cap E_j)$ belonging to a fixed component correspond to cosets in $(\gamma_i, ...\gamma_l..)/(\gamma_i, \gamma_j)$. The order of the vanishing of ω_ϕ on $\widetilde{V}_{m_1,\ldots,m_r}$ along E_k is equal to:

$$\Sigma_{i=1}^{i=r}(j_i - m_i + 1)\frac{m_1 \cdots \hat{m}_i \cdots m_r \cdot a_{k,i}}{g_{k,1} \cdots g_{k,r} s_k} + \frac{m_1 \cdots m_r \cdot ord_{E_k}(\pi^*(\phi))}{g_{k,1} \cdots g_{k,r} \cdot s_k} + \tag{58}$$
$$+\frac{c_k \cdot m_1 \cdots m_r}{g_{k,1} \cdots g_{k,r} \cdot s_k} + \frac{m_1 \cdots m_r}{g_{k,1} \cdots g_{k,r} \cdot s_k} - 1$$

where $g_{k,i} = g.c.d.(m_i, a_{k,i})$ and $s_k = g.c.d.(..., \frac{m_i}{g_{k,i}}, ...)$.

A consequence of (58) is that ω_ϕ has an order of pole equal to one (resp. zero) along the component E_k of the above resolution if and only if for such ϕ one has equality in (56) (resp. (55) is satisfied).

Proposition 5.3 *(cf. [54]) 1. Let $\mathcal{A}''$ be an ideal of log-quasiadjunction. There is a unique polytope $\mathcal{P}(\mathcal{A}'')$ such that a vector $(\frac{j_1+1}{m_1}, ..., \frac{j_r+1}{m_r}) \in \mathcal{P}(\mathcal{A}'')$ if and only if the ideal $\mathcal{A}''(j_1, .., j_r|m_1, .., m_r)$ contains $\mathcal{A}''$.*

2. The set of vectors (54) for which $\mathcal{A}(j_1, .., j_r|m_1, .., m_r) \neq \mathcal{A}''(j_1, .., j_r|m_1, .., m_r)$ is a dense subset in the boundary of the polytope having as its closure a union of faces of such a polytope. The closure of the set of vectors (54) for which $\mathcal{A}'(j_1, .., j_r|m_1, .., m_r) \neq \mathcal{A}''(j_1, .., j_r|m_1, .., m_r)$ is also a union of certain faces of such a polytope.

3. The ideal $\mathcal{A}(j_1, .., j_r|m_1, ..., m_r)$ (resp. $\mathcal{A}'(j_1, .., j_r|m_1, ..., m_r)$ and $\mathcal{A}''(j_1, .., j_r|m_1, ..., m_r)$) is independent of the array $(j_1, .., j_r|m_1, .., m_r)$ as long as the vector (54) varies within the interior of the same face of quasiadjunction.

We shall call the above faces *the faces of quasiadjunction* (resp. *weight one faces of quasiadjunction*). $\mathcal{A}_\Sigma$ will denote $\mathcal{A}(j_1, ..., j_r|m_1, ..., m_r)$ with corresponding vector (54) belonging to the interior of a face of quasiadjunction Σ (similarly for $\mathcal{A}'_\Sigma$ and $\mathcal{A}''_\Sigma$).

(*) i.e. a subset in $\mathbf{R}^r$ given by a set of linear inequalities $L_s \geq k_s$. We say that an affine hyperplane in $\mathbf{R}^r$ supports a codimension one face of a polytope if the intersection of this hyperplane with the boundary of the polytope has dimension $r - 1$. A face of a polytope is the intersection of a supporting face of the polytope with the boundary. A codimension one face of a polytope in $\mathbf{R}^r$ is a polytope of dimension $r - 1$. By induction one obtains faces of arbitrary codimension for original polytope (for $r = 3$ those are called edges and vertices). The boundary of the polytope is the union of its faces.

In the case $r = 1$ and when $f(x, y)$ is weighted homogeneous one can use the description of the adjoint ideals given by M.Merle and B.Tessier (cf. [61] and section 2.2.5). The polytopes of quasiadjunction are in $\mathbf{R}$ and hence are just constants. They are the constants of quasiadjunction introduced in [42]. It was shown in [58] that they are the elements of Arnold-Steenbrink spectrum which belong to the interval (0,1).

