MEROMORPHIC FUNCTIONS WITHOUT REAL CRITICAL VALUES AND RELATED BRAIDS

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Abstract. We consider an open subset of the Hurwitz space consisting of meromorphic functions of a given degree defined on closed Riemann surfaces of a given genus and having no real critical values. To each function in this open set we assign a braid in the braid group of the underlying closed surface and characterise all braids which might appear using this construction. We introduce the equivalence relation among these braids such that the braids corresponding to the meromorphic functions from the same connected component of the above Hurwitz space are equivalent while non-equivalent braids correspond to distinct connected components. Several special families of meromorphic functions, some applications, and further problems are discussed.

To Seryozha Natanzon, in memoriam

1. Introduction

In what follows, we discuss the connected components of the spaces of meromorphic functions on Riemann surfaces with only simple critical values none of which is real. The main motivation of our study comes from a large assortment of problems in mathematical physics related to the perturbation theory of linear operators, see e.g. [11, 24, 25] and references therein. In this theory, one typically considers perturbations of the form $A + tB$ where A is the initial and B is the perturbing operators while $t \in \mathbb{C}$ is the perturbation parameter.

Under some additional assumptions on A and B one obtains the (analytic) spectral curve $\Upsilon \subset \mathbb{C}^2$ with coordinates (λ, t) where λ is the spectral parameter. The restriction of Υ to any fixed value of t gives the spectrum of the operator $A + tB$ (assumed discrete). Projection of the spectral curve $\Upsilon \subset \mathbb{C}^2$ onto the t-axis defines a meromorphic function on Υ whose branching points are called the *level crossings* of the pencil $A+tB$; they are exactly the values of parameter t for which the spectrum of the pencil is non-simple. Special cases of such meromorphic functions will be the main object of our study. Numerous papers in theoretical and mathematical physics discuss level crossings and their properties for various concrete quantum mechanical and other systems; in particular, they determine the convergence radius for the perturbation expansion. One of the most recent examples of such study can be found in [9] containing a nice introduction to the subject.

If A and B are self-adjoint operators (satisfying with some additional technical and genericity assumptions) one can show that the projection of the spectral curve Υ onto the t-axis is a meromorphic function without real critical values. This phenomenon baptised as *avoided level crossings* has been observed already at the dawn of quantum mechanics in the 1920's. Avoided level crossings mean that the spectrum of the pencil $A + tB$ is simple for all real values of the parameter t. It occurs in many pencils of non-self-adjoint operators as well.

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In mathematical language most of problems about level crossings can be formulated as the questions about real and complex pencils of divisors on algebraic curves. Our main goal below is to study the notion of avoided level crossings from the perspective of algebraic geometry and, especially, braid theory using, in particular, the mapping class groups of the underlying surfaces.

For a start, consider the simplest case of pencils of binary forms. Given a pair $P_1(u,v)$ and $P_2(u,v)$ of binary forms of degree d with complex coefficients, consider the real pencil

$$\mathcal{P}(u,v,\alpha:\beta) := \alpha P_1(u,v) + \beta P_2(u,v) \tag{1.1}$$

where $(\alpha:\beta)$ are homogeneous coordinates on $\mathbb{R}P^1$. One can easily observe that if P_1 and P_2 are generic then, for every point $(\alpha:\beta)\in\mathbb{R}P^1$, the form $\mathcal{P}(u,v,\alpha:\beta)$ will have d distinct simple roots in $\mathbb{C}P^1\simeq S^2$. Furthermore, the space Θ_d^0 of generic real pencils is disconnected and the physics problems we mentioned above lead, in particular, to the question of enumeration of these components.

A natural invariant of such connected component can be obtained as follows. Identiying $\mathbb{RP}^1$ with an oriented circle and considering the roots of $\mathcal{P}(u,v,\alpha:\beta)=0$ for each $(\alpha:\beta)$ we obtain a loop in the space of bivariate homogeneous polynomials of degree d without multiple roots, i.e. a spherical braid. The following natural questions arise:

– do different components of Θ_d^0 correspond to non-equivalent braids?

– which braids in the braid group of S^2 appear as the braids of such pencils?

Complexification of the above real pencil can be viewed as a map $\mathbb{CP}^1\to\mathbb{CP}^1$ assigning to $(u,v)\in\mathbb{CP}^1$ the parameter $(\alpha:\beta)$ such that (u,v) is one of the roots of $\mathcal{P}(u,v,\alpha:\beta)=0$. Our braid then coincides with the pre-image of $\mathbb{RP}^1\subset\mathbb{CP}^1$ in the sense that the union of its strands is exactly the pre-image of the real line.

The above construction of the braid as well as the mentioned questions about connected components have an obvious extension to the case when the source of the map is a Riemann surface of an arbitrary genus. The corresponding analogs of the sets Θ_d^0 are the open subsets $\mathcal{H}_{g,d}^{nr}\subset\mathcal{H}_{g,d}$ of the (small) Hurwitz space which, by definition, consists of all degree d meromorphic functions on closed Riemann surfaces of genus g with simple critical values none of which is real. To a connected component of $\mathcal{H}_{g,d}^{nr}$ we assign the equivalence class of the braid of a close surface of genus g; cf. [2] and Section 2 for basic definitions and for an appropriate equivalence relation.

Precise description of the above equivalence relation uses the action of the mapping class group of a closed surface of genus g exceeding 1 on the conjugacy classes of the braid group of the surface on d strings; in the case of surfaces of genus 0 or 1 we use its quotient by the center. This action comes from the classical exact sequence relating the mapping class group of a surface with that of this surface with additional marked points (cf. section 2.3). Braids in the braid group of the surface which we construct are very special and belong to natural abelian subgroups of the braid groups of surfaces which we call the “boundary braids”. We give the characterization of boundary braids in terms of the geometry of the surface, in terms of generators of the braid group of surfaces and the corresponding to these braids classes of the mapping class group of a surface with marked points in terms of Nielsen-Thurston classification (cf. Section 2.4).

Our main results show that connected components of $\mathcal{H}_{g,d}^{nr}$ are enumerated by the equivalences classes of such braids and we give a characterization of the braids corresponding to such components.

These results on the enumeration of the connected components of the above open subspace of the Hurwitz space which is defined in terms of its real structure can be compared with those of the article [4] in which the main object of study is a cellular decomposition of a (different) open subset of $\mathcal{H}_{g,d}$ formed by the so-called *lemniscate generic functions*. These are meromorphic functions having all critical values with distinct absolute values assumed different from zero and infinity; this situation can be also related to the real structure of the Hurwitz space. Connected components appearing below have more complicated fundamental groups than contractible cells of [4]; they are rather homeomorphic to Hurwitz spaces of different degrees and genera.

The content of the article is as follows. In § 2 we introduce the subgroups of boundary braids to which the braids of meromorphic functions naturally belong and recall some basic material about the braid groups as well as the mapping class groups on surfaces. We describe the boundary braids in terms of the (standard) generators of the braid group, viewing braids in terms of the relevant configuration spaces. Further we give a description of the classes in the mapping class group corresponding to the boundary braids by means of the Nielsen–Thurston theory. In § 3 we address the enumeration of the connected components of $\mathcal{H}^{nr}_{g,d}$ by relating them to the classification of our braids of meromorphic functions. Our main result is a description of the connected components of $\mathcal{H}^{nr}_{g,d}$ as the orbits of the mapping class group of a closed surface acting on the conjugacy classes of the subgroups of boundary braids. (However in the case of surfaces of genus zero and one, the action is on the conjugacy classes of the quotient of the braid group by the respective centers; this specifies the equivalence class of the braid since each coset of the center determines the braid we attach to a point in the Hurwitz space.) In § 4 we discuss special classes of meromorphic functions f. For example, we look at meromorphic functions induced by generic projections of plane curves. Finally, in § 5 we suggest some further directions of study and make additional remarks about the role of planarity in this circle of questions, see Problem 7. We formulate several problems more closely related to mathematical physics in which one considers the restrictions of connected components of $\mathcal{H}^{nr}_{g,d}$ to specific families of Riemann surfaces such as plane curves coming from real pencils of matrices etc. Restrictions of our connected components to such families may split further; in order to understand and enumerate these connected components novel techniques/ideas should be applied. It seems likely that in the case of projections of plane real curves techniques related to the Hilbert 16-th problem might be useful. We plan to investigate this subject further; as an example of similar activity we explicitly calculate the braids obtained from real plane curves, but without real points, see § 4.

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2. Boundary braids in the braid groups of surfaces and systems of ovals.

In this section we describe the class of braids which will later appear in our enumeration of connected components of the spaces of meromorphic functions without real critical values. These braids are defined as the elements of certain abelian subgroups of the braid groups of surfaces which we call *the subgroups of boundary* *braids*. We describe boundary braids algebraically, i.e. in terms of the (standard) generators of the braid groups of surfaces, and also as the elements of the mapping class groups using the Nielsen-Thurston classification of the latter.

2.1. **Boundary braids.** Recall that the braid group $B_n(F,k)$ of n strands of a surface F with k punctures and with (possibly nonempty) boundary is defined as the fundamental group of the configuration space of n points on F (cf. [2], [8]). More precisely,

$$B_n(F,k) = \pi_1(((F-[k])^n \setminus \Delta)/Sym_n, f) \qquad (2.1)$$

where, as above, $[k] \subset F$ is a subset consisting of k fixed points, and the diagonal Δ is defined as

$$\Delta = \{(x_1,\dots,x_n)|\ x_i \in F\setminus[k],\ i=1,\dots,n,\ \text{and}\ \ \exists i \neq j \text{ such that } x_i = x_j\}.$$

In (2.1) the quotient is taken with respect to the free action of the symmetric group Sym_n on n letters acting by permuting the components in the Cartesian product $(F\setminus[k])^n$; the base point f of the fundamental group is a Sym_n-orbit of some point in $(F\setminus\Delta)^n$.

Let D^* be a punctured disk and $\beta_i \in B_{m_i}(D^*) = B_{m_i}(D,1)$, $i=1,\dots,k$ be a collection of braids in D^*. Define $d := \sum_1^k m_i$. For each $1 \le i \le k$, select a homeomorphism between D^* and the annulus which is the closure of the collar of the i-th connected component of the boundary of F. Such collection of homeomorphisms induces the homomorphism $\Phi_{m_1,\dots,m_k} : \prod_i B_{m_i}(D^*) \to B_d(F,k)$.

