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Local Topology of Reducible Divisors

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Abstract. We show that the universal abelian cover of the complement to a germ of a reducible divisor on a complex space Y with isolated singularity is $(\dim Y - 2)$-connected provided that the divisor has normal crossings outside of the singularity of Y. We apply this result to obtain a vanishing property for the cohomology of local systems of rank one and also study vanishing in the case of local systems of higher rank.

1. Introduction

The topology of holomorphic functions near an isolated singular point is a classical subject (cf. [23], [4]). Among the main results are the existence of Milnor fibration and the connectivity of the Milnor fiber yielding a very simple picture for the latter: it has the homotopy type of a wedge of spheres. Starting with the case of a germ of holomorphic function on $\mathbb{C}^N$ considered by Milnor ([23]), these results were eventually extended to the germs of holomorphic functions on analytic spaces (cf. [12], [19]).

In [22], it was shown that if the divisor of a holomorphic function on $\mathbb{C}^N$ is *reducible* then the results on the connectivity of Milnor fibers (cf. [23], [17]) can be refined. This refinement is based on the observation that the Milnor fiber is homotopy equivalent to the infinite cyclic cover of the total space of the Milnor fibration. So the classical connectivity results by Milnor and Kato-Matsumoto can be restated in terms of the connectivity of this cyclic cover.

In the case when the divisor of a holomorphic function is reducible, it is the associated universal abelian cover which has interesting connectivity properties generalizing the connectivity properties in the cyclic cover case, see Theorem 3.2 below. The present paper studies the case of reducible divisors on *arbitrary* isolated singularities.

More precisely the situation we consider is the following. Let $(Y, 0) \subset (\mathbb{C}^N, 0)$ be the intersection of a ball of a sufficiently small radius about the origin 0 with a $(n+1)$-dimensional irreducible complex analytic space with an isolated singularity at 0. Let $(D_j, 0) \subset (Y, 0)$ for $j = 1, \dots, r$ be r irreducible Cartier divisors on $(Y, 0)$. We set $X = \cup_{i=1,r} D_i$, $M = Y \setminus X$ and regard M as the complement of the hypersurface arrangement $\mathcal{D} = (D_j)_{j=1,r}$. Since $(Y, 0)$ is irreducible, the complement M is connected and so we can unambiguously talk about the fundamental group $\pi_1(M)$ without mentioning a base point.

In this paper we investigate the topology of this complement M. In Section 2 we generalize a case of the Lê-Saito result in [18] asserting that if $(Y, 0)$ is a smooth germ and X is an isolated non-normal crossing divisor (see the definition below), then the fundamental group $\pi_1(M)$ is abelian. Our proof is based on an idea used in [24] in the global case and is much shorter than the proof in [18].

In Section 3 we consider the case when the hypersurface arrangement $\mathcal{D}$ is an arrangement based on a hyperplane arrangement $\mathcal{A}$ in the sense of Damon [3]. We show that the (co)homology of M is determined up-to degree $(n-1)$ by the hyperplane arrangement $\mathcal{A}$. The key fact here is the functoriality of the Gysin sequence and the splitting of the Gysin sequence associated to a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of hyperplane arrangements into short exact sequences. Note that the proof of Theorem 3.1 can be done *only in this paper more general setting*, i.e., the deletion and restriction argument cannot be performed in the setting of [22] when $(Y, 0)$ is a smooth germ.
Combining the results above and following the approach in [22], we show that the universal abelian cover $\tilde{M}$ of M is homotopically a bouquet of spheres of dimension n which is the refinement of [23] and [12] we mentioned earlier.

In the last two sections we prove vanishing results for the (co)homology of the complement M with coefficients in a local system $\mathcal{L}$ on M. The case when the rank of $\mathcal{L}$ is equal one is treated in Section 4 and in this context we give a description for the dimension of the non zero homology groups $H_*(M, \mathcal{L})$. The general case when rank $\mathcal{L} \geq 1$ is treated in Section 5 where we allow a more general setting for the ambient space $(Y, 0)$ and for the divisor $(X, 0)$. The vanishing result in this case follows the general philosophy in [7], but the use of perverse sheaves as in [2] is unavoidable. Note that in our case the space M may be singular so one cannot use the technique of integrable connections to get vanishing results. A new point in our proof is the need to use the *interplay between constructible complexes of sheaves on real and complex spaces.* Indeed, real spaces occur in the picture in the form of links of singularities.

