BRAID MONODROMY AND ALEXANDER POLYNOMIALS OF REAL PLANE CURVES

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Abstract. We describe symmetries of the braid monodromy decomposition for a class of plane curves defined over reals including the real curves with no real points and proving new divisibility relations for Alexander invariants of such curves.

In Memory of John W.Wood.

1. Preface and statement of results

Alexander polynomial of a projective curve embedded into a smooth algebraic surface (cf. [17]) is an invariant of the fundamental group of the complement to the curve. It measures a degree of non-commutativity of the group and can be expressed in terms of geometric data of the surface and the curve, including the local types and position of singularities of the curve on the surface. An interesting problem is understanding which polynomials can appear as the Alexander polynomials of the fundamental groups of the complements: this is a very special case of the fundamental problem of understanding the quasi-projective groups.

Divisibility theorems give strong restrictions on the class of polynomials which can occur as the Alexander polynomials of projective curves. One such result (cf.[15],[17]) asserts that Alexander polynomial of the fundamental group of the complement to an ample curve C divides the product of the Alexander polynomials of links of all singularities of the curve. In particular, if such C has ordinary nodes and cusps as the only singularities than the global Alexander polynomial has a form $(t-1)^a(t^2-t+1)^b$. Focusing on the curves in a complex projective plane rather than general smooth algebraic surfaces (as we will do in this paper) one can easily see that $a+1$ is the number of irreducible components of C (cf. [17]) but the range of multiplicities b is far from clear. In the case of curves in $\mathbb{CP}^2$, the global Alexander polynomial also divides the Alexander polynomial of the link in the 3-sphere which is the boundary of a small tubular neighborhood of a line with the link being defined as the intersection of the curve with this 3-sphere. If this "line at infinity" is transversal to the curve C then the corresponding link is the Hopf link with d components where d is the degree of C and the Alexander polynomial of the latter is $(t-1)(t^d-1)^{d-2}$. One obtains that for curves with nodes and cusps the multiplicity of the factor (t^2-t+1) is at most $d-2$ if $6|d$ and is zero otherwise (for a similar divisibility relation on surfaces more general than $\mathbb{CP}^2$ see [17]). [1]

[1] A different divisibility relations one obtains if the line at infinity is selected to be non-transversal to the curve or contains the singularities. This gives divisibility by the Alexander polynomial of the complement to affine curve which is a complement in $\mathbb{CP}^2$ to the union of the curve and the line. The Alexander polynomial of this affine curve may be different than

This bound is much weaker than what so far was observed in examples. At the moment, it is unknown if the multiplicity of a primitive root of unity of degree 6 in the Alexander polynomial of a curve, having as its singularities ordinary cusps and nodes only, has a bound independent of d. The largest known multiplicity is 4 for a curve of degree 12 with 39 cusps (cf. [2]). It is known that if the Mordell Weil ranks of isotrivial elliptic threefold (or isotrivial elliptic surfaces) are bounded, the multiplicities of the factors of the Alexander polynomials of curves in this class, also are bounded independently of degree (cf. [2] for proof of both assertions).

In this note, we discuss a new type of divisibility relations for the Alexander polynomials for the complements to curves in $\mathbb{CP}^2$ for which the defining equations have only real coefficients [2]. The presence of a real structure imposes restrictions on the braid monodromy of the curve. The latter is an invariant of a curve $C \subset \mathbb{CP}^2$ and its projection onto a complex line $N \subset \mathbb{CP}^2$, given as the homomorphism $\pi_1(N \setminus Cr, b) \to B_d$. Here b is a base point, Cr is the subset of N consisting of points over which the fiber of projection of C has cardinality less than the degree d of C, and B_d is Artin's braid group. If one views B_d as the mapping class group of a disk with boundary and d marked points then the braid monodromy assigns to a loop in $N \setminus Cr$ the class of the diffeomorphism is given by the trivialization of the fibration of the pair $(\mathbb{CP}^2 \setminus p, C)$, over this loop (here p is the center of projection onto N, cf. [21], [17] for a more recent exposition or section 3 below).

The main results of the paper are the Theorem 3.1, its corollary given as Proposition 6.4, the Theorem 5.1, and its corollary the Proposition 6.7.