Definition 1. Let F be a surface with $k \ge 1$ punctures and let $\beta_i \in B_{m_i}(D^*)$ be a rotation about the puncture of D^* by an angle which is an integer multiple of $\frac{2\pi}{m_i}$ [1] A boundary braid of $d := \sum m_i$ strands on a punctured surface F is a braid in the abelian subgroup of $Br_d(F,k)$ formed by the braids $\Phi_{m_1,..,m_k}(\beta_1,\dots,\beta_k) \in Br_d(F)$ with $(\beta_1,\dots,\beta_n) \in \prod_i B_{m_i}(D^*)$.

The full boundary braid is the braid in the subgroup of boundary braids where β_i is the positive rotation by $\frac{2\pi}{m_i}$, (i.e. β_i is the positive generator of $B_{m_i}(D^*)$).

Denote by $BB_{m_1,\dots,m_k}(F)$ the subgroup of boundary braids in $B_d(F,k)$ and we shall sometimes refer to a conjugate of this subgroup in $B_d(F,k)$ as a subgroup of boundary braids as well.

Definition 2. A *boundary braid in a closed oriented surface* E is the image of a boundary braid in a punctured surface F under the homomorphism induced by a homeomorphism of the surface F onto the interior of a proper subsurface of E, (cf. [21], Sect. 2).

Alternatively, a braid in $B_d(E)$ is a boundary braid if and only if there exists a collection of *oriented simple closed curves* $\alpha_1,\dots,\alpha_k$ in E such that $E \setminus \cup_i \alpha_i$ has at least two connected components and this braid is the image of a boundary braid of one of these components. In particular, one can view a boundary braid as the result of a rotation of points equidistantly placed along the curves α_i.

Definition 3. A collection of ovals (i.e. simple closed curves) in E is called a *separating oriented collection* if

a) connected components of the complement to the union of ovals in E are split into two disjoint classes called the *positive* and the *negative classes* respectively in such a way that each oval belongs to the boundaries of exactly two connected components, one in each class,

[1] i.e. the loop β_i in (2.1) with $F = D^*$ corresponds to the rotation of the set of m_i points positioned equidistantly on a circle in D^* centered at the puncture mapping this set of m_i points onto itself

b) we assume that orientations of the ovals are consistent with the above splitting into the positive and the negative connected components in the following sense:

i) the orientation of the closure of each connected component of the complement to the union of ovals in the positive class is induced from the orientation of the ambient closed surface;

ii) the orientation of each oval is such that together with the normal vector pointing inside the component from the positive class to which this oval belongs induce the positive orientation of this connected component.

From Definitions 2 – 3 we obtain that each separating oriented collection of ovals corresponds to the subgroup of boundary braids in the braid group of the surface E. Moreover, the subgroups of boundary braids contain semigroups of positive braids, corresponding to rotations of the ovals in the position direction. We will refer to a braid in this abelian subgroup of $B_d(E)$ as a *boundary braid corresponding to the chosen separating oriented collection of ovals.*

In what follows, we need a result about the maps of braid groups induced by embeddings. Let $F \subset E$ be a proper *subsurface of a closed surface* E. Assume now that $p_1, \dots, p_n \in F$ and $p_{n+1}, \dots, p_m \in E \setminus F$.

Proposition 1 (see [21])**.** *Let F be a connected subsurface of E such that none of the connected components of $\overline{E \setminus F}$ is a disk or each connected component contains the points $p_i, i = 1, \dots, m$. Then the homomorphism*

$$B_n(F) \to B_m(E) \tag{2.2}$$

is injective.

The next important proposition describes the structure of the subgroups of boundary braids.

Proposition 2. *For any k-tuple of positive integers $m_1, m_2, \dots, m_k$, $k \geq 1$, the group $BB_{m_1,\dots,m_k}$ is a free abelian subgroup of $Br_d(F, k), d = \sum_1^k m_i$ of rank k. For a closed oriented surface E, the subgroup of boundary braids is free abelian whose rank equals the number of ovals, except for the case $E = S^2$ with $k = 1$. In the latter situation, $BB_m(S^2) = \mathbf{Z}_{2m}$.*

Proof. Recall that there exists a natural surjection $B_d(F, k) \to Sym_d$ sending a braid to the induced permutation of the initial points of the strands. Its kernel $PB_d(F, k)$ (called the *pure braid group*) is the fundamental group of the configuration space $Conf_d(F, k)$ of *ordered* collections of d points in the punctured surface F with the set of punctures $[k]$, (cf. [2], [8]). In other words,

$$Conf_d(F, k) = \left[\prod_1^d (F \setminus [k])\right] \setminus \Delta,$$

where Δ is the collection of d-tuples having at least two coinciding components. Moreover, the group Sym_d acts freely on $Conf_d(F, k)$ and the group $B_d(F, k)$ is the fundamental group of the quotient $Conf_d(F, k)/Sym_d$.

The map $\Phi := \Phi_{m_1,\dots,m_k} : \prod_j B_{m_j}(D^*) \to B_d(F, k)$ can be viewed as the homomorphism of the fundamental groups induced by the map of the quotients of configuration spaces:

$$\prod_{i=1}^k Conf_{m_i}(D^*)/Sym_{m_i} \to Conf_d(F, k)/Sym_d. \tag{2.3}$$

Let $Rot_{m_1,\dots,m_k}$ be the subgroup of $\prod_j B_{m_j}(D^*)$ consisting of the braids $(\beta_1, \dots, \beta_k)$ where each β_j is some power of the rotation by the angle $\frac{2\pi}{m_j}$. Obviously, $Rot_{m_1,\dots,m_k}$

is a free abelian group with k generators. Therefore, to show the freeness of $BB_{m_1,\dots,m_k} = \Phi(Rot_{m_1,\dots,m_k})$ and the fact that its rank equals k, it suffices to show that the intersection of this image with $PB_d(F,k)$ is a free abelian group of rank k. Notice that in the diagram

$$\begin{array}{ccc} \Phi(Rot_{m_1,\dots,m_k}) \cap PB_d(F,k) & \to & PB_d(F,k) \\ \downarrow & & \downarrow \\ \Phi(Rot_{m_1,\dots,m_k}) & \to & B_d(F,k) \\ & & \downarrow \\ & & Sym_d \end{array} \qquad (2.4)$$

the vertical arrows are injective and have the subgroups of finite index as their images. Now if $\Phi(Rot_{m_1,\dots,m_k})$ either has torsion or has rank smaller than k then the latter diagram implies that $\Phi(Rot_{m_1,\dots,m_k}) \cap PB_d(F,k)$ will have rank strictly smaller than k.

The group $\Phi(Rot_{m_1,\dots,m_r}) \cap PB_d(F,k)$ is the image in $B_d(F,k)$ of the finite index subgroup of $Rot_{m_1,\dots,m_k}$ generated by the full twists of m_i points in the disk D^* corresponding to the i-th puncture. To show that this group is free abelian of rank k it suffices to check that the image of the subgroup $\Phi(Rot_{m_1,\dots,m_r}) \cap PB_d(F,k)$ in the abelianization of the pure braid group $PB_d(F,k)$ is a free abelian group of rank k. Let us identify the abelianization of the fundamental group of a connected topological space X with the first homology group $H_1(X,\mathbb{Z})$.

As a model of D^* let us take the disk in $\mathbb{C}^*$ centered at 0 and with radius 2. As an m-tuple of points moving in D^* we take the points $\omega_m^j, j = 0,\dots,m-1$ where $\omega_m = exp(\frac{2\pi i}{m})$ is the primitive root of unity of degree m. The class of the full twist in $B_m(D^*)$ or $PB_m(D^*)$ is represented in the abelianization $H_1(D^*)^m \setminus \Delta, \mathbb{Z})^2$ of $PB_m(D^*)$ by the homology class of the simple loop $(\dots,\omega_m^j e^{\frac{2\pi\sqrt{-1}t}{m}},\dots)$. Here $j = 0,1,\dots,m-1$ and $0 \le t \le 1$. The map

$$H_1((D^*)^m \setminus \Delta) \to H_1((D^*)^m,\mathbb{Z}) = \mathbb{Z}^m$$

induced by the inclusion sends this class to the class $(1,\dots,1) \in H_1((D^*)^m,\mathbb{Z})$. Hence its image in $H_1(D^*,\mathbb{Z})$ under the map $H_1((D^*)^m,\mathbb{Z}) \to H_1(D^*,\mathbb{Z})$ sending a homology class to the sum of all its coordinates equals m times the class positive generator of $H_1(D^*,\mathbb{Z})$, i.e. to the boundary of a small disk in D^* centered at the puncture.

Now consider a similar composition of the maps:

$$H_1((F \setminus [k])^d \setminus \Delta,\mathbb{Z}) \to H_1((F \setminus [k])^d,\mathbb{Z}) \to H_1((F \setminus [k])^k,\mathbb{Z}) \qquad (2.5)$$

induced by the inclusion and addition in the homology respectively. We claim that for $k > 1$, the images (of the classes) of each of k generators of $Rot_{m_1,\dots,m_k}$ span a free abelian group in the latter homology group. Indeed, notice that $H_1(F \setminus [k]) \simeq \mathbb{Z}^{k-1} \oplus \mathbb{Z}^{2g}$, where $\mathbb{Z}^{k-1}$ has k generators each being the class of the boundary of a small disk centered at the respective punctures subject to one relation. By the Künneth formula,

$$H_1((F \setminus [k])^k) = \oplus_1^k H_1(F \setminus [k]) = \oplus_1^k(\mathbb{Z}^{k-1} \oplus \mathbb{Z}^{2g}).$$

Now for $j = 1,\dots,k$, the class of the braid which makes a complete turn about the j-th puncture and which is trivial near the others, is given by the element having all vanishing components except for the entry m_j in the j-th direct summand. This class is non-trivial unless $k = 1, g = 0$ in which case we have the braid corresponding to the rotation of m uniformly distributed along a circle points which clearly has infinite order (as the full twist in the Artin braid group).