The polytopes of quasiadjunctions are subsets of a unit cube $\mathcal{U}$ with the coordinates corresponding to the components of the link. We shall view it also as the fundamental domain for the Galois group $H^1(S^3 - L, \mathbf{Z})$ of the universal abelian cover $H^1(S^3 - L, \mathbf{R})$ of the group $H^1(S^3 - L, \mathbf{R}/\mathbf{Z})$ of the unitary characters of $H_1(S^3 - L, \mathbf{Z})$ (i.e. the maximal compact subgroup of $Char(H_1(S^3 - L, \mathbf{Z})) = H^1(S^3 - L, \mathbf{C}^*)$). $exp : \mathcal{U} \to Char(H_1(S^3 - L, \mathbf{Z}))$ will denote the restriction of $H^1(S^3 - L, \mathbf{R}) \to H^1(S^3 - L, \mathbf{R}/\mathbf{Z})$ on $\mathcal{U}$.

For any sub-link $\tilde{L}$ of L, i.e. a link formed by components of L, we have surjection $\pi_1(S^3 - L) \to \pi_1(S^3 - \tilde{L})$ induced by inclusion. Hence $CharH_1(S^3 - \tilde{L}, \mathbf{Z})$ is a subtorus of $CharH_1(S^3 - L, \mathbf{Z}))$ (in coordinates in the latter torus corresponding to the components of L it is given by equations of the form $t_\alpha = 1$ where subscripts correspond to components of L absent in $\tilde{L}$). Moreover, since the homology of the universal abelian cover $H_1(\widetilde{S^3 - L})$ surjects onto $H_1(\widetilde{S^3 - \tilde{L}})$, it follows that $V_i(S^3 - \tilde{L})$ belongs to a component of $V_i(S^3 - L)$ (cf. [52]). We shall call a character of $\pi_1(S^3 - L)$ (or a connected component of $V_i(S^3 - L)$) *essential* if it does not belong to a subtorus $CharH^1(S^3 - \tilde{L})$ for any sublink $\tilde{L}$ of L.

Let $L_{m_1,..,m_r}$ be the link of singularity (53) or equivalently the cover of S^3 branched over the link L and having a quotient $H_{m_1,..,m_r} = \mathbf{Z}/m_1\mathbf{Z} \oplus .... \oplus \mathbf{Z}/m_1\mathbf{Z}$ of $H_1(S^3 - L, \mathbf{Z})$ as its Galois group. We shall view $CharH_{m_1,..,m_r}$ as a subgroup of $CharH_1(S^3 - L, \mathbf{Z})$. The group $H_{m_1,..,m_r}$ acting on $H^1(L_{m_1,..,m_r})$ preserves both Hodge and weight filtrations.

Theorem 5.4 *(cf. [54]) An essential character $\chi \in Char(H_1(S^3 - L, \mathbf{Z}))$ is a character of the representation of $H_{m_1,..,m_r}$ acting on $F^1(H^1(L_{m_1,..,m_r}))$ if and only if it factors through the Galois group $H_{m_1,..,m_r}$ and belongs to the image of a face of quasiadjunction under the exponential map.*

The multiplicity of χ in this representation of the Galois group is equal to $dim\mathcal{A}''_\Sigma/\mathcal{A}_\Sigma$ where $\mathcal{A}''_\Sigma$ (resp. $\mathcal{A}_\Sigma$) is the ideal of log-quasiadjunction (resp. ideal of quasiadjunction) corresponding to a vector (54) belonging to the face of quasiadjunction Σ.

A character χ is a character of the representation of the Galois group of the cover on $W_0(H^1(L_{m_1,..,m_r}))$ if and only if it factors through the Galois group $H_{m_1,..,m_r}$ and it belongs to the image under the exponential map of a weight one face of quasiadjunction.

5.4 Multiplier ideals and log-canonical thresholds.

The ideals and polytopes of quasiadjunction are closely related to recently studies multiplier ideals (cf. [65], [38]) and log-canonical thresholds.

For a $\mathbf{Q}$-divisor D on a non singular manifold X its multiplier ideal $\mathcal{J}(D)$ (cf. ([65]) can be defined as follows. Let $f : Y \to X$ be an embedded resolution of D and $f^*(D) = -E$. Then $\mathcal{J}(D) = f_*(\mathcal{O}_Y(K_Y - f^*(K_X) - \lfloor E \rfloor))$ where $\lfloor E \rfloor$ is round-down of a $\mathbf{Q}$-divisor. In this terminology one can define the ideals of quasiadjunction as follows. For an array $(\gamma_1, .., \gamma_r), (\gamma_i \in \mathbf{Q})$ let $D_{\gamma_1,..,\gamma_r}$ be given by equation $f_1^{\gamma_1} \cdots f_r^{\gamma_r}$. Then $\mathcal{J}(D_{\gamma_1,..,\gamma_r}) = \mathcal{A}(j_1, .., j_r | m_1, ..., m_r)$ where $\gamma_i = 1 - \frac{j_i+1}{m_i}$ for $i = 1, .., r$. This follows immediately from (55).