In Section 3, we describe symmetry in the structure of this homomorphism depending on the real structure of the curve. It appears that certain operators introduced by Garside (cf. [9]) play important role in the description of this symmetry and vice versa, the study of braid monodromy of real curves gives geometric interpretation to some of Garside's identities. In particular, if the projection is defined over $\mathbb{R}$ and the intersection of the finite set Cr with the real locus of N is empty then the braid monodromy takes the class in $\pi_1(N \setminus Cr, b)$ (where b is real) represented by the loop corresponding to $\mathbb{RP}^1$ to the Garside word Δ. Braids corresponding to such loops were considered in [18] in the related context of Hurwitz schemes. If the real part of the critical set $Cr \cap \mathbb{RP}^1 \neq \emptyset$, then $\pi_1(N \setminus Cr, b)$ contains three canonical loops: the one containing only critical points in the real part of the critical set and two loops containing all critical points in each of two connected components of $N \setminus \mathbb{RP}^1$. We describe constraints on corresponding braids and solve the equations in the braid group to obtain an explicit form of the braids corresponding to these canonical loops.

In section 4.1 we prove that the fundamental group of a curve over $\mathbb{R}$ is a quotient of the fundamental group of a link which is one of the closed braids attached to the curve in section 3. The argument here is purely topological and does not use an algebro-geometric structure, and can be used in a different, for example symplectic, context.

In Section 5 we show that the global Alexander polynomial of a curve over $\mathbb{R}$ divides the Alexander polynomial of a link which is the closure of a braid associated

the Alexander polynomial of the projective curve. A slightly better than $d-2$ bound on the multiplicity is given in [2] Cor.3.13.

[2]in the case of reducible curves, this allows for irreducible components not to have $\mathbb{R}$ as the field of definition; we do not make any assumptions about the reality of the critical values of projections used to construct the braid monodromy as was customary in previous works, cf. [14].

with the real structure and discussed in two previous sections. We calculate these Alexander polynomials in some cases and make the divisibility relations explicit. For example for real curves without real points at all, the Alexander polynomial divides $(t^d-1)^{\frac{d-2}{2}}(t^{\frac{d}{2}}+1)(t-1)$. This gives $\frac{d}{2}-1$ as a new bound on the multiplicity of the factor t^2-t+1 in the Alexander polynomial of a curve with nodes and cusps over $\mathbb{R}$ and no real points. Moreover, we show that this bound is sharp at least for such sextics. The last section also contains a discussion of braid monodromy of arrangements defined over reals, i.e. such that the equation of the union of all lines is defined over $\mathbb{R}$, but having only finitely many real points. Such arrangements are perhaps of interest on their own. Note that the effect of complex conjugation on braid monodromy was considered earlier in [3] (in connection with a study of MacLane arrangements).

Part of this work was done while the author participated in "Braids" program at ICERM in the spring of 2022. The author thanks B.Guerville-Balle, A.Degtyarev, the anonymous referee for very useful comments and references in connection with the earlier version of this note, and most importantly Juan Gonzalez-Meneses for the answer to the author's question (cf. [11] and Remark 3.8).

2. Complex conjugation and braid groups

Let $\mathcal{P}_N = \{P_1, \cdots P_N\}$ be an invariant under conjugation subset in $\mathbb{C}$. Let $P_0 \in \mathbb{R}$. Complex conjugation induces on free group $\pi_1(\mathbb{C} \setminus \mathcal{P}_N, P_0)$ an automorphism of order 2. In the following system of generators, this automorphism has a particularly simple form.

Recall that a good ordered system of generators of $\pi_1(\mathbb{C} \setminus \mathcal{P}_N, P_0)$ is given by N loops each consisting of a segment I_i running from P_0 to the vicinity of one of the points P_i, followed by a counterclockwise loop running along the boundary of a small circle centered at P_i and then returning to P_0 along I_i. Moreover, it is assumed that these loops are non-intersecting and ordered by the counterclockwise ordering of their intersection points with a small circle centered at P_0.