[2]as above, Δ denotes the diagonal consisting of collections of points in D^* having at least two equal components.

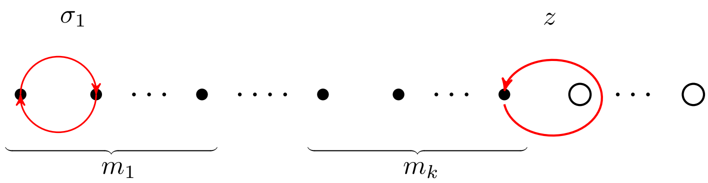

FIGURE 1. Illustration of the generators σ_1 and z from Proposition 3

For $k \geq 2$, the case of closed surfaces follows from that of punctured surfaces and Proposition 1. If $k = 1$ and one of components of the complement to a considered collection of ovals is a disk, then assuming that the second component is not a disk, but the rotating points are in the disk, we also obtain that the group of boundary braids is infinite cyclic. Finally, in the remaining case $k = 1$ when both components of the complement to the oval are disks, we obtain the cyclic group of order $2m$ since the full twist of m points in the braid group of the 2-dimensional sphere has order 2, (cf. [8]).

□

Remark 1. The group of boundary braids is a subgroup of the braid group of the disconnected surface $\sqcup_k D^*$. We can show the injectivity of its image in $B(F, k)$, but this fact does not immediately follow from similar results of [21] since the argument in loc. cit. is only applicable to embeddings of connected surfaces, (cf. also 1).

The following is an immediate consequence of Proposition 2.

Corollary 1. *The full boundary braid has infinite order in $Br_d(F)$ unless $F = S^2$ and the braid corresponds to a single oval. In the latter case its order is $2d$.*

2.2. **Boundary braids in terms of generators.** Let F be an orientable surface of genus g with k punctures. One can view F as a polygon P with $4g$ sides and with k punctures $[k] \subset P$ represented by small disks deleted from the polygon. We mark d points in P, whose movements will form the braids in the braid group $B_d(F, k)$ of d strands and place the points and punctures along a horizontal segment in P in such a way that d points come first and are ordered from left to right; they are followed by k punctures (shown by small disks below). One identifies the opposite sides of the polygon with appropriate orientation so that they correspond to $2g$ generators of $\pi_1(F, b)$ where b is the image of the vertex, see Fig. 2.

Recall the definition of *a good ordered system of generators* of $\pi_1(D \setminus [N], p_0)$, where D is a disk (which will be taken in P), $[N]$ is a subset of D consisting of N points (or punctures) positioned along a segment and $p_0 \in \partial D$, (cf. [15]). Such system corresponds to an ordered collection of loops; each loop is obtained by at first moving from p_0 along a straight segment connecting it to a point on the boundary of one of non-intersecting disks centered at all N selected points in D, then traversing the boundary of the respective disk in the counterclockwise direction and finally returning back to p_0 along the same straight segment.

In these notations we have the following result.

Proposition 3 (cf. [1]). *The set of standard generators of $B_d(F, k)$ consists of the following three groups:*

(i) σ_i, $i = 1, \dots, d-1$ *(Artin's generators of the braid group of a disk);*

(ii) a_i, b_i, $i = 1, \dots, g$, *(braids in which only the point p_1 moves through the walls of the pairs of identified sides of P while other points p_i, $i > 1$ are fixed; on the*

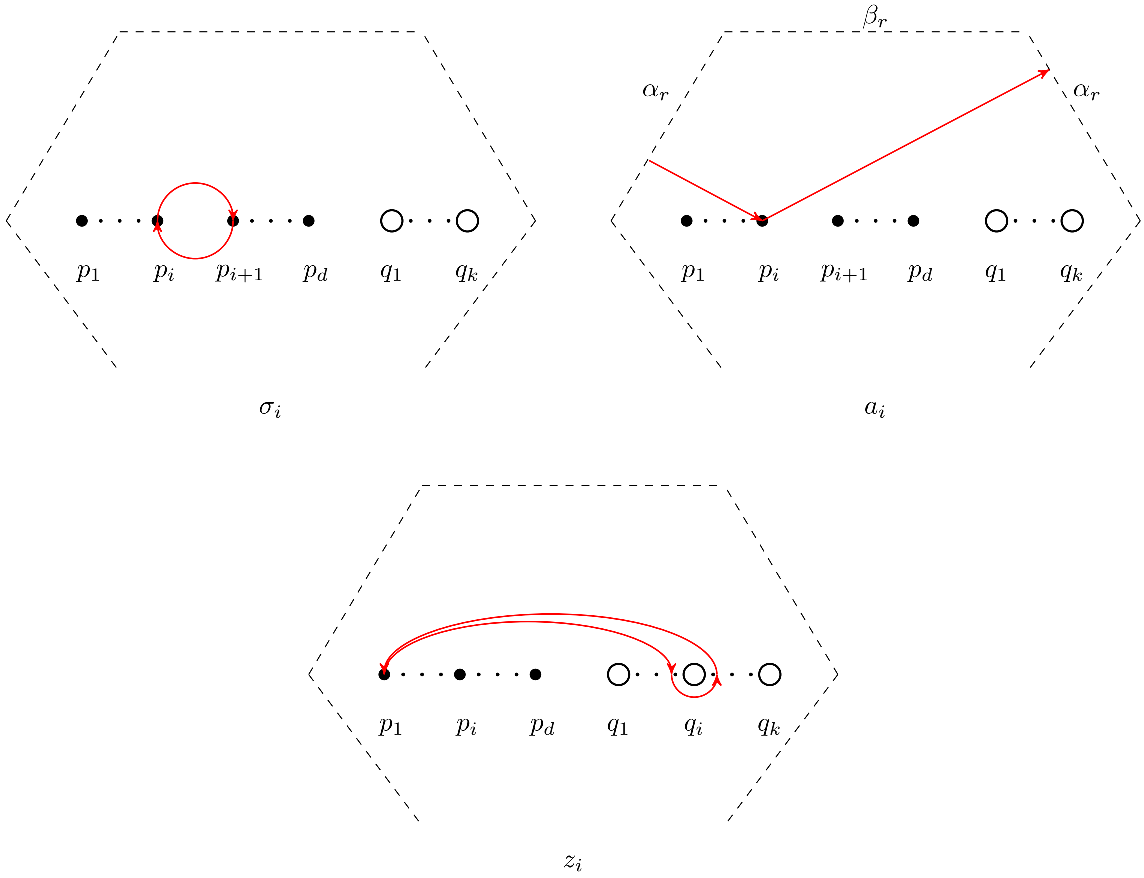

FIGURE 2. Generators of the braid group $B_d(F,k)$.

closed surface these generators correspond to the motion along one of $2g$ standard generators of the fundamental group of a closed surface of genus g).

(iii) z_i, $i = 1, \dots, k-1$ *(braids in which only the point p_1 is moving along the loops encircling one of the first $k-1$ punctures of a good ordered system of generators of the fundamental group of the complement to the punctures in the polygon P; the braids represented by similar loops encircling one of the remaining marked points are conjugates of z_i by some of Artin's generators σ_i's).*

Remark 2. For the sake of completeness, we presented in § 6 below the set of relations among the latter standard generators. However these relations will not be explicitly used in the present article.

The following Proposition gives an algebraic characterization of the boundary braids in terms of the standard generators from Proposition 3. This description is parallel to that of the braids corresponding to loops around projections of singular points of plane curves via braid monodromy, (cf.[15]). We use the following relation between good ordered systems $(w_1, \dots, w_{k-1})$ and $(w'_1, \dots, w'_{k-1})$ given by the Hurwitz moves:

$$\tau_i : (w_1, \dots, w_{k-1}) \to (w_1, \dots, w_{i-1}, w_i w_{i+1} w_i^{-1}, w_i, w_{i+1}, \dots, w_{k-1}) \quad i = 1, \dots, k-2. \tag{2.6}$$

Proposition 4. *Let $p_1^1, \dots, p_{m_1}^1, p_1^2, \dots, p_{m_2}^2, \dots, p_1^k, \dots, p_{m_k}^k$ be a collection of $d = \sum_1^k m_i$ moving points in the braids group $B_d(E,k)$ of a closed surface E with k*

punctures $q_1, \dots, q_k$. *Assume that they are positioned inside the polygon* P *used in Theorem 2 and aligned along a straight segment (which can be thought of as a segment on* $\mathbb{R} \subset \mathbb{C}$*). Assume that the latter points are ordered left-to-right as* $p^1_1, \dots, p^1_{m_1}, p^2_1, \dots, p^2_{m_2}, \dots, p^k_1, \dots, p^k_{m_k}, q_1, \dots, q_k$. *Let* $\sigma^1_1, \dots, \sigma^k_{m_k-1}$ *be the standard generators of the braid group of* $d = \sum_1^k m_i$ *strands corresponding to the clockwise half-twists of pairs of consecutive points and let* z *be the braid corresponding to the motion of the rightmost point* $p^k_{m_k}$ *around the puncture* q_1 *in the counterclockwise direction. Then the following facts hold.*

a) *In the above notation, in the subgroup* $B_d(k) \subset B_d(E, k)$ *generated by the braids* $\sigma^1_1, \dots, \sigma^k_{m_k-1}$*, together with* $z_1, \dots, z_k$ *considered in Proposition 3, the above braid* z *is conjugate to the braid* z_1*. Additionally, the boundary braid obtained as the clockwise rotation of the collection of points* $p^k_1, \dots, p^1_{m_k}$ *about the puncture* q_1 *is given by*

$$z^{-1}\sigma^k_{m_{k-1}} \dots \sigma^k_1. \tag{2.7}$$

(Here the composition is written left-to-right).

b) *The boundary braid corresponding to the rotation of the collection of points* $p^i_1, \dots, p^i_{m_i}$ *about the* j*-th puncture* q_j *for* $(i, j) \neq (k, 1)$ *has the same form (2.7) in which* z *is conjugate to the generator* z_j *in* $B_d(k)$ *and the factors* σ^k_l, $l = 1, \dots, m_k$ *are replaced by the factors* $\tilde\sigma^i_1, \dots, \tilde\sigma^i_{m_i-1}$ *obtained by application of a sequence of Hurwitz moves (2.6) to* $\sigma^i_1, \dots, \sigma^i_{m_i-1}$.