To describe the relation with the log-canonical thresholds, recall ([35]) that a pair (X, D) where X is normal and D is a $\mathbf{R}$-divisor such that $K_X + D$ is $\mathbf{R}$-Cartier is called log-canonical at $x \in X$ if for any birational morphism $f : Y \to X$, with Y normal, in the decomposition

$$K_Y = f^*(K_X + D) + \sum_E a(E, X, D)E \tag{59}$$

for each irreducible E having center at x one has $a(E, X, D) \geq -1$. This coefficient is called *discrepancy* of divisor D on X along E.

Proposition 5.5 *(cf. [54]) The local ring $\mathcal{O}_O$ of a singularity $f_1 \cdots f_r = 0$ at the origin O of $\mathbf{C}^2$ considered as the ideal in itself is an ideal of log-quasiadjunction. Let $\mathcal{P}$ be the corresponding polytope of log-quasiadjunction. Let D_i be the divisor in $\mathbf{C}^2$ with the local equation $f_i = 0$ near the origin.*
Then for $\{(\gamma_1, ..., \gamma_r)\} \in \mathbf{R}^r$ the divisor $\gamma_1 D_1 + ... + \gamma_r D_r$ is log-canonical at $(0, 0) \in \mathbf{C}^2$ if and only if $(1 - \gamma_1, .., 1 - \gamma_r)$ belongs to the polytope $\mathcal{P}$.

To see why this is the case, let us consider the polytope given by inequalities (55) in which one puts $e_k(\mathcal{A}'') = 0$, i.e.

$$a_{k,1}x_1 + ... + a_{k,r}x_r \geq a_{k,1} + ... + a_{k_r} - c_k - 1 \tag{60}$$

Let $(j_1, .., j_r | m_1, .., m_r)$ be such that the corresponding vector (54) belongs to the boundary of this polytope. Then $1 \in \mathcal{A}''(j_1, .., j_r | m_1, .., m_r)$ and hence the ideal $\mathcal{A}''(j_1, .., j_r | m_1, .., m_r)$ is the local ring of the origin (i.e is not proper).

If $\pi : V \to \mathbf{C}^2$ is an embedded resolution then the discrepancy of $f_1^{\gamma_1} \cdots f_r^{\gamma_r}$ along E_k is:

$$c_k - (a_{k,1}\gamma_1 + ... + a_{k,r}\gamma_r)$$

i.e. the discrepancy along each E_k is not less than -1 if and only if $(1-\gamma_1, ..., 1-\gamma_r)$ satisfies (60).

As an example to this proposition one can consider the ordinary cusp $x^2 + y^3$ the log-canonical threshold is $\frac{5}{6}$ and the constant of quasiadjunction is $\frac{1}{6}$ (cf. section 2.2.5).

The polytopes of quasiadjunction are "pieces" of the zeros of the (multivariable) Alexander polynomail and in this sense are analogs of the spectrum.

Problem 5.6 *Find a generalization of the semicontinuity of spectrum of a singularity*

For some results in this direction cf. [54].

5.5 Hodge decomposition for INNCs.

The calculation from the last section can be partially extended to INNC. Namely we extend the calculation which involve the forms of top degree and hence will obtain at least a part of the components of characteristic varieties.

We shall start with the definition:

Definition 5.7 *(cf. [54]) Let $f_i = 0$ be the equation of divisor D_i and let $\pi : \tilde{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ be a resolution of the singularities of $\bigcup D_i$ (i.e. the proper preimage of the latter is a normal crossings divisor). Let $\mathcal{V}_{m_1,...,m_r}$ be the singularity (41) having $V_{m_1,...,m_r}$ as its link. Let $\tilde{V}$ be a normalization of $\tilde{\mathbf{C}}^{n+1} \times_{\mathbf{C}^{n+1}} \mathcal{V}_{\Updownarrow\infty,..,\Updownarrow\nabla}$ (cf. (41)) The ideal of quasiadjunction of type $(j_1, ..., j_r|m_1, ..., m_r)$ is the ideal $\mathcal{A}(j_1, ..., j_r|m_1, ..., m_r)$ of germs $\phi \in \mathcal{O}_{\prime,\mathbf{C}^{\setminus+\infty}}$ such that the $(n+1)$-form:*

$$\omega_\phi = \frac{\phi z_1^{j_1} \cdot ... \cdot z_r^{j_r} dx_1 \wedge ... \wedge dx_{n+1}}{z_1^{m_1-1} \cdot ... \cdot z_r^{m_r-1}} \tag{61}$$

on the non singular locus of $V_{m_1,...,m_r}$ after the pull back on $\tilde{V}$ extends over the exceptional set.