2. Fundamental group of the complements to INNC

Let $(Y, 0) \subset (\mathbb{C}^N, 0)$ be as above an $(n+1)$-dimensional irreducible complex analytic space germ with an isolated singularity at the origin. Let $(D_j, 0) \subset (Y, 0)$ for $j = 1, \dots, r$ be r irreducible Cartier divisors on $(Y, 0)$, i.e., each D_j is given (with its reduced structure) as the zero set of a holomorphic function germ $f_j : (Y, 0) \to (\mathbb{C}, 0)$. When the local ring $\mathcal{O}_{Y,0}$ is factorial, then any hypersurface germ in $(Y,0)$ is Cartier. This is the case for instance when $(Y,0)$ is smooth or an isolated complete intersection singularity (ICIS for short) with dim $Y \geq 4$, see [11]. See also Example 2.1. Here and in the sequel we identify germs with their (good) representatives.

In particular the local homotopy groups of Y and $M = Y \setminus X$ are well defined as $\pi_j(L_Y)$ and $\pi_j(L_Y \setminus L_X)$, where the links L_Y and L_X are as defined below. We assume in this section that the following condition holds.

(**C1**) The divisor $X = \cup_{i=1}^{i=r} D_i$ has only normal crossing singularities on Y except possibly at the origin. We say in this case that X is an isolated non normal crossing divisor (for short INNC) on $(Y,0)$.

In particular each germ $(D_j,0)$ has an isolated singularity at the origin as well. Since the $(r+1)$-tuple $(Y, D_1, \ldots, D_r)$ has a conical structure (cf. [9]) we have an isomorphism:

$$\pi_1(L_Y \setminus L_X) = \pi_1(M \cap \partial B_\epsilon) \to \pi_1(M) \tag{1}$$

where L_Y (resp. L_X) denotes the link of Y (resp. of X), i.e., the intersection of Y (resp. of X) with the boundary ∂B_ϵ of a small ball B_ϵ about 0. In particular we get an epimorphism

$$\pi_1(M) = \pi_1(L_Y \setminus L_X) \to \pi_1(L_Y)$$

induced by the inclusion $L_Y \setminus L_X \to L_Y$.

Theorem 2.1. *For $n \geq 2$, the kernel of the surjection $\pi_1(M) \to \pi_1(L_Y)$ is contained in the center of the group $\pi_1(M)$. In particular, if L_Y is simply connected, then the fundamental group $\pi_1(M)$ is abelian.*

Proof. First notice that if dim$Y > 3$ and if H is a generic linear subspace passing through 0 such that the codimension of H in $\mathbb{C}^N$ is dim$Y - 3$, then, by Lefschetz hyperplane section theorem (cf. [9], p. 26 and p. 155), we have an isomorphism

$$\pi_1(M \cap H) \to \pi_1(M). \tag{2}$$

Hence it is enough to consider the case dim$Y = 3$ only (though the arguments below work for any dimension ≥ 3).

Next notice that $\kappa = \mathrm{Ker}(\pi_1(L_Y \setminus L_X) \to \pi_1(L_Y))$ is the normal subgroup spanned by the set of elements in the fundamental group $\pi = \pi_1(L_Y \setminus L_X)$ represented by the loops δ_i each of which is the boundary of a fiber over a non singular point of a small closed tubular neighborhood $T(D_i)$ of the submanifold $D_i \cap L_Y$ in the manifold L_Y. Indeed a loop representing an element γ in the kernel κ is the image of the boundary of a 2-disk under a map $\phi : D^2 \to L_Y$ which is isotopic to an embedding (since dim$L_Y \geq 5$) and which we may assume to be transversal to all the submanifolds $D_i \cap L_Y$. Now δ_i are the ϕ-images of loops in D^2 each of which is composed of a path α_i going from the point in D^2 corresponding to the base point $p \in L_Y$ to the vicinity of a point $y \in D^2$ corresponding to a point in $\phi(D^2) \cap D_i$, a small loop about y and back along α_i^{-1}. So it is enough to show that all these loops δ_i (note that there may be several of them for a given i) belong to the center of $\pi_1(L_Y \setminus L_X)$.

Let $T(D_i)$ be a tubular neighborhood of $D_i \cap L_Y$ in L_Y as above. We claim that for any i ($i = 1, \ldots, r$) there is a surjection:

$$\pi_1(T(D_i) \setminus L_X) \to \pi_1(L_Y \setminus L_X). \tag{3}$$

Notice that assuming the surjectivity in (3) we can conclude the proof as follows. Since the divisors D_i's have normal intersections in L_Y, the space $T(D_i) \setminus L_X$ is homotopy equivalent to the total space of a locally trivial circle fibration over $D_i \setminus \cup_{j \neq i} D_j$. The fiber δ_i' of this fibration, which is a loop based at a point p', is in the center of $\pi_1(T(D_i) \setminus L_X, p')$.
Indeed, if $\alpha : S^1 \to T(D_i) \setminus L_X$ is any loop, we set $\beta = \pi \cdot \alpha$ with $\pi : T(D_i) \setminus L_X \to D_i \setminus \cup_{j \neq i} D_j$ the the corresponding projection. Then the commutativity $\alpha\delta' = \delta'\alpha$ follows from the triviality of the pull-back of the normal bundle $\pi : T(D_i) \setminus L_X \to D_i \setminus \cup_{j \neq i} D_j$ under β. This triviality in turn follows from the triviality of any complex line bundle over a circle S^1.
Therefore the surjectivity in (3) yields that the class of δ' commutes with any element in $\pi_1(L_Y \setminus L_X, p')$ and hence with any element in $\pi_1(L_Y \setminus L_X, p)$.