Proof. Formula (2.7) follows immediately from the algebraic form of rotations in Artin's braid group (see e.g. [8], Sec. 9.2). It gives the required algebraic form in the case when the rotating collection of points and the puncture about which this collection is rotating are adjacent to each other.

In the remaining cases, applying a diffeomorphism of the disk containing all the collections of points $\mathbf{p} = \{p^i_{i'}\}$, $\mathbf{q} = q_1, \cdots, q_k$ which moves $p^i_1, \dots, p^i_{m_i}$ to the right in such a way that they will be located to the right of all remaining points in $\mathbf{p}$ and to the left of all points in $\mathbf{q}$ leads to the relation (2.7) in which the standard generators corresponding to the moving points p^i_j are used as σ's. (The correspondence between the action of the diffeomorphism group of the disk on the braids and the Hurwitz moves allows us to express these new generators in terms of the original ones.) □

2.3. **Action of the mapping class group on boundary braids.** Denote by g the genus of a closed oriented surface E and by $Mod(E)$ the mapping class group of E; we let $Mod(E, d)$ be the mapping class group of E with d marked points.

Recall that one has the exact sequence:

$$Br_d(E) \to Mod(E, d) \to Mod(E) \to 0, \tag{2.8}$$

(cf. [8], Theorem 9.1 or [2]).

If either $E \simeq S^2$ is a 2-sphere or $E \simeq T^2$ is a 2-torus, then we get

$$0 \to \mathbf{Z}_2 \to Br_d(S^2) \to Mod(S^2, d) \to 0 \tag{2.9}$$

and

$$0 \to C \to Br_d((T^2) \to Mod(T^2, d) \to SL_2(\mathbb{Z}) \to 0$$

respectively, where C is the center of $Br_d(T^2)$ and $\mathbf{Z}_2$ is the center of $Br_d(S^2)$, (cf. [8], p. 245 or [2], Theorem 4.3).

In case when the surface E has a negative Euler characteristic, the left homomorphism in the sequence (2.8) is injective, i.e.

$$0 \to Br_d(E) \to Mod(E, d) \to Mod(E) \to 0. \tag{2.10}$$

In particular, the mapping class group $Mod(E)$ acts on the conjugacy classes of $Br_d(E)$ (and on its quotients by the centers in the cases of non-negative Euler characteristic (2.9)).

On the other hand, the mapping class group $Mod(E)$ also acts on the isotopy classes of separating collections of ovals. The next proposition compares its action on the conjugacy classes of braids or the subgroups of boundary braids and on the isotopy classes of separating oriented collections of ovals, (cf. Definitions 2, 3).

Proposition 5. *Two subgroups of boundary braids (resp. the two full boundary braids) with respect to two collections of ovals belong to the same orbit of the mapping class group if and only if the isotopy classes of oriented collections of ovals belong to the same orbit of the mapping class group. In case of S^2, the conjugacy class of a boundary braid is determined by the isotopy class of the oriented collection of ovals.*

Proof. Indeed, let ϕ be a diffeomorphism of a closed surface E such that for two systems of ovals $\{\alpha_i\}$ and $\{\beta_i\}$, one has $\phi(\alpha_i) = \beta_i$ for every i. Let F be one of the connected components of $E \setminus \cup_i \alpha$. Then ϕ induces the map between the braid groups of each connected component of F and the respective connected component of $\phi(F)$. Note that since the points at which the image of the braid in F and the braid in $\phi(F)$ are based (cf. definition of a braid in § 1) do not necessarily correspond to each other, the identification of the subgroups of boundary braids requires a choice of a path in the configuration space connecting the points at which these braids are based. This circumstance leads to ambiguity up to conjugation.

And vice versa, if two full boundary braids δ_1, δ_2 corresponding to two subsurfaces F_1, F_2 of a surface E with negative Euler characteristic lie in the conjugacy classes belonging to the same orbit of $Mod(E)$, then the elements of $Mod(E, d)$ corresponding to the braids δ_i are conjugate by an element $\gamma \in Mod(E, d)$. Then a diffeomorphism representing γ sends the collection of ovals determining the group of boundary braids containing δ_1 to a collection of ovals determining the subgroup of the boundary braids containing δ_2. This observation settles the claim. □

2.4. **Boundary braids and Nielsen–Thurston classification.** Here we will describe the images of the boundary braids in the mapping class group in terms of the Nielsen–Thurston classification, see (2.8). (We will use [8] as the main reference for this material).

Recall that to an element $[f]$ of a mapping class group containing a diffeomorphism $f \in Diff^+(E, d)$ one associates a *reduction system*, i.e. a system of (defined up to isotopy) pairwise non-intersecting simple closed curves $c_i,\ i = 1, \dots, N$ in $E \setminus [d]$ such that

(i) each c_i is essential, i.e. non–contractible in $E \setminus [n]$;
(ii) $f(c_i) = c_i$.
(iii) on each connected component of the complement to a canonical reduction system, the restriction of the diffeomorphism is either periodic or pseudo–Anosov.

Moreover, one obtains *the canonical reduction system* by taking the intersection among all maximal reduction systems. (We refer to [8], Sect. 13.2.2 for the details of this construction). A diffeomorphism $f \in Diff^+(E, d)$ is called *reducible* if its canonical reduction system is non–empty. By the Nielsen–Thurston classification, any diffeomorphism $f \in Diff^+(E, d)$ is either periodic, reducible, or pseudo–Anosov, (cf. [8], Theorem 13.2).

The following result describes boundary braids in terms of the Nielsen–Thurston theory.

Proposition 6. *A braid in $Br_d(E)$ is a boundary braid if and only if the corresponding mapping class induces the identity class on each connected component of the complement to its canonical reduction system different from an annulus and it induces a periodic class on each connected component homeomorphic to an annulus. In particular, boundary braids are pseudo–periodic, (cf. terminology used in* [14]*).*

Proof. Let β be a boundary braid corresponding to a collection of braids $\beta_i \in Br_{m_i}(D_i^*)$, $i = 1, \dots, k$. The images of the boundaries of punctured disks D_i^* in F composed with the embedding into E from Definitions 1 and 2 give collection of simple closed curves c_i, $i = 1, \dots, k$. Those of the latter curves which are not null–homotopic in $E \setminus [d]$ form a reduction system which is canonical. Furthermore, the restriction of a homeomorphism representing β on E is trivial on all connected components of $E \setminus \cup c_i$ which are different from either of the punctured disks. (Here $\cup c_i$ is the union of all c_i's in the canonical reduction system). On each of the punctured disks the corresponding diffeomorphism is a rotation above the respective puncture having a finite order in $Diff^+(D^*)$ (i.e. twice punctured sphere, cf. [8], Sect. 7.1.1).

Vice versa, given a canonical reduction system of the diffeomorphism corresponding to a braid $\beta \in Br_d(E)$, let $F_i, i = 1, \dots, k$ be the collection of components of the complement diffeomorphic to an annulus. By assumption of Proposition, the diffeomorphism induces a finite order diffeomorphism in $Diff^+(D^*, m_i) = Diff^+(S^2, m_i + 2)$ fixing two points. Such a diffeomorphism is isotopic to a rotation about the axis containing two punctures ([8], sect. 7.1.1), i.e. corresponds to a braid β_i in $BB_{m_i}(D^*)$. Since the diffeomorphism corresponding to β is isotopic to identity on the complement to $F_i, i = 1, \dots, k$ the braid β is just the product of the braids β_i.

□

3. Meromorphic functions without real critical values.

Recall the problems from the introduction we address in this paper:

Problem 1. Describe the equivalence class of the closed spherical braid attached to a pencil of binary forms as well as the set of the equivalence classes of closed d-stranded spherical braids which might occur for (generic) pencils of binary forms of degree d.

Problem 2. Enumerate connected components of Θ_d^0 of the space of pencils of binary (1.1) such that $\mathcal{P}(u, v, \alpha : \beta) = 0$ has no multiple roots for all $(\alpha : \beta) \in \mathbb{RP}^1$.

Problem 3. Describe the equivalence class of the braid attached to a meromorphic function as well as the set of equivalence classes of all d-stranded braids which might occur on Riemann surfaces of genus g from meromorphic functions of degree d.

Problem 4. Enumerate connected components of $\mathcal{H}_{g,d}^{nr} \subset \mathcal{H}_{g,d}$.

In this section we address all these questions. We start by setting up the notations. Given a meromorphic function $f : E \to \mathbb{C}P^1 \supset \mathbb{R}P^1$ of degree d without real critical values, let $N_f := f^{-1}(\mathbb{R}P^1) \subset E$ be the pre-image of the real line in $\mathbb{C}P^1$. Notice that since f has no real critical values, N_f is a collection $\{O_1, O_2, \dots, O_\ell\}$ of smooth simple closed disjoint curves endowed with orientation corresponding to the positive direction of $\mathbb{R} \subset \mathbb{R}P^1$. The set $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ splits into the upper and the lower half-planes $\mathbb{H}^+$ and $\mathbb{H}^-$ respectively. Each connected component of $E \setminus N_f$ is mapped by f either onto $\mathbb{H}^+$ or onto $\mathbb{H}^-$ and hence the set of connected components splits into *positive and negative classes* (cf. Definition 2) denoted E^+ and E^- respectively. In particular, N_f is *a separating oriented collection of ovals.*

Each oval O_j is assigned a positive integer m_j which is the degree of the restriction $f : O_j \to \mathbb{R}P^1$. Since by our assumption of absence of real critical values, f restricted to O_j has no critical points, we obtain that m_j equals the number of times the oval O_j wraps around $\mathbb{R}P^1$ under f. Obviously, $m_1 + m_2 + \cdots + m_\ell = d$. We will call *a labeled collection of ovals* a collection in which each oval is assigned a positive integer.