The l-th ideal of log-quasiadjunction $\mathcal{A}_l(\log E)(j_1, ..., j_r|m_1, ..., m_r)$ is the ideal of $\phi \in \mathcal{O}_{0,\mathbf{C}^{n+1}}$ such that the the pull back of the corresponding form ω_ϕ on $\tilde{V}$ is log-*form on $(\tilde{V}, E)$ having weight at most l.*

We have the following:

Proposition 5.8 *(cf. [54], [55]) There exist a collection of subsets $\mathcal{P}_\kappa, (\kappa \in \mathcal{K})$ in the unit cube*

$$\mathcal{U} = \{(x_1, ..., x_r)|0 \leq x_i \leq 1\}$$

in $\mathbf{R}^r$ and a collection of affine hyperplanes $l_i(x_1, ..., x_r) = \alpha_i$ such that each $\mathcal{P}_\kappa$ is the boundary of the polytope consisting of solutions to the system of inequalities:

$$l_i(x_1, ..., x_r) \geq \alpha_i$$

and such that

$$(\frac{j_1 + 1}{m_1}, ..., \frac{j_r + 1}{m_r}) \in \mathcal{U} \tag{62}$$

belongs to $\mathcal{P}_\kappa$ if and only if

$$\dim \mathcal{A}(\log E)(j_1, ..., j_r|m_1, ..., m_r)/\mathcal{A}(j_1, ..., j_r|m_1, ..., m_r) \geq 1 \tag{63}$$

Moreover

$$\dim \mathcal{A}_l(\log E)(j_1, ..., j_r|m_1, ..., m_r)/\mathcal{A}_{l-1}(j_1, ..., j_r|m_1, ..., m_r) \geq k \tag{64}$$

if only if (54) belongs to a collection of certain faces $\mathcal{P}^{k,l}_{\kappa,\iota}(\iota \in \mathcal{I}^{k,l})$ of polytopes $\mathcal{P}_\kappa$.

Now the exponents of the polytopes of quasiadjunction land in the characteristic varieties. More precisely we have:

Theorem 5.9 *(cf. [55]) A character of* $\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap (\bigcup_{1 \le i \le r} D_i)$ *acting on* $W_l(F^n H^n(V_{m_1,....,m_r}))$ *via the action of the Galois group has the eigenspace of dimension at least* k *if and only if it has the form:*

$$(exp 2\pi\sqrt{-1}a_1, ..., exp 2\pi\sqrt{-1}a_r)$$

where $(a_1, ..., a_r)$ *belongs to one of the faces* $\mathcal{P}^{k,l}_{\kappa,\iota}$ *of a polytope* $\mathcal{P}_\kappa$ *of quasiadjunction of* $\bigcup D_i$*. In particular, the Zariski closures of exponents of polytopes of quasiadjunction are components of characteristic varieties. These components are the translated subgroups by points of finite order*

We conjecture that all components are the translated subgroups by points of finite order.

Conjecture 5.10 *Characteristic variety is a union of translated subtori of* $\mathrm{Spec}\mathbf{C}[\pi_1(\partial B_\epsilon - \partial B_\epsilon \cap X)]$ *with each translations given by a point of finite order.*

An interesting problem is to calculate them in terms of resolution.

6 Homotopy groups of the complements to hypersurfaces in projective space and linear systems determined by singularities

In this section we want to discuss the characteristic varieties associated with hypersurfaces which are divisors with isolated non normal crossings in a projective space. An interesting case occurs already when all hypersurfaces have degree one i.e. the case of arrangements of hyperplanes. The advantage of the case of INNC is that one does not have the problems associated with complexity of the fundamental group since the fundamental groups for such arrangements are abelian (unless we are dealing with an arrangement of lines). The theory of such arrangement is still highly non trivial and is far from being well understood. Note that a more general case of divisors with normal crossings in general projective manifolds (rather then in $\mathbf{P}^{n+1}$) is considered in [56]. The main results and conjectures of this section show how the local characteristic varieties plus certain linear systems associated with the points of non normal crossings determine the global characteristic varieties. This generalizes the results on the Alexander polynomial discussed earlier.