To show the surjectivity (3), let us consider a generic holomorphic function g on Y sufficiently close to f_i so that $L_Y \cap \{g = 0\} \subset T(D_i)$. We have the decomposition

$$\begin{array}{ccc} \pi_1(L_Y \cap \{g = 0\} \setminus L_X) & & \\ \downarrow & \searrow & \\ \pi_1(T(D_i) \setminus L_X) & \to & \pi_1(L_Y \setminus L_X) \end{array} \tag{4}$$

corresponding to the factorization of the embeddings. This yields that the horizontal map is surjective provided the map:

$$\pi_1(L_Y \cap (g = 0) \setminus L_X) \to \pi_1(L_Y \setminus L_X) \tag{5}$$

is surjective. But this follows from [14]. □

Note that this result in the case when $Y = \mathbb{C}^{n+1}$ is a consequence of a theorem of Lê Dung Trang and K.Saito (cf. [18]).

Example 2.1.
(i) If $(Y, 0)$ is an ICIS with dim $(Y, 0) \geq 3$, then it follows from [12] that the link L_Y is simply-connected.

(ii) If $V \subset \mathbb{P}^m$ is a locally complete intersection such that $n = \dim V > \operatorname{codim} V$, then the morphism $\pi_2(V) \to \pi_2(\mathbb{P}^m) = \mathbb{Z}$ induced by the inclusion $V \to \mathbb{P}^m$ is an epimorphism by the generalized Barth Theorem, see [9], p. 27. It follows that the associated affine cone $(Y, 0) = (CV, 0)$ has a simply-connected link.

If we assume that V is smooth and that $n = \dim V > \operatorname{codim} V + 1$, then the divisor class group $C\ell(\mathcal{O}_{Y,0})$ is trivial, i.e., any divisor on this germ $(Y, 0)$ is Cartier. This follows from the exact sequence in [15], Exercise II.6.3 comparing the divisor class groups in the local and the global settings, the usual isomorphism $C\ell(V) =$

$H^1(V, \mathcal{O}_V^*)$, see [15], II.6.12.1 and II.6.16 and the GAGA results allowing to use the exponential sequence, see [15], Appendix B, to relate topology to $H^1(V, \mathcal{O}_V^*)$.

If $E = \cup_{j=1,r} E_j$ is a normal crossing divisor on the smooth variety V, then the associated cone $X = \cup_{j=1,r} CE_j$ is an INNC divisor on the cone $(Y, 0)$.
The epimorphism $\pi_1(M) \to \pi_1(V \setminus E)$ can then be used to show that this last fundamental group is abelian.

3. Homology of the complements to reducible divisors

Assume in this section that the germ $(Y, 0)$ is an ICIS with $\dim Y = n + 1$ and let $\mathcal{A} = \{H_i\}_{i=1,\dots,r}$ be a central hyperplane arrangement in $\mathbb{C}^m$. Suppose given an analytic map germ $f : (Y, 0) \to (\mathbb{C}^m, 0)$ such that the following condition holds.

(**C2**) For any edge $L \in L(\mathcal{A})$ with codim $L = c$, the (scheme-theoretic) pull-back $D_L = f^{-1}(L)$ is an ICIS in $(Y, 0)$ of codimension exactly c for $c \leq n$ and $D_L = \{0\}$ for $c \geq n + 1$.

This condition (**C2**) is equivalent to asking that $f : (Y, 0) \to (\mathbb{C}^m, 0)$ is transverse to $\mathcal{A}$ off 0 in the sense of Damon, see [3], Definition 1.2. In his language, $X = \cup_{i=1,r} D_i$ is a nonlinear arrangement of hypersurfaces based on the central arrangement $\mathcal{A}$, with $D_i = f^{-1}(H_i)$. Note that for $n \geq 2$ all the germs D_j are irreducible by [12], but on the other hand the condition (**C1**) may well fail in this setting.

Consider the complements $M = Y \setminus X$ and $N = \mathbb{C}^m \setminus \cup_{i=1,r} H_i$, and note that there is an induced mapping $f : M \to N$. Our result is the following

Theorem 3.1. *With this notation,*

$$f_* : H_j(M) \to H_j(N)$$

is an isomorphism for $j < n$ and an epimorphism for $j = n$. Similarly

$$f^* : H^j(N) \to H^j(M)$$

is an isomorphism for $j < n$ and a monomorphism for $j = n$. In particular, the algebra $H^*(M)$ is spanned by $H^1(M)$ up-to degree $(n-1)$.