Since Problems 1 – 2 are the special cases of Problems 3 – 4, we will concentrate on settling the latter questions. To address Problem 3 consider the (small) Hurwitz space $\mathcal{H}_{g,d}$ of meromorphic functions $f : E \to \mathbb{C}P^1$ of a given degree d on smooth compact Riemann surfaces of genus g with only simple critical values=branching points. Recall that the number of these critical values equals $2d + 2g - 2$. (We consider meromorphic functions up to triangular equivalence, i.e. functions $f_i : E_i \to \mathbb{C}P^1, i = 1, 2$ are called equivalent if and only if there exists a biholomorphic map $\phi : E_1 \to E_2$ such that $f_1 = f_2 \circ \phi$.)

Recall that $\mathcal{H}_{g,d}$ is a smooth quasi-projective complex manifold, (cf. [22]). In fact, since all the critical points of meromorphic functions are assumed to be simple, it is a finite unbranched cover of the configurations space of unordered distinct tuples of points in $\mathbb{C}P^1$ of cardinality $2d + 2g - 2$. Denote by $\mathcal{H}^{nr}_{g,d} \subset \mathcal{H}_{g,d}$ the subset of meromorphic functions without real critical values. ($\mathcal{H}^{nr}_{g,d}$ is a Zariski open subset of $\mathcal{H}_{g,d}$ viewed as a quasi-projective manifold over $\mathbb{R}$ and hence is smooth.) For obvious reasons, it is an open complex submanifold of $\mathcal{H}_{g,d}$.

Definition 4. Let $f : E \to \mathbb{C}P^1$ be a meromorphic function from $\mathcal{H}^{nr}_{g,d}$ and let $N_f := f^{-1}(\mathbb{R}P^1) \subset E$ be the corresponding labeled collection of ovals. The isotopy class of the map $S^1 \to (E^d \setminus \Delta)/Sym_d$ given by $t \to f^{-1}(t) \subset t \times E$ considered as an element in $Br_d(E)$ is called the *braid of* f.

Clearly, a braid in Definition 4 is *the full boundary braid* in the subgroup of boundary braids corresponding to the system of ovals N_f considered in § 2, (cf. Def. 1).

The braid of a meromorphic function can be described in terms of the mapping class group–valued monodromy of a curve on an algebraic surface, (cf. [13]). Recall that a pencil of curves $p : Z \to \mathbb{C}P^1$ on a complex algebraic surface Z and a curve $C \subset Z$ with the set of critical values $Cr \subset \mathbb{C}P^1$ of both p and its restriction to C corresponds the homomorphism of braid monodromy from $\pi_1(\mathbb{C}P^1 \setminus Cr, b)$ into the mapping class group $Mod(E, d)$ where E is a generic fiber of p and d is the intersection index of a fiber of p and C. Specifically, for a loop γ representing a class in $\pi_1(\mathbb{C}P^1 \setminus Cr, b)$ one selects a trivialization of the locally trivial fibration of pair $(p^{-1}(\gamma), p^{-1}(\gamma) \cap C) \subset (Z, C)$ i.e. a map $[0, 1] \times (p^{-1}(b), p^{-1}(b) \cap C) \to (p^{-1}(\gamma), p^{-1}(\gamma) \cap C)$ which for any $t \in [0, 1]$ restricts to a diffeomorphism of $t \times (p^{-1}(b), p^{-1}(b) \cap C)$ onto its image. Then to γ one associate the diffeomorphism of $p^{-1}(b)$ taking the image of a point in $(0, x) \in 0 \times p^{-1}(b) \in [0, 1] \times p^{-1}(b)$ to the image in $p^{-1}(b) \subset p^{-1}(\gamma)$ of the point $(1, x) \in [0, 1] \times p^{-1}(b)$ (cf. [13] for details of this construction). UNCLEAR.

Note that each trivialization of the pair $(p^{-1}(\gamma), p^{-1}(\gamma) \cap C)$ over γ induces a diffeomorphism of the pair $p^{-1}(b), p^{-1}(b) \cap C$ i.e. induces an element in $Mod(p^{-1}(b), d)$ where $d = \text{Card}(p^{-1}(b) \cap C)$. Moreover, a trivialization of the locally trivial fibration $p^{-1}(\gamma) \to \gamma$ by the same construction as outlined above induces a diffeomorphism of $p^{-1}(b)$. The class of this diffeomorphism in $Mod(p^{-1}(b))$ is the image of the class corresponding to γ in $Mod(p^{-1}(b), d)$ by right map in sequence (2.8).

If p is the projection of a direct product with a curve, then the mapping class group–valued monodromy of the pair (Z, C) has as the image a subgroup of the kernel of the right map in the sequence (2.8) since the mapping class group–valued monodromy of fibration of direct product $Z = E \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is trivial. UNCLEAR. In other words, we have the monodromy map with values in $B_d(E)$; in cases when the left map in (2.8) is not injective, the values are in the quotient of the corresponding braid group.

Lemma 1. *Let $f : E \to \mathbb{C}P^1$ be a meromorphic function taken from $\mathcal{H}^{nr}_{g,d}$. Set $Z = E \times \mathbb{C}P^1$ and let $C \subset Z$ be the graph of f. Then the braid of the meromorphic function f is the image of the class of the loop represented by the real line in $\pi_1(\mathbb{C}P^1 \setminus Cr)$ under the mapping class group–valued monodromy of the graph C corresponding to the projection $\nu : Z \to \mathbb{C}P^1$.*

Proof. Note that the critical values of the restriction of the projection ν coincides with the critical set of f. Further, since $f \in \mathcal{H}^{nr}_{g,d}$, the real line in $\mathbb{C}P^1$ defines a loop in $\pi_1(\mathbb{C}P^1 \setminus Cr, 0)$. Comparison of the definitions immediately implies the claim. □

3.1. **Connected components of $\mathcal{H}^{nr}_{g,d}$ and systems of ovals.** The following Proposition gives a condition for two separating oriented collections of ovals to be in the same orbit of the mapping class group in terms of topology of connected components of their complements.

Proposition 7. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two oriented collections of ovals on a surface E and let $E_i^+, E_i^-, i = 1, 2$ be corresponding positive and negative classes of components of the complement to each collection, (cf. Definition 3). The isotopy classes of $\mathcal{C}_1$ and $\mathcal{C}_2$ belong to the same orbit of the mapping class group $Mod(E)$ if and only if there exist one-to-one correspondences between the connected components of E_1^+ and E_2^+ (resp. E_1^- and E_2^-), preserving the topological type of connected components* [3] *and a one-to-one correspondence between the ovals of $\mathcal{C}_1$ and $\mathcal{C}_2$ such that the incidence relation between components and ovals is preserved.*

Proof. Given collections of ovals $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_d$, let ϕ be the element of the mapping class group such that $\phi(\alpha_i) = \beta_i$. Then there exists a diffeomorphism of E representing the class of ϕ satisfying the same relation for the ovals and which induces a diffeomorphsim between corresponding elements of pairs of surfaces into which α_i (resp. β_i) split E. Similarly, the one-to-one correspondence allows us to make a choice of diffeomorphisms between connected components of the elements of pairs into which E is split by each collection of ovals. This choice of diffeomorphisms of ovals extends to a global diffeomorphism of the surface representing the required element of the mapping class group. □

The following lemma describes critical points of generic maps of a surface with a boundary which send this surface to a fixed disk and its boundary to the circle bounding this disk. The space of such generic maps turns out to be connected.

Lemma 2. *Given a smooth connected surface W of genus $g \geq 0$ with $k \geq 1$ boundary components $O_1, O_2, \dots, O_k$ equipped with positive integers (multiplicities) $m_1, m_2, \dots, m_k$, $d = \sum_{j=1}^{k} m_j$ assigned to each component, there exists a meromorphic function $\phi : W \to \bar{D}$ of degree d with simple critical values at a given set of $d + 2g + k - 2$ points in an open disk D with the following additional properties:*

(i) *ϕ sends each connected component of the boundary of W locally diffeomorphically on the boundary of $\bar{D}$ where $\bar{D}$ denotes the closure of D;*

(ii) *the restriction of ϕ onto each O_i has degree m_i, $i = 1, \dots, k$.*

[3] i.e. preserving the genus and the number of boundary components

Finally, the set of all meromorphic functions ϕ with the above properties is connected.

Proof. Lemma 2 has been settled in § 3.1, Lemmas 1-3 of [19]. Alternatively, the connectedness of the latter space of meromorphic functions can be deduced from the uniqueness of Hurwitz action on the factorizations of products of cycles of lengths $m_1, \dots, m_k$ in the symmetric group Sym_d into transpositions, (cf. [12]). (The number of critical points is immediate from the Riemann-Hurwitz formula.) □

For the next Proposition, consider the lift of the action of the group of orientation–preserving diffeomorphisms of a closed surface E on oriented collections of ovals to its action on the covering space of the latter space consisting of *labeled* oriented collections of ovals, see Lemma 2. Namely, we define the action of $\phi \in Diff^+(E)$ as $\phi(O_i, m_i) := (\phi(O_i), m_i)$. In fact, we consider the action of the *isotopy classes* of diffeomorphisms, i.e. the action of the mapping class group of E on the isotopy classes of collections of labeled ovals.

Proposition 8. *In the above notation, connected components of $\mathcal{H}^{nr}_{g,d}$ are in one–to–one correspondence with the orbits of the mapping class group acting on the isotopy classes of oriented collections of ovals $\{O_1, O_2, \dots, O_\ell\}$ on a fixed surface of genus g equipped with positive multiplicities $\{m_1, m_2, \dots, m_\ell\}$ adding up to d.*

Proof. For any two meromorphic functions $f_i : C_i \to \mathbb{C}P^1, i = 1, 2$ within a given connected component of $\mathcal{H}^{nr}_{g,d}$, a choice of trivialization over a path connecting f_1 with f_2 produces a diffeomorphism between C_1 and C_2. This diffeomorphism maps the collections of ovals on C_1 with multiplicities given by the degrees f_1 restricted to each connected component of the preimage of the real axis onto that of C_2 and f_2. It also matches the connected components of pre-images $f_i^{-1}(H^+), i = 1, 2$ (resp. $f_i^{-1}(H^-), i = 1, 2$). After selecting diffeomorphisms between C_1 and C_2 with a fixed surface E, we obtain two isotopy classes of labeled oriented ovals in E which are mapped to one another by an element of the mapping class group of E.