Without loss of generality, we assume that the points in $\mathcal{P}_N$ are labeled so that $P_1, \cdots P_k \in H^+$, $P_N, \cdots P_{N+1-k} \in H^-$, P_i is the complex conjugate of $P_{N+1-i}, i = 1, \cdots, k$ [3], $P_{k+1}, \cdots P_{N-k}$ form an increasing sequence of real numbers i.e. $P_{k+1} < \cdots < P_{N-k} < P_0$. Let $\alpha_+ : [0,1] \to H^+$ be an embedding of the interval $[0,1]$ such that $\alpha^+(0) = P_1, \alpha^+(1) = P_k$ and $P_i \in \alpha^+[0,1], i = 1, \cdots, k$ and the sequence $(\alpha^+)^{-1}(P_i) \in [0,1]$ is increasing. Let I be the interval in $\mathbb{C}$ which is the union of the interval $\alpha_+([0,1])$, followed by the line segment P_k, P_{k+1}, then followed by the part of the real axis running from P_{k+1} to P_{N-k}, then followed by the segment $[P_{N-k}, P_{N+1-k}]$ and having as the final part the interval $\overline{\alpha_+([0,1])}$. Then $I \supset \mathcal{P}_N$ and the above order in $\mathcal{P}_N$ is induced by an orientation of I. In the case when $N = 2k$ and $\mathcal{P}_N \cap \mathbb{R} = \emptyset$, $P_{k+1} = \bar{P}_k$ and in the above construction of I, instead of the segments with real endpoints we use the segment connecting P_k and P_{k+1} [4].

[3] here H^+ (resp. H^-) denote the upper (resp. lower) half plane in $\mathbb{C}$

[4] the order of the points in $\mathcal{P}_N$ in H^+ can be selected arbitrarily but, as follows from this description, is consistent with the orientation of $\alpha_+([0,1])$ and the order of points in H^+ determines the order of the conjugate points in H^-. In practice, it is convenient to order the points in H^+ according to the order of imaginary parts of the points.

We select a good ordered system of generators $x_1, \cdots, x_{N-k}$ of the fundamental group $\pi_1(H^+_\epsilon \setminus \bigcup_1^{N-k} P_i, P_0)$ of the complement to the set $P_1, \cdots P_{N-k}$ in a ϵ-neighborhood H^+_ϵ of the closure of H^+ in $\mathbb{CP}^1$ and extend it to a good ordered system $x_1, \cdots, x_N$ adding loops, as the set each being the conjugate of a loop among the first k loops in the already selected system but using orientation and the order given by the above definition of a good ordered system. With these notations, the involution on $\pi_1(\mathbb{C} \setminus \mathcal{P}_N, P_0)$ induced by the complex conjugation $\gamma \to \bar{\gamma}$ of the oriented loops is given by (writing from the right to the left):

(1)

$$\bar{x}_i = x^{-1}_{N+1-i}, i = 1, ...k \quad \bar{x}_i = x_{N-k} \cdots x_{i+1} x_i^{-1} x^{-1}_{i+1} \cdots x^{-1}_{N-k}, \quad i = k+1, \cdots N-k$$

Note that if at most one of $P_i,\ i = 1, \cdots N$ is real, then the action is just $\bar{x}_i = x^{-1}_{N-i}$. In particular, one has

$$\overline{x_N \cdots x_1} = (x_N \cdots x_1)^{-1}$$

Let $Diff(D^2, \mathcal{P}_N)$ be the group of diffeomorphisms of a conjugation invariant disk in $\mathbb{C}$ containing $\mathcal{P}_N$ and taking the set $\mathcal{P}_N$ into itself. Let $Diff^+(D^2, \mathcal{P}_N, \partial D^2)$ be its subgroup consisting of diffeomorphisms which are orientation preserving and constant on the boundary of the disk. The latter is a normal subgroup of the former and the same is the case for the groups of connected components of each of these groups. The group $\pi_0(Diff^+(D^2, \mathcal{P}_N, \partial D^2)$ is Artin's braid group B_N. The complex conjugation is an orientation-reversing element in $Diff(D^2, \mathcal{P}_N)$ and conjugation by this element, acting as an inner automorphism of $\pi_0(Diff(D^2, \mathcal{P}_N))$, acts as the outer automorphism of the normal subgroup $B_N = \pi_0(Diff^+(D^2, \mathcal{P}_N, \partial D^2)$. We denote this automorphism as $\beta \to \bar{\beta}, \beta \in B_N$. The group B_N is a subgroup of the group of automorphisms of $\pi_1(\mathbb{C} \setminus \mathcal{P}_N, P_0)$ and from the above definition, one has

$$\bar{\beta}(x) = \overline{\beta(\bar{x})} \quad x \in \pi_1(\mathbb{C} \setminus \mathcal{P}_N, P_0), \beta \in B_N \tag{2}$$