Proof. For $n = 1$ everything is clear, so we can assume in the sequel $n > 1$.

The proof is by induction on r. For $r = 1$ the result follows since M can be identified to the total space of the Milnor fibration, whose Milnor fiber is a bouquet of n-dimensional spheres by work of Hamm, see [12].

Assume now that $r > 1$ and apply the deletion and restriction trick, see more on this in [25], p. 4. Namely, let $\mathcal{A}' = \{H_i\}_{i=2,\dots,r}$ and $\mathcal{A}'' = \{H_1 \cap H_i\}_{i=2,\dots,r}$. Then both $\mathcal{A}'$ and $\mathcal{A}''$ are central arrangements with at most $(r-1)$ hyperplanes.

Since $L(\mathcal{A}') \subset L(\mathcal{A})$, it is clear that $f : (Y, 0) \to (\mathbb{C}^m, 0)$ satisfies the condition **C2** with respect to $\mathcal{A}'$. Moreover, $L(\mathcal{A}'') \subset L(\mathcal{A})$ (with a 1-shift in codimensions) and $f : (D_1, 0) \to (\mathbb{C}^{m-1}, 0)$ satisfies the condition **C2** with respect to $\mathcal{A}''$.

Let M', M'', N', N'' be the corresponding complements. Since M'' (resp. N'') is a smooth hypersurface in M' (resp. N') we have the following ladder of Gysin sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{j-1}(M'') \arrow[r] \arrow[d, "f_*"'] & H_j(M) \arrow[r] \arrow[d, "f_*"'] & H_j(M') \arrow[r] \arrow[d, "f_*"'] & H_{j-2}(M'') \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_{j-1}(N'') \arrow[r] & H_j(N) \arrow[r] & H_j(N') \arrow[r] & H_{j-2}(N'') \arrow[r] & \cdots
\end{tikzcd}$$

It is a standard fact in hyperplane arrangement theory that the bottom-right morphism $H_j(N') \to H_{j-2}(N'')$ is zero. Usually this result is stated for cohomology, see [25], p.191, but since both homology and cohomology of hyperplane arrangement complements are torsion free, the vanishing holds for homology as well.

An easy diagram chasing, using the induction hypothesis, shows that for $j \leq n$ the top-right morphism $H_j(M') \to H_{j-2}(M'')$ is zero as well. This implies the claim, again by an easy diagram chasing and using the induction hypothesis.

The proof for the cohomological result is completely dual. □

Using Theorems 2.1 and 3.1 and Example 2.1,(i) we get the following result.

Corollary 3.1. *Assume that $n \geq 2$ and that the divisor X satisfies the condition* **C1**. *Then*

$$\pi_1(M) = H_1(M) = H_1(N)$$

is a free abelian group of rank $r = |\mathcal{A}|$.

Remark 3.1. Note that M is a Stein manifold, and hence $H_j(M) = 0$ for $j > n+1$ by [13]. It follows that there are only two Betti numbers $b_j(M)$ to compute, namely for $j = n, n+1$. Indeed, for $j < n$ the Betti number $b_j(M) = b_j(N)$ is known by the results in [25], Theorem 5.93. Moreover by the additivity of Euler characteristics, see [8], it follows that $\chi(M) = \chi(Y) - \chi(X) = 0$. This gives a relation between the two top unknown Betti numbers of M.

Similarly to [22], the above results yield the following.

Theorem 3.2. *Let $(Y, 0)$ be a an isolated complete intersection singularity of dimension $n + 1 \geq 3$. Let $X = \cup_{i=1,r} D_i$ be a union of Cartier divisors of Y which have normal crossing outside of the origin. Then the universal abelian cover $\tilde{M}$ of $M = Y \setminus X$ has the homotopy type of a bouquet of spheres of dimension n.*

Proof. The proof is similar to the proof of Thm. 2.2 in [22]. Firstly, let us consider the exact homotopy sequence corresponding to the map $f : M \to \mathbb{C}^{*r}$, obtained by using the equations $f_i = 0$ for the divisors D_i. The isomorphism $\pi_1(M) \to \pi_1(\mathbb{C}^{*r})$ which follows from Theorems 2.1 and 3.1 and the known fact $\pi_j(\mathbb{C}^{*r}) = 0$ for $j > 1$ yield that $\pi_2(\mathbb{C}^{*r}, M) = 0$ and $\pi_j(\mathbb{C}^{*r}, M) = \pi_{j-1}(M)$ for $j > 2$ (here we assume that f is replaced by an embedding, which is of course possible up-to homotopy type). Moreover we can show exactly as in [22] that the action of $\pi_1(M)$