To show the opposite implication in Proposition 8, suppose that we have a surface E endowed with labeled oriented collection of ovals with multiplicities adding up to d. Using Lemma 2 we can construct a meromorphic function of each connected component of the complement in E to this collection of ovals with prescribed degrees of covering on each boundary components. Then we can glue these "partial" meromorphic functions into a global meromorphic function $f : E \to \mathbb{C}P^1 \supset \mathbb{R}P^1$ of degree d.

Now let $\mathcal{C}_i, i = 1, 2$ be two collections of ovals with multiplicities in the same orbit of the group of diffeomorphisms. An orientation preserving diffeomorphism taking one onto the other induces the correspondence between connected components in positive (respectively negative) class matching the multiplicities of the boundary components. The connectedness claim in Lemma 2 shows that two meromorphic functions $f_i, i = 1, 2$ in $\mathcal{H}^{nr}_{g,d}$ having the same combinatorial information can be deformed one into the other inside $\mathcal{H}^{nr}_{g,d}$.

□

3.2. Connected components of $\mathcal{H}^{nr}_{g,d}$ and boundary braids.

Each oriented collection of ovals determines a subgroup of boundary braids and vice versa. Our main Theorem 1 relates the connected components of $\mathcal{H}^{nr}_{g,d}$ and the orbits of the mapping class group acting on abelian subgroups of the braid group representing boundary braids.

Theorem 1. *The map sending each connected component of $\mathcal{H}^{nr}_{g,d}$ to the respective orbit of the mapping class group $Mod(E)$ of a closed surface E of genus g acting* *on the conjugacy classes of the full boundary braids of the group $Br_d(E)$ via the sequences (2.8), (2.9) is injective.*

Proof. Let $\beta_i, i = 1, 2$ be the braids of two meromorphic functions f_1 and f_2 such that $\beta_2 = \phi(\alpha\beta_1\alpha^{-1})$ for an appropriate element $\phi \in Mod(E)$ of the mapping class group of a surface E of genus g and a braid $\alpha \in Br_d(E)$. Since β_i are the boundary braids they correspond to two systems of oriented label collections of ovals. It follows from Proposition 5, that these collections are mapped one onto the other by an element of the mapping class group as well. Now since both f_1 and f_2 have the same degrees as coverings of $\mathbb{R}P^1 \subset \mathbb{C}P^1$, Proposition 8 implies that from this relation between the collections of ovals follows that the functions lie in the same connected component of $\mathcal{H}^{nr}_{g,d}$. □

4. Special classes of meromorphic functions

4.1. **Hurwitz spaces $\mathcal{H}^{nr}_{0,d}$ of rational functions.** The next claims can be easily derived from the earlier results of [16, 17], but they also follow from the above more general Theorem 1 and Proposition 8.

Proposition 9. *Consider the space Rat^{nr}_d of all rational functions of degree d with all non-real critical values. Then connected components of Rat^{nr}_d are in on-to-one correspondence with the equivalence classes consisting of a collection of disjoint ovals $\{O_1, O_2, \ldots, O_\ell\}$ in $\mathbb{C}P^1$ together with a collection of positive multiplicities $\{m_1, m_2, \ldots, m_\ell\}$ adding up to d.*

Similar results can be found in e.g. [3, 19].

Proposition 10. *For $F \simeq \mathbb{C}P^1 \simeq S^2$, the representative of the conjugacy class of the braid corresponding to a given collection of ovals in $\mathbb{C}P^1$ with the positive multiplicities is obtained by the same construction as in Proposition 8.*

Proof. The argument follows that in Proposition 8. □

Proposition 11. *In case $F \simeq \mathbb{C}P^1 \simeq S^2$, the conjugacy classes of the collections of ovals (without multiplicities) are in one-to-one correspondence with (the isomorphism classes of planar) directed graphs on S^2 and the conjugacy classes of the collections of ovals (without multiplicities) are in one-to-one correspondence with (the isomorphism classes of planar) directed graphs on S^2 equipped with positive integer weights of their vertices.*

Proof. Given a collection of the ovals in S^2, we assign a vertex to each connected component of the complement to this collection and to each oval we assign the oriented edge directed away from the vertex representing connected component mapped to H^+. □

Remark 3. Apparently, Proposition 11 can de extended to surfaces of all genera, but the respective combinatorial gadgets are not very illuminating, comp. [6].

4.2. **Case of special real meromorphic functions.** Assume now that a Riemann surface E is equipped with an anti-holomorphic involution σ (complex conjugation) and that a meromorphic function $f : E \to \mathbb{CP}^1$ is equivariant with respect to σ and complex conjugation $[u : v] \to [\bar{u}, \bar{v}]$ on $\mathbb{CP}^1$. Such functions f are classically referred to as *real meromorphic functions*. Connected components of the spaces of generic rational and generic meromorphic functions have been earlier studied in [19, 6] respectively.

We say that a real meromorphic function $f : E \to \mathbb{CP}^1$ is *special* if it has no real critical values. Notice that generic real meromorphic functions might have simple real critical values which implies that condition of speciality substantially restricts the class of real meromorphic functions under consideration. However special meromorphic functions of a given degree form full-dimensional subsets among all real meromorphic functions of the same degree.

Proposition 12. *Space of special real meromorphic functions of degree d on a surface of genus g is real manifold of dimension equal to complex dimension of the whole (small) Hurwitz space.*

Proof. The (small) Hurwitz space is an etale cover of the configuration space of $b = 2d - 2g - 2$ points and has complex dimension b. For real meromorphic functions, non-real critical values come in conjugate pairs and hence for any special real meromorphic function, the number of critical points in the upper half-plane equals $d - g - 1$. A connected component of the space of special real meromorphic functions containing a function f consists of functions determined by the set of critical points in the upper half-plane and having the same monodromy factorization as f. (The critical points in the lower half-plane and the respective monodromy are uniquely determined via complex conjugation.) Clearly, the complex conjugation of $\mathbb{C}P^1$ lifts to the complex conjugation of the domain of each meromorphic function obtained through this construction. This connected component is an etale cover of the configuration space of $d - g - 1$ points in the upper half-plane and hence has real dimension $2d - 2g - 2$. □

Now we study the monodromy of special real meromorphic functions, arising restrictions on the system of ovals, and some examples of boundary braids occurring for special real meromorphic functions. Given such a function f, notice that the equivariance and the absence of its real critical values implies that σ (resp. complex conjugation) acts freely on the set of critical points (resp. critical values) of f, i.e. these actions have no fixed points. Moreover, as already mentioned earlier, Hurwitz monodromy has a natural split into the monodromies corresponding to the critical values lying in the upper resp. the lower half-planes.

In the case when a meromorphic function $f : E \to \mathbb{CP}^1$ is real, a good ordered system of generators of $\pi_1(\mathbb{CP}^1 \setminus Cr, b)$ can be selected as a good ordered system of generators of the fundamental group of any disk containing $b \cup Cr$ and compatible with complex conjugation, (cf. Section 2.2). Here Cr is the set of critical values of f and $b \in \mathbb{R}P^1$. Indeed let $\gamma_1, \dots, \gamma_N$ be a good ordered system of generators of $\pi_1(\mathbb{H}^+ \setminus (Cr \cap \mathbb{H}^+), b)$ and let $\bar{\gamma}_i$ be the loop conjugate to γ_i, $i = 1, \dots, N$ and lying in the lower half-plane $\mathbb{H}^-$. Then

$$\gamma_1, \dots, \gamma_{N-1}, \gamma_N, \bar{\gamma}_N^{-1}, \bar{\gamma}_{N-1}^{-1}, \dots, \bar{\gamma}_1^{-1} \tag{4.1}$$

is a good ordered system of generators of $\pi_1(\mathbb{CP}^1 \setminus Cr, b)$.

Now let us compare the braids introduced earlier with those corresponding to the upper and the lower half-planes. As in Section 2, set $E^+ = f^{-1}(\mathbb{H}^+)$, $E^- = f^{-1}(\mathbb{H}^-)$, and $B = f^{-1}(b)$, $\operatorname{card} B = \deg(f)$. As before, E^+ and E^- will be equipped with the orientations induced by the complex structure so that the complex conjugation $E^+ \to E^-$ reverses the orientation. Consider the corresponding maps of the braid groups:

$$\begin{cases} i^+ : Br(E^+, B) \to Br(E, B) \\ i^- : Br(E^-, B) \to Br(E, B) \end{cases}$$

induced by the embedding of subsurfaces.

It was already mentioned that unless E^- (resp. E^+) is a disk these maps are injections, (cf. Proposition 1).

Note that for any $\gamma \in \pi_1(\mathbb{H}^+ \setminus Cr, b)$ (resp. $\gamma \in \pi_1(\mathbb{H}^- \setminus Cr, b)$), its preimage in E^+ (resp. in E^-) induces a path in the configuration space of subsets of cardinality $\operatorname{card} B$ in E^+, i.e. a braid in $Br(E^+, B)$ (resp. in $Br(E^-, B)$). In particular, we obtain the homomorphisms:

$$M^+ : \pi_1(\mathbb{H}^+ \setminus Cr, b) \to Br(E^+, B) \tag{4.2}$$

and

$$M^- : \pi_1(\mathbb{H}^- \setminus Cr, B) \to Br(E^-, B)$$

called *Hurwitz braid monodromy*[4] where both maps are restrictions of the monodromy

$$M : \pi_1(\mathbb{CP}^1 \setminus Cr, b) \to Br(E, B).$$

The above braid coincides with $M^+(\gamma_N \cdot \ldots \cdot \gamma_1)$, where $\gamma_N \cdot \ldots \cdot \gamma_1$ is the element of the fundamental group corresponding to the loop represented by the real axis $\mathbb{R}P^1 \subset \mathbb{C}P^1$. It can also be expressed as $M^-(\bar{\gamma}_N \cdot \ldots \cdot \bar{\gamma}_1)$. Using decomposition of the loop corresponding to the real axis in $\bar{H}$ we immediately obtain the following.