6.1 Homotopy groups of the complements to INNC

Let us consider the a divisor D in projective space $\mathbf{P}^{n+1}$ which is a divisor with isolated non normal crossings. This situation includes as its special cases:

a) Arbitrary reduced curves in $\mathbf{P}^2$

b) Hypersurfaces in $\mathbf{P}^{n+1}$ with isolated singularities and hypersurfaces in $\mathbf{C}^{n+1}$ with isolated singularities and transversal to the hyperplanes at infinity.

c) Arrangements of hyperplanes in $\mathbf{P}^{n+1}$ such that each intersection of hyperplanes having codimension $k \neq n+1$ belongs to exactly k hyperplanes of the arrangement.

The starting point is the the following vanishing of the homotopy groups generalizing already discussed result from [48]:

Theorem 6.1 *(cf. [56]) Let $X = \mathbf{P}^{n+1}$ and D be a divisor having finitely many non-normal crossings. Assume that one of the components has degree 1. Then $\pi_i(\mathbf{P}^{n+1} - D) = 0$ for $2 \le i \le n-1$. If all intersections are the normal crossings, then $\pi_n(\mathbf{P}^{n+1} - D) = 0$ and hence $\mathbf{P}^{n+1} - D$ is homotopy equivalent to the wedge of the $n+1$-skeleton of the torus $(S^1)^k$ and several copies of S^{n+1}.*

One also has a similar vanishing for the homology of local systems.

Theorem 6.2 *(cf. [56]) Let $\chi \in \mathrm{Char}\pi_1(\mathbf{P}^{n+1}-D)$ be a character of the fundamental group different from the identity and let $\mathbf{C}_\chi$ be $\mathbf{C}$ considered as $\mathbf{C}[\pi_1(\mathbf{P}^{n+1} - D)]$ module via the character χ. Then*

$$H_i(\mathbf{P}^{n+1} - D, \chi) = 0 \ (i \le n-1)$$

$$H_n(\mathbf{P}^{n+1} - D, \chi) = \pi_n(\mathbf{P}^{n+1} - D) \otimes_{\mathbf{C}[\pi_1(\mathbf{P}^{n+1}-D)]} \mathbf{C}_\chi$$

The main problem for INNC hence is to understand the first non trivial homotopy group $\pi_n(\mathbf{P}^{n+1} - D)$. Similarly to the local case the starting point is the following:

Definition 6.3 *(cf. [56]) The k-th characteristic variety $V_k(\pi_n(\mathbf{P}^{n+1} - D))$ of the homotopy group $\pi_n(\mathbf{P}^{n+1}-D)$ is the zero set of the k-th Fitting ideal of $\pi_n(\mathbf{P}^{n+1}-D)$, i.e. the zero set of minors of order $(n-k+1) \times (n-k+1)$ of Φ in a presentation*

$$\Phi : \mathbf{C}[\pi_1(\mathbf{P}^{n+1} - D)]^m \to \mathbf{C}[\pi_1(\mathbf{P}^{n+1} - D)]^l \to \pi_n(X) \to 0$$

of $\pi_1(\mathbf{P}^{n+1} - D)$ module $\pi_n(\mathbf{P}^{n+1} - D)$ via generators and relations. Alternatively (cf. theorem 6.2) outside of $\chi = 1$, $V_k(\pi_n(\mathbf{P}^{n+1} - D))$ is the set of characters $\chi \in \mathrm{Char}[\pi_1(\mathbf{P}^{n+1} - D)]$ such that $\dim H_n(\mathbf{P}^{n+1} - D, \chi) \ge k$.

6.2 Jumping loci on quasiprojective varieties

A remarkable fact is that the characteristic varieties of the complements have a very simple structure (unlike in the similar situations outside of algebraic geometry). We did see this already in the case of links of curve singularities and in the case of charactersstic variety V_1 in general local case. The local systems on *a non singular projective variety* correspond to holomorphic bundles which are topologically trivial. The jumping loci for the cohomology of such bundles are unions of translates of abelian subvarieties of the Picard variety. These results, having long history, are due to Catanese, Beauville, Green-Lazarsfeld, Simpson and Deligne. We shall use the following quasi-projective version dealing with the cohomology of local systems which will allow us eventually to describe the characteristic varieties.