on $\pi_j(\mathbb{C}^{*r}, M)$ is trivial. Hence we can apply the relative Hurewich theorem to the pair $(\mathbb{C}^{*r}, M)$ and note that we have a vanishing of the relative homology of this pair as a consequence of the previous theorem. Looking now at $f : M \to \mathbb{C}^{*r}$ as a homotopy fibration with fiber $\tilde{M}$, we get the vanishing of the homotopy groups of the universal abelian cover $\tilde{M}$ of M up to dimension $n-1$. On the other hand, the existence of the Milnor fibration of $g = f_1 \dots f_r : (Y, 0) \to (\mathbb{C}, 0)$ (theorem of Hamm in [12]) yields that M admits a cyclic cover which has the homotopy type of a CW complex of dimension n (i.e., the Milnor fiber F of the hypersurface X in Y). Hence the universal abelian cover $\tilde{M}$, which is the universal abelian cover of this Milnor fiber F has the homotopy type of an n-complex. Therefore the universal abelian cover $\tilde{M}$ is homotopy equivalent to the wedge of spheres S^n. □

4. Homology of local systems (rank one case)

Let $(Y, 0)$ be a germ of an isolated complete intersection singularity and let $X = \bigcup_{i=1,r} D_i$ be a divisor which has normal crossings outside of the origin, i.e., we place ourselves again in the setting of Theorem 3.2. The above notation is still used here.

Let $\rho : \pi_1(M) \to \mathbb{C}^*$ be a character of the fundamental group or equivalently a local system $\mathcal{L}$ of rank one on M. The space M, being a Stein space of dimension $(n+1)$, has the homotopy type of an $(n+1)$ complex and hence $H_j(M, \mathcal{L}) = H_j(M, \rho) = 0$ for $j > n+1$.

The main result of this section is the following:

Theorem 4.1. *Let $\rho : \pi_1(M) \to \mathbb{C}^*$ be a non trivial character and let $\mathcal{L}$ be the associated rank one local system on M. Then:*

(i) $H_j(M, \mathcal{L}) = 0$ *for* $j \neq n, n+1$;

(ii) dim $H_n(M, \mathcal{L})$ *is the largest integer k such that ρ belongs to the zero set V_k of the kth Fitting ideal of the $\mathbb{C}[\pi_1(M)]$- module $\pi_n(M) \otimes_{\mathbb{Z}} \mathbb{C}$.*

(iii) *The largest integer k such that the trivial character of $\pi_1(M)$ belongs to V_k is equal to*

$$\dim \operatorname{Ker} (\Lambda^{n+1} H^1(M) \to H^{n+1}(M)) + \dim H_n(M) - \binom{r}{n}.$$

Proof. Recall the spectral sequence for the cohomology of local systems (cf. [1]) Thm. 8.4. Let $C_*^\rho(\tilde{M})$ be the chain complex on which $H_1(M, \mathbf{Z})$ acts from the right via $g(x) = \rho(g)xg$ $(g \in H_1(M, \mathbb{Z}), x \in C_*(\tilde{M}, \mathbb{C}))$ where $x \to xg$ is the action via deck transformations. Let $H_q^\rho(\tilde{M})$ be the homology of this complex. We have a spectral sequence:

$$E^2_{p,q} = H_p(H_1(M, \mathbb{Z}), H_q^\rho(\tilde{M})) \Rightarrow H_{p+q}(M, \rho).$$

Recall that M has the homotopy type of an $(n+1)$-complex and $\tilde{M}$ has the homotopy type of a bouquet of spheres of dimension n, see 3.2.

The group $H^\rho_q(\tilde{M})$ carries the canonical structure of $H_1(M,\mathbb{Z})$-module coming from the corresponding module structure on chains. We have the isomorphism: $H^\rho_0(\tilde{M}) = \mathbb{C}_\rho$ where $\mathbb{C}_\rho$ is the one-dimensional representation of $H_1(M,\mathbb{Z})$ given by ρ. Indeed, if x is a generator of $C^\rho_0(\tilde{M})$ as $H_1(M,\mathbb{Z})$-module then $x - x\cdot g = 0$ in $H^\rho_0(\tilde{M})$. On the other hand $x - x\cdot g = x - \rho(g)g^{-1}x$. Hence $gx = \rho(g)x$ in $H^\rho_0(\tilde{M})$.