Proposition 13. *Let $\gamma_1, \ldots, \gamma_N$ be a good order system of generators of $\pi_1(\mathbb{H}^+ \setminus Cr, b)$. The identity*

$$\gamma_1 \ldots \gamma_N = \bar{\gamma}_1 \ldots \bar{\gamma}_N \tag{4.3}$$

holds in $\pi_1(\mathbb{CP}^2 \setminus Cr, b))$ which implies that

$$i^+(M^+(\gamma_1 \ldots \gamma_N)) = i^-(M^-(\bar{\gamma}_N \ldots \bar{\gamma}_1)).$$

Projecting smooth real plane algebraic curves onto $\mathbb{C}P^1$ from a real point in $\mathbb{C}P^2$ we obtain interesting examples of real meromorphic functions. If such a projection has no real critical points one has the following result (whose proof is standard).

Proposition 14. *Let $\Gamma \subset \mathbb{C}P^2$ be a real algebraic curve of degree d whose projection on the real x-axis has d real pre-images and no real critical points. Then for even d, the pre-image of the real axis in Γ is a union of $\frac{d}{2}$ circles each being a double covering the real axis. For d odd, in addition to $\left[\frac{d}{2}\right]$ "double" circles there is one more circle covering the real axis diffeomorphically.*

Note that Proposition 13 describes two factorizations of the braid in Proposition 14.

4.2.1. *Examples.* Let us give an explicit calculation of the braid corresponding to the real axis in a special case. Namely, let $C_d \subset \mathbb{C}P^2$ be the complex projective curve given by $x^d + y^d + z^d = 0$ usually referred to as the Fermat curve and let $\pi : C_d \to \mathbb{CP}^1$ be projection of C_d onto the real x-axis $\mathbb{R}P^1_x$ (i.e. the projection centered at $[0,1,0] \in \mathbb{CP}^2$).

Example 1. For $d = 2$, C_2 is a rational curve and the preimage $\pi^{-1}(\mathbb{R}P^1_x)$ of the real x-axis is given by

$$\pi^{-1}(\mathbb{R}P^1_x) = [x, \pm i\sqrt{1+x^2}, 1] \cup [1, \pm i, 0] \subset \mathbb{CP}^1,$$

i.e. it has no real points and π has two critical points $[\pm i, 0, 1]$. In particular, $\pi^{-1}(\mathbb{R}P^1_x)$ is a circle. (Projection on the y-axis is the complement to the interval $(-i, i)$ of the imaginary axis). $M(\gamma_1) = \sigma_1 \in \mathcal{B}_2(S^2)$ (resp. $M(\gamma_1) = \sigma_1^{-1}$) is positive (resp. negative) rotation of a pair of points on C_2. Relation (4.3) reduces to $\sigma = \sigma^{-1}$ in $Br(S^2, 2) \simeq \mathbb{Z}_2$.

[4] to distinguish from the braid monodromy introduced in [15]

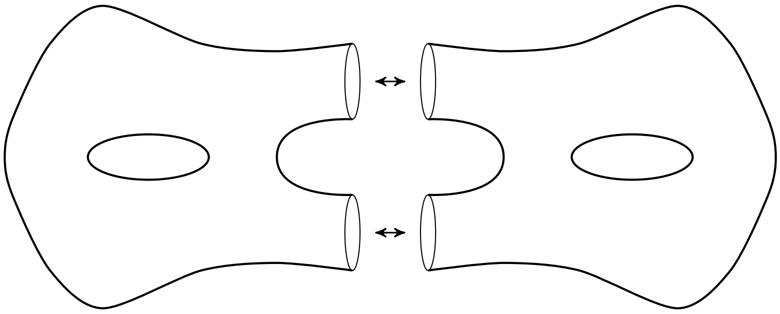

FIGURE 3. Splitting of C_4 into 2 connected components

Example 2. Now let d be an even integer greater than 2. Then C_d has no real points and d critical values of its projection onto the x-axis are given by $\omega_k := exp\left(\frac{(2k-1)\pi i}{d}\right)$. They belong to $\mathbb{H}^+$ (resp. to $\mathbb{H}^-$) for $k = 1, \dots, \frac{d}{2}$ (resp. for $k = \frac{d}{2}+1, \dots, d$).

The points of $C_d \subset \mathbb{C}P^2$ which project onto the origin $x = 0$ are of the form $(0, \omega_k, 1)$. The lift of the preimage of the positive (resp. the negative) x-semi-axis with the initial point $(0, \omega_k, 1)$ is given by $(x, \alpha(x)\omega_k, 1)$ where $\alpha(x) = (1 + x^d)^{\frac{1}{d}}$ and $x \geq 0$ for the positive semi-axis (resp. $x \leq 0$ for the negative semi-axis).

Notice that $\lim_{x\to\pm\infty}(x, \alpha(x)\omega_k, 1) = (1, \pm\omega_k, 0)$. Hence traversing the real axis first from 0 to ∞ along the positive semi-axis and then returning back to 0 along the negative semi-axis (in positive direction) interchanges the points $(0, \omega_k, 1)$ and $(0, \omega_k^{-1}, 1)$. Hence the preimage of the real axis $\mathbb{R}P^1_x$ consists of $\frac{d}{2}$ circles each covering $\mathbb{R}P^1_x$ twice, as pointed out in Proposition 14.

Hence C_d (which is a compact Riemann surface of genus $\frac{1}{2}(d-1)(d-2)$) splits by these $\frac{d}{2}$ circles into the union of two diffeomorphic surfaces with opposite orientation, see Fig. 3. Direct calculation shows that this split of C_d by the preimage of the real axis gives a decomposition of C_d into a union of two surfaces of genus $\frac{(d-1)^2}{2}$ punctured at $\frac{d}{2}$ points. Additionally, the braid we are interested in is the rotation by the angle π along the system of ovals providing the split.

4.3. **Deformations of meromorphic functions.** In this subsection we discuss what happens to the braid of a meromorphic function $f : E \to \mathbb{CP}^1$ having simple critical points and no real critical values when f is deformed in such a way that all the critical points remain simple, but exactly one of the critical values crosses the real axis $\mathbb{R}P^1$.

Our goal is to compare the braid of a function f with one of the critical values c_0 located close to the real line in the lower half-plane and the braid of the function $\tilde{f}$ which is obtained from f as the end function of a deformation starting with f and which moves the critical c_0 of f value across the real axis from the lower to the upper half-plane, while other critical values remaining in their respective half-planes during the deformation.

To fix our notations let us, without loss of generality, assume that c_0 lies inside a half-disk D_0 bounded by a small semicircle in the lower half-plane and the interval of the real axis. Let $\gamma \in \pi_1(\mathbb{CP}^1 \setminus Cr(f), b)$, $b \in \mathbb{R}P^1$ be the class represented by the loop given by the real axis where (as above) $Cr(f)$ is the set of critical values of f. Let $\delta_{c_0} \in \pi_1(\mathbb{H}^- \setminus Cr, b)$ be the class represented by the loop consisting of the interval in real axis connecting the base point $b \in \mathbb{R}$ with the right end of the semi-circle bounding the semi-disk containing c_0, and then followed by the interval in $\mathbb{R}$ connecting the left end of the semicircle and the base point b.

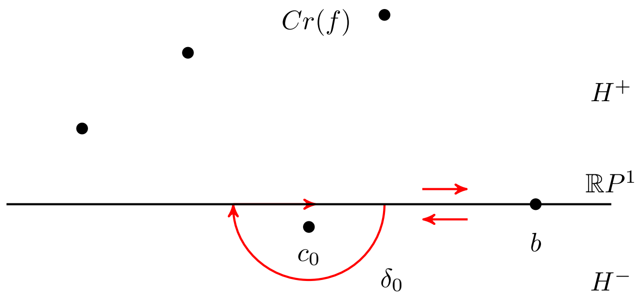

FIGURE 4. Deformation of a meromorphic function

Observe that $\mathbb{H}^+ \setminus Cr(\tilde{f})$ is a retract of $\mathbb{H}^+ \setminus Cr(f) \cup (D_0 \setminus c_0))$ [5] which implies the isomorphism $\pi_1(\mathbb{H}^+ \setminus Cr(\tilde{f}), b) \to \pi_1(\mathbb{H}^+ \setminus Cr(f) \cup (D_0 \setminus c_0))$ with this isomorphism taking the class of $\tilde{f}^{-1}(\mathbb{R}P^1) \in \pi_1(\mathbb{H}^+ \setminus Cr(\tilde{f}), b)$ to the class of δ_{c_0} followed by the class of $f^{-1}(\mathbb{R}P^1)$, see Fig. 4 where the orientation of the loop $\mathbb{R}P^1$ given by moving in negative direction. UNCLEAR

Let (i_1, i_2) be the transposition corresponding to δ_0 and let $(\tau_1, \dots, \tau_k)$ be the cycle decomposition of the permutation corresponding to the image of γ in the Hurwitz monodromy $\pi_1(\mathbb{CP}^1 \setminus Cr(f), b) \to Sym_d$. Recall that the ovals in $f^{-1}(\mathbb{R}P^1)$ correspond to the cycles τ_i and the braid for each cycle is represented by the rotation by the angle $\frac{2\pi}{l(\tau_i)}$, where $l(\tau_i)$ is the length of the cycle.

There are two distinct cases to consider:

(1) i_1 and i_2 belong to a single cycle τ_1;
(2) i_1 and i_2 lie in distinct cycles, say τ_1 and τ_2.

Case 1. We can assume that the cycle τ_1 is written in the form (i_1, a, i_2, b) for some ordered subsets of $[1, \dots, d]$. One has the following cycle decomposition

$$(i_1, i_2)(i_1, a, i_2, b) = (i_1, b)(i_2, a) \tag{4.4}$$

for the product $(i_1, i_2)\tau_1$ which should be read from left to right.

We claim that in this case the oval in $f^{-1}(\mathbb{R}P^1)$ corresponding to the cycle τ_1 splits in two disjoint ovals and the braid corresponding to the rotation by $\frac{2\pi}{l(\tau_1)}$ splits into the product of the braids corresponding to rotations by the angles $\frac{2\pi}{1+\text{card}(a)}$ and $\frac{2\pi}{1+\text{card}(b)}$. (Note that $l(\tau_1) = 2 + \text{card}(a) + \text{card}(b)$) respectively.