Theorem 6.4 *([6]) Let $\hat{X}$ be a projective manifold such that $H^1(\hat{X}, \mathbf{C}) = 0$. Let $\hat{D}$ be a divisor with normal crossings. Then there exists a finite number of unitary characters $\rho_j \in \mathrm{Char}\pi_1(\hat{X} - \hat{D})$ and holomorphic maps $f_j : \hat{X} - \hat{D} \to T_j$ into complex tori T_j such that the set $\Sigma^k(\hat{X} - \hat{D}) = \{\rho \in \mathrm{Char}\pi_1(\hat{X} - \hat{D}) | \mathrm{dim} H^k(\hat{X} - \hat{D}, \rho) \geq 1\}$ coincides with $\bigcup \rho_j f_i^* H^1(T_j, \mathbf{C}^*)$. In particular, Σ^k is a union of translated by unitary characters subgroups of $\mathrm{Char}\pi_1(\hat{X} - \hat{D})$.*

Hence we also obtain:

Corollary 6.5 *The characteristic variety $V_k(\pi_n(\mathbf{P}^{n+1} - D))$ is a union of translated subgroups S_j of the group $\mathrm{Char}\pi_1(\mathbf{P}^{n+1} - D)$ by unitary characters ρ_j:*

$$V_k(\pi_n(\mathbf{P}^{n+1} - D)) = \bigcup \rho_j S_j$$

In the case $k = 1$ the components of characteristic variety having a positive dimension correspond to the maps onto hyperbolic curves. This has many applications for example to calculations of characteristic varieties (cf. [52]), estimating the order of the group of automorphisms of the complements (cf. [7]), classification of arrangements of lines (cf.[53]) among others but we won't discuss them here.

6.3 The Hodge numbers of abelian covers of projective spaces and linear systems.

Let, as before, $D = \cup_{i=0,..r} D_i$ be a divisor in $\mathbf{P}^{n+1}$. We shall assume, to simplify the exposition, that one of components, say D_0 has degree equal to one and that there are D has non non normal crossings on D_0. Let $\pi_1(\mathbf{P}^{n+1} - D) \to \oplus \mathbf{Z}/m_i\mathbf{Z}$ be a surjective homomorphism and let $X_{\mathbf{m}}$ ($\mathbf{m} = (..., m_i, ...)$) be a normalization of a compactification of unbranched cover of $\mathbf{P}^{n+1} - D$ corresponding to this homomorphism. Let $f : X_{\mathbf{m}} \to \mathbf{P}^{n+1}$ be the corresponding projection.

Our goal is to calculate the Hodge number $h^{n,0}(X_{\mathbf{m}})$. Starting from D, we shall define global polytopes of quasiadjucntion so that with each face δ of the polytope is associated the ideal sheaf $\mathcal{J}_\delta$. The above Hodge number is equal to the number of lattice points is δ counted with the weight given by the dimension of linear system of hypersurface of degree given by δ and with local conditions given by the ideal sheaf $\mathcal{J}_\delta$.

Let is consider the unit cube $\mathcal{U} = \{(x_1, ..., x_r) \in \mathbf{R}^r | 0 \leq x_i \leq 1\}$ coordinate of which correspond to irreducible components $D_1, ..., D_r$ of the divisor D. We view $\mathbf{R}^r$ as the universal cover of the group $(S^1)^r$ of unitary characters of $\pi_1(\mathbf{P}^{n+1} - D)$ and $\mathcal{U}$ as the fundamental domain for the action of the covering group on the cover. With each point P in $\mathbf{P}^{n+1}$ where D has a non-normal crossing the definition 5.8 associates a polytope $\mathcal{P}_\kappa$ in the unit cube in $\mathbf{R}^s$ with coordinates corresponding to the components of D. Since one has the canonical projection $\pi : \mathbf{R}^r \to \mathbf{R}^s$, forgetting the coordinates corresponding to D_i's not containing P, each $\mathcal{P}_\kappa$ defines the polytope $\pi^{-1}(\mathcal{P}_\kappa)$ in $\mathcal{U}$ which we shall denote by the same letter. This defines a finite collection of polytopes $\mathcal{P}_{\kappa,P} \subset \mathcal{U}$.

Definition 6.6 *Consider the equivalence relation on points in $\mathcal{U}$ calling two points equivalent if the collections of polytopes $\mathcal{P}_{\kappa,P}$ containing these two points are identical. The equivalence class is called the global polytope of quasiadjucntion.*

A global face of quasiadjunction is a face of a global polytope of quasiadjunction.

Let $\mathcal{S}_\delta$ be the set of non normal crossings P of D for which there exist the polytopes $\mathcal{P}_{\kappa,P}$ contaning δ.