Since for $\rho \neq 1$ one has $H_p(H_1(M,\mathbb{Z}),\mathbb{C}_\rho) = 0$, it follows from the vanishing theorem in the last section that the term E_2 has only one horizontal row: $q = n$. This yields the claim (i). We have

$$H_0(H_1(M,\mathbb{Z}), H^\rho_n(\tilde{M})) = H^\rho_n(\tilde{M})_{Inv} = H_n(\tilde{M}) \otimes_{H_1(M,\mathbb{Z})} \mathbb{C}_\rho$$

On the other hand, taking tensor product with $\mathbb{C}_\rho$ in the resolution $\Phi : \Lambda^s \to \Lambda^t \to H_n(\tilde{M}) \to 0$ we obtain the resolution of $H_n(\tilde{M}) \otimes_{H_1(M,\ Z)} \mathbb{C}_\rho$ in which the matrix of $\Phi_\rho : \Lambda^s \otimes \mathbb{C}_\rho \to \Lambda^t \otimes \mathbb{C}_\rho$ is obtained from the matrix of Φ by replacing its entries by values of the entries at ρ. Hence if ρ belongs to the set of zeros of the kth Fitting ideal (and k is maximal with this property), then the corank of Φ_ρ is k. This yields the second claim.

Let us consider the exact sequence:

$$\begin{aligned} H_{n+1}(M) \to H_{n+1}(H_1(M,\mathbb{Z})) \to H_n(\tilde{M}) \otimes_{H_1(M,\mathbb{Z})} \mathbb{C} \\ \to H_n(M) \to H_n(H_1(M,\mathbb{Z})) \to 0 \end{aligned} \tag{6}$$

corresponding to the spectral sequence:

$$H_p(H_1(M,\mathbb{Z}), H_q(\tilde{M})) \Rightarrow H_{p+q}(M) \tag{7}$$

We have

$$\begin{aligned} &\dim\ \mathrm{Coker}(H_{n+1}(M) \to H_{n+1}(H_1(M,\mathbb{Z}))) \\ &\quad = \dim\ \mathrm{Ker} H_{n+1}(H_1(M,\mathbb{Z}))^* \to H_{n+1}(M)^*) \end{aligned} \tag{8}$$

The latter kernel (using Kronecker pairing identification $H^*_i = H^i$ over $\mathbb{C}$) is isomorphic to $\dim\ \mathrm{Ker} H^{n+1}(H_1(M)) \to H^{n+1}(M)$. Since $H_1(M,\mathbb{Z}) = \mathbb{Z}^r$ we have $\Lambda^i(H_1(M)) = H^i(H_1(M))$ with the isomorphism provided by the cup product. Hence the dimension in (8) is equal to

$$\dim\ \mathrm{Ker}(\Lambda^{n+1}(H_1(M)) \to H^{n+1}(M)).$$

Therefore, using the sequence (6) and the equality $\dim H_n(H_1(M)) = \binom{r}{n}$, we obtain

$$\begin{aligned} &\dim H_n(\tilde{M}) \otimes_{H_1(M,\mathbb{Z})} \mathbb{C} \\ &\quad = \dim\ \mathrm{Ker}(\Lambda^{n+1}(H_1(M)) \to H^{n+1}(M)) + \dim H_n(M) - \binom{r}{n}. \end{aligned}$$

This yields the last claim in Theorem 4.1. □

Notice that the claim 4.1 (iii) is a generalization of a result in [20] and that the space of local systems with non vanishing cohomology in the cases when $Y = \mathbb{C}^2$ and $Y = \mathbb{C}^{n+1}$ was studied in [21] and [22] respectively.

Remark 4.1. The morphism $\Lambda^{n+1}H^1(M) \to H^{n+1}(M)$ in the above Theorem is surjective in the following two cases.

(i) $n = 1$ and $(Y, 0) = (\mathbb{C}^2, 0)$, see [4], Corollary 2.20 ;

(ii) $(Y, 0) = (\mathbb{C}^{n+1}, 0)$ and X a central hyperplane arrangement, see [25], Corollary 5.88.

The following result is a generalization of Example (6.1.8) in [5], where Y was assumed to be a smooth germ and the proof uses properties of the vanishing cycle functor and a generalization of Prop. 4.6 from [22] where the case of X smooth was treated.

Corollary 4.1. *Let F be the Milnor fiber of the reduced germ $g : (Y, 0) \to (\mathbb{C}, 0)$ which defines the divisor X in Y. With the above assumptions, the monodromy action $h^j : H^j(F, \mathbb{C}) \to H^j(F, \mathbb{C})$ is trivial for $j \leq n - 1$.*

Proof. Let ρ_a be the representation sending each elementary loop to the same complex number $a \in \mathbb{C}^*$. Then it is well-known, see for instance [5], Corollary 6.4.9, that

$$\dim H^q(M, \rho_a) = \dim \operatorname{Ker} (h^q - aId) + \dim \operatorname{Ker} (h^{q-1} - aId).$$

The unipotence follows by applying this equality to $a \neq 1$.