Case 2. The transposed elements $i_1, i_2 \in [1, \dots, d]$ belong to a pair of disjoint cycles $\tau_1 = (i_1, a)$ and $\tau_2 = (i_2, b)$. Similarly to (4.4) one has

$$(i_1, i_2)(i_1, b)(i_2 a) = (i_1 a i_2 b). \tag{4.5}$$

In this case, two ovals corresponding to the cycles τ_1, τ_2 merge into one.

Proposition 15. *Let $f : E \to \mathbb{CP}^1$ be a function without real critical values. A deformation of f into the function $\tilde{f}$ during which one of the critical values of f crosses the real axis from the lower half plane to upper half plane results in the transformation of the system of ovals in which either one oval splits into two* (Case 1) *or two ovals merge into one* (Case 2). *The parameters $(m_1, \dots, m_r)$ specifying the subgroups of boundary braids $BB_{m_1,..m_r}$ are transformed as described above.*

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[5] a retraction moves the arc of semicircle to its diameter along the vertical lines and is constant outside a closed disk containing this semicircle

5. Outlook

We conclude with several additional questions dealing with the special classes of the families of meromorphic function similar to those which we have answered in this paper.

1. What arrangements of ovals with multiplicities a given meromorphic function $f : F \to \mathbb{C}P^1$ might get under possible choices of $\mathbb{R}P^1 \subset \mathbb{C}P^1$. Different choices of $\mathbb{R}P^1$ are different by a Möbius mapping. One should apparently use the dual plane where critical values are lines and $\mathbb{R}P^1$ is a point.

2. The next question is an analog of Problem 1 of § 3 in the case of square matrices and is important in mathematical physics, signal processing, and (applied) linear algebra, see e.g. [5, 10, 26]. Namely, given two $d\times d$-matrices A and B with complex entries, consider the projectivized real span $\mathcal{M}(A, B, \alpha : \beta) := \alpha A + \beta B$, where as above $(\alpha : \beta)$ are homogeneous coordinates on $\mathbb{R}P^1$. One can easily observe that if A and B are generic then for every point $(\alpha : \beta) \in \mathbb{R}P^1$, $\mathcal{M}(A, B, \alpha : \beta)$ will have d distinct (simple) eigenvalues in $\mathbb{C}$.

Therefore, for generic A and B, these eigenvalues depending on the parameter $(\alpha : \beta) \in \mathbb{R}P^1$ traverse a closed braid in $\mathbb{C}P^1 \times \mathbb{R}P^1$.

Problem 5. In the above notation, describe the equivalence class of the closed braid in terms of (A, B) as well as the set of the equivalence classes of all d-stranded braids which might occur when the latter construction is applied to (generic) pencils of complex-valued $d \times d$-matrices.

Observe that Problem 1 is a special case of Problem 5 if one considers pencils of diagonal matrices. Identifying the space of complex-valued $d\times d$-matrices with $\mathbb{C}^{d^2}$ and denoting the space of real pencils in $\mathbb{C}^{d^2}$ by $\mathcal{M}_d$, we define its subset $\mathcal{M}_d^0 \subset \mathcal{M}_d$ consisting of all real matrix pencils forming closed braids, i.e., such that for each value of $(\alpha : \beta) \in \mathbb{R}P^1$, all d eigenvalues are pairwise distinct.

Problem 6. Enumerate connected components of $\mathcal{M}_d^0$.

Again, braids corresponding to matrix pencils belonging to the same connected component of $\mathcal{M}_d^0$ are equivalent. Problem 6 asks whether different connected components can have equivalent closed braids.

The latter problems about matrix pencils can be translated into questions about projections of plane algebraic curves due to the following old result. Let us recall the classical theorem about determinantal representations, see e.g. [7].

Theorem A. Every homogeneous polynomial in three variables of degree d can be written as

$$f(x, y, z) = \det(Ax + By + Cz)$$

where A, B and C are symmetric $d \times d$-matrices. Here the coefficients of f and the matrix entries are complex numbers.

3. The last problem we suggest is a refinement of the Problem 3 from § 3 related to Problem 5 and projections of plane algebraic curves. A natural (sub)class of meromorphic functions is associated to plane curves and their projections. Namely, given a plane curve $\mathcal{C} \subset \mathbb{C}P^2$ and a point $p \in \mathbb{C}P^2$, we obtain a meromorphic function $\pi_p : \mathcal{C} \to \mathbb{C}P^1$ by projecting $\mathcal{C}$ onto the pencil of lines through p.

In fact, every meromorphic function $f : F \to \mathbb{C}P^1$ on any compact Riemann surface F can be realized as the composition $\pi_p \circ \nu : F \to \mathbb{C}P^1$ where $\nu : F \to \mathbb{C}P^2$ is a birational mapping of F onto the plane curve $\nu(F) \subset \mathbb{C}P^2$ and $\pi_p : \mathbb{C}P^2 \backslash p \to \mathbb{C}P^1$ is the projection from a point $p \in \mathbb{C}P^2$, see [20]. Obviously if $\deg f = d$ then $d' := \deg \mathcal{C} \geq d$. In fact every meromorphic function $f : F \to \mathbb{C}P^1$ where the genus of F is g can be realized by a projection of a plane curve $\mathcal{C}$ of degree (at most) $d' \geq g+2$ from an appropriate point p. For many meromorphic functions, $\mathcal{C}$ can be chosen of a much smaller degree, see examples in [20].

Definition 5. Define the **planarity defect** $\text{pdef}(f)$ of a given meromorphic function $f : F \to \mathbb{C}P^1$ as

$$\text{pdef}(f) := \min_{\nu}(\deg(\nu(F)) - \deg(f)$$

such that $f = \pi_p \circ \nu$, as above.

The space $\mathcal{H}_{g,d}$ of all meromorphic functions of degree d on Riemann surfaces of genus g can be stratified as

$$\mathcal{H}^0_{g,d} \subset \mathcal{H}^1_{g,d} \subset \cdots \subset \mathcal{H}^\ell_{g,d} \subset \cdots \subset \mathcal{H}^{M(g,d)}_{g,d} = \mathcal{H}_{g,d} \tag{5.1}$$

where $\mathcal{H}^\ell_{g,d}$ consists of all meromorphic functions in $\mathcal{H}_{g,d}$ whose planarity defect is at most ℓ. The exact value $M(g,d)$ is given by $\max\left(0, \left\lceil \frac{g-d+2}{2} \right\rceil\right)$, see Corollary 1.15 of [20].

Problem 7. What spherical braids can occur from the meromorphic functions of degree d on Riemann surfaces of genus g whose planarity defect is at most ℓ and without real critical values?

Observe that applying projective transformations in the above construction, we can once and forever fix the point $p \in \mathbb{C}P^2$ and choose the affine chart $\mathbb{C}^2 \subset \mathbb{C}P^2$ with p at infinity. We can also choose $\mathbb{R}P^1 \subset \mathbb{C}P^1$ in the pencil of lines through p. Now we can consider the set of plane curves $\mathcal{C} \subset \mathbb{C}P^2$ such that the meromorphic function $f : \widetilde{\mathcal{C}} \to \mathbb{C}P^1 \supset \mathbb{R}P^1$ induces by π_p has no real critical values. This construction together with the above one associates the set of admissible braids appearing in each term $\mathcal{H}^\ell_{g,d}$ of the above filtration. In particular, for the final term $\mathcal{H}^{M(g,d)}_{g,d} = \mathcal{H}_{g,d}$ the set of admissible braids is described in the previous section.

6. Appendix: Relations among the generators of the braids groups of surfaces

Theorem 2 (cf. [1])**.** *For $n > 1$, the braid group $B_n(F,k)$ of n strands on a surface F of genus $g \geq 0$ with $k \geq 0$ punctures has the presentation consisting of the braid and mixed relations given below.*

Braid relations:

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1. \end{cases}$$

Mixed relations:

$$(R1): \quad a_r\sigma_i = \sigma_i a_r, \quad b_r\sigma_i = \sigma_i b_r, \quad i \neq 1,\ r = 1, \ldots, g; \tag{6.1}$$

$$(R2): \quad (\sigma_1^{-1}a_r)^2 = (a_r\sigma_1^{-1})^2, \quad (\sigma_1^{-1}b_r)^2 = (b_r\sigma_1^{-1})^2, \quad 1 \leq r \leq g; \tag{6.2}$$

$$(R3): \begin{cases} [\sigma_1^{-1}a_s\sigma_1, a_r] = 1, \quad [\sigma_1^{-1}b_s\sigma_1, b_r] = 1, \quad s < r \\ [\sigma_1^{-1}a_s\sigma_1, b_r] = 1, \quad [\sigma_1^{-1}b_s\sigma_1, a_r] = 1, \quad s < r; \end{cases}$$

$$(R4) \quad [\sigma_1^{-1}a_r\sigma_1^{-1}, b_r] = 1, \quad 1 \leq r \leq g; \tag{6.3}$$

$$(R5) \quad [z_j, \sigma_i] = 1, \quad i \neq 1; j = 1, \ldots, k-1; \tag{6.4}$$

$$(R6) \quad [\sigma_1^{-1}z_i\sigma_i, a_r] = [\sigma_1^{-1}z_i\sigma_i, b_r] = 1, \quad i \neq 1; \quad j = 1, \dots, p-1, 1 \leq r \leq g; \tag{6.5}$$

$$(R7) \quad [\sigma_1^{-1}z_j\sigma_1, z_l] = 1, \quad j = 1, \dots, p-1, j < l; \tag{6.6}$$

$$(R8) \quad [\sigma_1^{-1}z_j\sigma_1^{-1}, z_j] = 1, \quad j = 1, \dots, p-1. \tag{6.7}$$

In other words, if $g \geq 1$ one uses all the above relation and in the case $g = 0$ one has generators $\sigma_1, .., \sigma_{n-1}, z_1, .., z_{p-1}$ subject only to braid relations and relations $R(5), R(7), R(8)$.

In the case of a closed surface F, $B_n(F)$ is generated by $\sigma_1, ..., \sigma_{n-1}, a_1, b_1, ..., a_g, b_g$ with braid relations, the relations $R(1) - R(4)$ and the relation:

$$[a_1, b_1^{-1}] \cdot ... \cdot [a_g, b_g^{-1}] = \sigma_1 \cdot .... \cdot \sigma_{n-1}^2 \cdot ... \cdot \sigma_1. \tag{6.8}$$

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