The ideal sheaf corresponding to δ is a sheaf $\mathcal{J}_\delta \subset \mathcal{O}_{\mathbf{P}^{n+1}}$ such that $\mathcal{O}_{\mathbf{P}^{n+1}}/\mathcal{J}_{delta}$ is supported at $\mathcal{S}_\delta$ and such that the stalk at P is the ideal which is the intersection of local ideals of quasiadjunction corresponding to local polytopes containing δ.

Clearly such an equivalence class is a polytope i.e. consists of points satisfying a system of linear inequalities. Also the collections S_δ of non normal crossings are defined entierly by the local data of D. The Hodge number $h^{n,0}(X_{\mathbf{m}})$ depends on additional piece of information.

Theorem 6.7 *(cf. [52], [56]) Let D as above and let d_i be the degree of the irreducible component D_i. For each $\chi \in \mathrm{Char} \oplus \mathbf{Z}/m_i\mathbf{Z} \subset \mathrm{Char}\pi_1(\mathbf{P}^{n+1} - D)$ let $\delta(\chi)$ be the global face of quasiadjunction containing $\frac{1}{2\pi\sqrt{-1}}\log(\chi) \in \mathcal{U}$. Let l be such that the hyperplane $d_1x_1 + ... + d_rx_r = l$ ($l \in \mathbf{Z}$) contains δ. Then*

$$h^{n,0}(X_{\mathbf{m}}) = \sum_{\chi \in \mathrm{Char} \oplus \mathbf{Z}/m_i\mathbf{Z}} \mathrm{dim} H^1(\mathbf{P}^{n+1}, \mathcal{J}_{delta}(l-n-1)$$

A proof in cyclic case and in the case of curves and generalizing Zariski's approach ([81]) is given in [49] and [52] and the case of INNC is similar. Alternatively, one can also use the approach in [24].

Example 6.8 *For an irreducible curve of degree d with nodes and the ordinary cusps as the only singularities the global polytope of quasiadjunction coincide with the local one of the cusp. The only face of quasiadjunction is $x = \frac{1}{6}$. The contributing hyperplane is given by $dx = \frac{d}{6}$ and its level is $\frac{d}{6}$. The sheaf of quasiadjunction corresponding to this face of quasiadjunction is the ideal sheaf having stalks different from the local ring only at the points of $\mathbf{P}^2$ where the curve has cusps and the stalks at those points are the maximal ideals of the corresponding local rings.*

For characters not on the global faces of quasiadjunction one still can define the ideal sheaves looking at the polytopes containing the lifts of the characters $\frac{1}{2\pi\sqrt{-1}}\log(\chi) \in \mathcal{U}$ into universal cover of the torus of unitary character and also the integer l such that $d_1x_1 + ... + d_rx_r = l$ contains the lift. However the corresopnding group $H^1(\mathbf{P}^{n+1}, \mathcal{J}_{delta}(l-n-1)$ will be vanishing. For plane curves with nodes and cusps one obtains the following classical result (for the most part already discussed earlier).

Corollary 6.9 *(Zariski's theorem) Let C be a plane curve of degree d having nodes and cusps as the only singularities. Let $\mathcal{J}$ be a subsheaf of the sheaf of regular* *functions whose sections belong to the maximal ideals at the points in* $\mathbf{P}^2$ *which are cusps of* C. *If* $k > \frac{5d}{6}$ *then*

$$H^1(\mathbf{P}^2, \mathcal{J}(k-3)) = 0$$

If $6|d$ *then* $H^1(\mathbf{P}^2, \mathcal{J}(\frac{5d}{6}-3)) = h^{1,0}(X_d)$ *is equal to the irregularity of a a resolution of singularities of a d-fold cyclic cover of* $\mathbf{P}^2$ *branched over* C.

6.4 Mixed Hodge structure on homotopy groups

The theorem 6.2 suggests an additional structure on the characteristic varieties coming from the mixed Hodge structure on the cohomology of local systems. This is an analog of discussed earlier in local case the Hodge decomposition of characterstic varities. The MHS on the cohomology of local systems can be understood by interpreting the cohomology of local systems having finite order as the eigenspaces of the Galois group acting on the abelian covers as follows.

Theorem 6.10 *Let* G *be a finite group and* $g : \pi_1(X) \to G$ *be a surjection. Let* $\chi \in \mathrm{Char}\pi_1(X)$ *which is the pull back of a character of* G. *Assume that* $\pi_i(X) = 0$ *for* $2 \le i \le n-1$. *Finally let* X_G *be the unbranched cover of* X *corresponding to* g. *Then the eigenspace* $H^n(X_G)_\chi$ *is isomorphic to the homology of* $H^n(\mathbf{C}_\chi)$ *of the local system* $\mathbf{C}_\chi$ *corresponding to* χ. *In particualar, the cohomology classes in* $H^n(\mathbf{C}_\chi)$ *acquire the Hodge type.*

If X is quasiprojective and non singular, so is X_G and hence $H^n(X_G)$ admits the mixed Hodge structure with the weights $n, ..., 2n$.