To obtain triviality of the monodromy action, notice that due to the Milnor's fibration theorem, the Milnor fiber F is homotopy equivalent to the infinite cyclic cover of M. Hence, it is a quotient of the universal abelian cover $\tilde{M}$ by the action of the kernel of $\pi : \pi_1(M) \to \mathbf{Z}$ where π sends each elementary loop to 1. Let us consider the corresponding spectral sequence:

$$H^p(\operatorname{Ker}\pi, H^q(\tilde{M}, \mathbb{C})) \Rightarrow H^{p+q}(F, \mathbb{C}) \tag{9}$$

for this action of the group $\operatorname{Ker}\pi = \mathbb{Z}^{r-1}$ on the universal abelian cover $\tilde{M}$. Notice that this is a spectral sequence of $\mathbb{C}[t, t^{-1}]$ modules where the action on $H^p(\operatorname{Ker}\pi, H^q(\tilde{M}))$ is the standard action of the generator of $\pi_1/\operatorname{Ker}\pi$ and the action of t on cohomology of the Milnor fiber is the monodromy action. Since, by Theorem 3.2, $H^q(\tilde{M}) = 0$ for $1 \leq q < n$ we have $n - 1$ zero-rows in the term E_2 and hence the isomorphism $H^j(F, \mathbb{C}) = H^j(\operatorname{Ker}\pi, H^0(\tilde{M}))$ for $1 \leq j \leq n-1$. Since the map of the classifying spaces $(S^1)^r \to S^1$ corresponding to the homomorphism π has trivial monodromy, the action of $\pi_1/\operatorname{Ker}\pi$ on $H^j(\operatorname{Ker}\pi, \mathbb{C})$ is trivial for any j in the range $0 \leq j \leq n - 1$ and the claim follows. $\square$

Remark 4.2. One can obtain the triviality of the monodromy action also using mixed Hodge theory, at least for $j < n - 1$. See for details [6], Theorem 0.2. Note that the above proof shows that $\dim H^j(F) = \binom{r-1}{j}$ for $j \leq n - 1$ (cf. [22]).

5. Homology of local systems (higher rank case)

In this section we work with weaker assumptions on the germs $(Y, 0)$ and $(D_j, 0)$ as above. Indeed, we simply need that M has only locally complete intersection singularities (which is weaker than asking $(Y, 0)$ to be an isolated singularity) and that there is a $\mathbb{Q}$-Cartier divisor, say D_1, among the divisors D_j such that the INNC condition for X holds only along D_1. In particular, the singularities of the divisors $D_j \setminus D_1$ for $j > 1$ can be arbitrary.

To start, note that if $D_1 \setminus \{0\}$ is contained in the smooth part of the space $Y \setminus \{0\}$, then one has an elementary loop δ_1 which goes once about the irreducible divisor D_1 (in a transversal at a smooth point). It follows that a rank m local system $\mathcal{L}$ on $Y \setminus X$ which corresponds to a representation

$$\rho : \pi_1(Y \setminus X) \to GL_m(\mathbb{C})$$

gives rise to a monodromy operators $T_1 = \rho(\delta_1)$. Of course, both δ_1 and T_1 are well-defined only up-to conjugacy. The following result should be compared to the vanishing part of Theorem 0.2 in [6].

Theorem 5.1. *Let $\mathcal{L}$ be a local system on M such that*

(i) *M is a locally complete intersection and D_1 is an irreducible $\mathbb{Q}$-Cartier divisor, i.e., there is an integer m such that mD_1 is the zero set of a holomorphic germ;*

(ii) *$D_1 \setminus \{0\}$ is contained in the smooth part of the space $Y \setminus \{0\}$ and X has only normal crossings along $D_1 \setminus \{0\}$;*

(iii) *the corresponding monodromy operator T_1 has not 1 as an eigenvalue.*

Then $H^k(M, \mathcal{L}) = 0$ for $k < n$ and for $k > n + 1$.

Proof. For this proof we assume that the (good) representatives for our germs $Y, D_j, \ldots$ exist as closed analytic subspaces in an open ball B of radius 2ϵ centered at the origin. This implies in particular that Y is a Stein space, as well as $Y \setminus X$, which is the complement of the zero set of a holomorphic function on Y. Such a Stein space has the homotopy type of a CW complex of dimension at most $(n+1)$ by [13], and this already gives $H^k(Y \setminus X, \mathcal{L}) = 0$ for $k > n + 1$.

These representatives are good in the sense that all the germs $Y, D_j, \ldots$ have a conic structure inside the ball B such that the corresponding retractions are the same for all these germs. We represent the links $L_Y, L_X, L_{D_j}, \ldots$ as the intersections of these representatives inside B with a sphere S of radius ϵ. In such a way we have an inclusion $i_\epsilon : L_Y \to Y$ and a retraction $r_\epsilon : Y^* \to L_Y$, with $Y^* = Y \setminus \{0\}$, which induces inclusions and retractions for the other germs.

The main tool for the proof below is the theory of constructible (resp. perverse) sheaves. For all necessary background material on this subject we refer to [16] and [5].