It is immediate that (2) implies the following for the action of the conjugation on the standard generators $s_1, \cdots s_{N-1}$ of B_N [5], given by the orientation of I from P_1 to P_N i.e. the counterclockwise Dehn half-twists corresponding to the subintervals of I connected consecutive points in $\mathcal{P}_N$, with the order along I:

$$\bar{s}_i = s^{-1}_{N-i} \quad i = 1, .., k-1, N-k+1, \cdots, N-1; \tag{3}$$

$$\bar{s}_i = s_i^{-1} \quad i = k+1, \cdots N-k-1 \tag{4}$$

Conjugates of generators in the remaining pair are given by [6]

(5)

$$\bar{s}_k = s_{N-k} \cdots s_{k+1} s_k^{-1} s^{-1}_{k+1} \cdots s^{-1}_{N-k}; \quad \bar{s}_{N-k} = s_{N-k-1} \cdots s_{k+1} s_k^{-1} s^{-1}_{k+1} \cdots s^{-1}_{N-k-1}$$

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[5]these generators depend on a choice of I or, more specifically, the choice of the part of I containing the points in H^+ since the rest of interval I described above canonically. Recall, that generators of the braid group, viewed as $\pi_1(Diff^+(D^2, \mathcal{P}_N, \partial D^2))$ are the half-twists corresponding to simple proper arcs connecting consecutive points of $\mathcal{P}_N$ (cf. [8], 9.3.1). The choice of I with orientation provides both the ordering and the arcs between consecutive points. The action of the braid group on the free group in these generators is given by $s_i(x_i) = x_i^{-1} x_{i+1} x_i, s_i(x_{i+1}) = x_i, s_i(x_j) = x_j, \forall j \neq i, i+1$. In particular $s_i(x_N \cdots x_1) = x_N \cdots x_1$.

[6]$\bar{s}_k$ is the clockwise Dehn twist about the segment $\overline{[P_k, P_{k+1}]} = [P_{N-k+1}, P_k]$ and $\bar{s}_{N-k}$ is the clockwise Dehn twist about the segment $[P_{N-k}, P_k]$

Conjugation on the braid group depends on the number of real points in the set $\mathcal{P}_N$ and has particularly form (3) if $\mathrm{Card}(\mathcal{P}_N \cap \mathbb{R}) \le 1$ or (4) if $\mathcal{P}_N \subset \mathbb{R}$ (the latter is the case considered in [3]).

Following Garside (cf. [9] sec. 1.2 and 2.1) we will use the involution $\mathfrak{R} : s_i \to s_{N-i}$ and the anti-homomorphism $rev : B_N \to B_N$ which is rewriting a word in generators s_i of the braid group or their inverses in reversed order. One has $rev(gh) = rev(h)rev(g), \forall g, h \in B_N$. We will use similar operations $\mathfrak{R}x_i = x_{N+1-i}$ and rev on generators $x_1, .., x_N$ of a free group. In particular, if $\mathrm{Card}\mathcal{P}_N \cap \mathbb{R} \le 1$ (equivalently $k = \frac{N}{2}$ or $k = \frac{N-1}{2}, N\ odd$) i.e. the action is given by (3) then

$$\bar{s}_i = \mathfrak{R}(s_i)^{-1} \quad \bar{x}_i = \mathfrak{R}(x_i)^{-1} \tag{6}$$

It follows from [9] that with such restriction on k, the action of complex conjugation on Garside word satisfies $\bar{\Delta} = \Delta^{-1}$. Also, note that the complex conjugation and inner automorphisms generate $AutB_N$ cf. [7]: the automorphism ϵ_n in that paper is the product of $\mathfrak{R}$ and complex conjugation; in the case $\mathcal{P}_N \in \mathbb{R}$, the automorphism ϵ_n is the complex conjugation cf. (4).