Definition 6.11 *(cf. [56]) Let* $\mathbf{P}^{n+1} - D$ *be a complement to an INNC in* $\mathbf{P}^{n+1}$. *For a local system* χ *pf finite order let* $h^{p,q,n})_\chi$ *be the dimension of the space of cohomology classes in* $H^n(\mathbf{C}_\chi)$ *having the Hodge type* (p, q). *The following subset of* $V_k(\pi_n(\mathbf{P}^{n+1} - D))$:

$$\mathcal{P}_k^{p,q,n} = \{\chi | h_\chi^{p,q,n} \ge k\}$$

is called the component of the characteristic variety of type (p, q, n)

One has $\mathcal{P}_k^{p,q,n} \neq \emptyset$ only if $n \le p+q \le 2n$ and $\bigcup_{p,q,n} \mathcal{P}^l = V_1(\pi_1(\mathbf{P}^{n+1} - D))$.

6.5 A relation between the Hodge numbers of branched and unbranched abelian covers

We want to use the theorem 6.7 to detect some components of characteristic varieties of the homotopy groups. Here is the relation between branched and unbranched covers which we shall need since the theorem 6.7 works in compact case.

Theorem 6.12 *(cf. [56]) Let* $\chi \in Char(\pi_1(\mathbf{P}^{n+1} - D))$ *be a character of a finite quotient* G *of* $\pi_1(\mathbf{P}^{n+1} - D)$. *Let* $\bar{U}_G$ *be a* G*-equivariant non-singular compactification of* U_G *and let* $H^{p,q}(\bar{U}_G)_\chi$ *be the* χ*-eigenspace of* G *acting on* $H^{p,q}(\bar{U}_G)$. *Then*

$$h^{n,0,n}(\mathbf{C}_\chi) = h^{n,0}(U_G)_\chi = h^{n,0}(\bar{U}_G)_\chi$$

6.6 Main theorem and Open Problems

Combining this together we obtain the following, extending results on Alexander polynomials and the case of reducible curves in [52]:

Theorem 6.13 *(cf. [56]) Let $D \subset \mathbf{P}^{n+1}$ be a union of hypersurfaces $D_0, D_1, ..., D_r$ of degrees $1, d_1, ..., d_r$ respectively, which is a divisor with isolated non-normal crossings. Let $\mathcal{F}$ be a face of global polytope of quasi-adjunction, i.e. a face of an intersection of polytopes of quasi-adjunction corresponding to a collection $\mathcal{S}$ of non-normal crossings of D. Let $d_1x_1 + ... + d_rx_r = l$ be a hyperplane containing the face of quasiadjunction $\mathcal{F}$. If $H^1(\mathcal{A}_{\mathcal{F}} \otimes \mathcal{O}(l-3)) = k$, then the Zariski closure of $exp(\mathcal{F}) \subset \mathrm{Char} H_1(\mathbf{P}^{n+1} - D)$ belongs to a component of $V_k(\pi_n(\mathbf{P}^{n+1} - D))$.*

There is a generalization to INNC divisors on arbitrary projective simply-connected varieties. I refer to [56] for conjectures. Here is a short list of the open problems even in the case of divisors in $\mathbf{P}^{n+1}$.

Problem 6.14 *Are there components of characteristic variety $V_k(\pi_1(\mathbf{P}^{n+1} - D))$ which are not Zariski closures of $\mathcal{P}_{n,0,n}$?*

Problem 6.15 *Find methods for detecting the sets $\mathcal{P}_{p,q,n}$ with $(p, q) \neq (n, 0)$*

A difficulty here is that one cannot work with arbitrary compactification since the Hodge numbers $h^{p,q}$ are not birational invariants. It would be good to have techniques which will allow to work directly with the complement and avoiding to some extent the compactification.

Problem 6.16 *Generalize the main theorem to projective algebraic varieties and beyond the cases when $\mathcal{O}(D_i) = \mathcal{L}^{m_i}$.*

See discussion of this in [56]

Problem 6.17 *Find additional interesting examples beyond the one described in [56].*

Problem 6.18 *Solve realization problem for characteristic varieties i.e. describe how many components and what are their dimensions depending on the numerical data of the divisor D on X.*

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