Let $i : Y \setminus X \to Y \setminus D_1$ be the inclusion and set $\mathcal{F}^* = Ri_*\mathcal{L} \in D_c^b(Y \setminus D_1)$, $\mathcal{F}_1^* = \mathcal{F}^*|(L_Y \setminus L_{D_1})$. The constructible sheaf complex $\mathcal{F}^*$ has constant cohomology sheaves along the fibers of the retraction r_ϵ (since the topology is constant along such a fiber). It follows, as in Lemma 2.7.3 in [16], that

$$H^k(Y \setminus X, \mathcal{L}) = \mathbb{H}^k(Y \setminus D_1, \mathcal{F}^*) = \mathbb{H}^k(L_Y \setminus L_{D_1}, \mathcal{F}_1^*).$$

Let $j_1 : L_Y \setminus L_{D_1} \to L_Y$ be the inclusion and note that

$$Rj_{1*}\mathcal{F}_1^* = Rj_{1!}\mathcal{F}_1^*$$

exactly as in [7] and [2], the key points being the assumptions (ii) and (iii) in the above statement. Since the link L_Y is compact, it plays the role of the compact algebraic variety in [7] and [2], and we get

$$\mathbb{H}^k(L_Y \setminus L_{D_1}, \mathcal{F}_1^*) = \mathbb{H}_c^k(L_Y \setminus L_{D_1}, \mathcal{F}_1^*)$$

for any integer k.

The new difficulty we encounter here is that $L_Y \setminus L_{D_1}$ is not a Stein space (not even properly homotopically equivalent to a Stein space as the retraction $r_\epsilon : Y \setminus D_1 \to L_Y \setminus L_{D_1}$ is not proper!), hence the vanishing for the last hypercohomology group is not obvious.

We proceed as follows. We apply first Poincaré-Verdier Duality on the real semialgebraic set $L_Y \setminus L_{D_1}$ and get

$$\mathbb{H}_c^k(L_Y \setminus L_{D_1}, \mathcal{F}_1^*)^\vee = \mathbb{H}^{-k}(L_Y \setminus L_{D_1}, D_\mathbb{R}\mathcal{F}_1^*).$$

Here $D_\mathbb{R}\mathcal{F}_1^*$ is the dual sheaf of $\mathcal{F}_1^*$ in this real setting. Note that we can also consider the (complex) dual sheaf $D\mathcal{F}^* \in D_c^b(Y \setminus D_1)$. It follows that

$$D\mathcal{F}^*|(L_Y \setminus L_{D_1}) = D_\mathbb{R}\mathcal{F}_1^*[1]$$

since the inclusion $L_Y \setminus L_{D_1} \to Y \setminus D_1$ is normally nonsingular in the sense of [10] (this is what corresponds to a non-characteristic embedding in the sense of [16], Definition 5.4.12 in the case of singular spaces).

This yields the following isomorphisms

$$\begin{aligned}\mathbb{H}^{-k}(L_Y \setminus L_{D_1}, D_\mathbb{R}\mathcal{F}_1^*) &= \mathbb{H}^{-k}(L_Y \setminus L_{D_1}, D\mathcal{F}^*[-1]|(L_Y \setminus L_{D_1}))\\ &= \mathbb{H}^{-k-1}(L_Y \setminus L_{D_1}, D\mathcal{F}^*|(L_Y \setminus L_{D_1})).\end{aligned}$$

Here we are again in the presence of a constructible sheaf complex, namely $D\mathcal{F}^*$, whose cohomology sheaves are constant along the fibers of the retraction r_ϵ. This implies that

$$\begin{aligned}\mathbb{H}^{-k-1}(L_Y \setminus L_{D_1}, D\mathcal{F}^*|(L_Y \setminus L_{D_1})) &= \mathbb{H}^{-k-1}(Y \setminus D_1, D\mathcal{F}^*)\\ &= \mathbb{H}_c^{k+1}(Y \setminus D_1, \mathcal{F}^*)\end{aligned}$$

the last isomorphism coming from Poincaré-Verdier Duality on the algebraic variety $Y \setminus D_1$.

Now it is time to note that the shifted local system $\mathcal{L}[n+1]$ is a perverse sheaf on the locally complete intersection variety M and hence $\mathcal{F}^*[n+1]$ is a perverse sheaf on the variety $Y \setminus D_1$ since the morphism i is Stein and quasi-finite. It follows that

$$\mathbb{H}_c^{k+1}(Y \setminus D_1, \mathcal{F}^*) = \mathbb{H}_c^{k-n}(Y \setminus D_1, \mathcal{F}^*[n+1]) = 0$$

for any $k < n$ by Artin's Vanishing Theorem in the Stein setting, see [16] Proposition 10.3.3 (iv) and Theorem 10.3.8 (ii). □

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