3. Braid monodromy of curves over $\mathbb{R}$

Recall the definition of braid monodromy in the context of real curves. Let $P \in \mathbb{RP}^2$, $p_{\mathbb{R}} : \mathbb{RP}^2 \setminus P \to N_{\mathbb{R}}$ be the projection from P onto a line $N_{\mathbb{R}}$ and $\mathcal{L}_{\mathbb{R}}$ be the corresponding pencil of lines in $\mathbb{RP}^2$. For each of these objects, the corresponding complexification will be denoted by the same letter but with $\mathbb{R}$ changed to $\mathbb{C}$. Complex conjugation acts on the set of $\mathbb{C}$-points of each of these sets, having the set of real points as the fixed point set. The fiber of projection $p_{\mathbb{C}}$ over $c \in N_{\mathbb{C}}$ will be denoted $L_{\mathbb{C},c}$.

Let $b \in N_{\mathbb{R}}$ be a point selected so that $L_{\mathbb{C},b}$ is transversal to the complexification $C_{\mathbb{C}}$ of a curve $C_{\mathbb{R}}$. Let $Cr \subset N_{\mathbb{C}}$ be the subset of the points c such that $\mathrm{Card}(p^{-1}(c) \cap C_{\mathbb{C}}) < d, d = \deg(C)$.

Let $\gamma(t)$ be a loop in $N_{\mathbb{C}}$ with initial and endpoints being at $b \in N_{\mathbb{R}}$ and situated in the upper half plane of $N_{\mathbb{C}}$. Let $\bar{\gamma}(t) = \overline{\gamma(t)}$ be its conjugate. Consider a trivialization of projection of the pair: $p : (p_{\mathbb{C}}^{-1}(\gamma), C_{\mathbb{C}} \cap p_{\mathbb{C}}^{-1}(\gamma)) \to \gamma$ i.e. a continuous map of pairs $\Phi : (I \times \mathbb{C}, I \times [d]) \to (p_{\mathbb{C}}^{-1}(\gamma), C_{\mathbb{C}} \cap p_{\mathbb{C}}^{-1}(\gamma))$ (here $I = [0,1]$ and $[d]$ is a fixed subset of $\mathbb{C}$ of cardinality d) such that

- (i) Φ is compatible with projections of its source and target onto I and γ respectively and in particular $\Phi(0, z) = \Phi(1, z) \in p_{\mathbb{C}}^{-1}(b)$ for any $z \in \mathbb{C}$.
- (ii) Restrictions of Φ onto $[0,1) \times \mathbb{C}$ and $(0,1] \times \mathbb{C}$ are homeomorphisms onto their targets.
- (iii) The trivialization is constant outside of a disk in $L_{\mathbb{C},b}$ containing $L_{\mathbb{C},b} \cap C$ (in particular, for any $x \in L_{\mathbb{C},b}$ outside of this disk and $z \in \mathbb{C}$ such that $\Phi(0, z) = x$ one has $\Phi(1, z) = x$).

The monodromy along the loop $\gamma(t)$ is a diffeomorphism of the pair $(L_{\mathbb{C},b}, C_{\mathbb{C}} \cap L_{\mathbb{C},b})$ into itself sending $x \in L_{\mathbb{C},b}$ to $\Phi(1, z(x))$ where $z(x)$ is the solution to $\Phi(0, z(x)) = x$. For a trivialization satisfying $(i), (ii), (iii)$, the braid corresponding to the isotopy class of such diffeomorphism via identification of Artin's braid group with the mapping class group of a disk with marked points will be denoted as $\beta(\gamma)$.

Definition 3.1. *The braid monodromy of a plane curve is the homomorphism* $\pi_1(N_{\mathbb{C}} \setminus Cr, b) \to B_d$ *which assigns to the class of a loop the braid in Artin's braid* *group corresponding to the diffeomorphism given by the monodromy obtained from a trivialization over the loop as described above.*

Following [21] *we present braid monodromy as a factorization of the word* Δ^2 *written as the product of braids representing the value of braid monodromy on a sequence of a good ordered system of generators of the fundamental group of the complement* $\pi_1(N_{\mathbb{C}} \setminus Cr, b)$*.*

Complex conjugation acts on trivializations as follows. Clearly, since C is defined over $\mathbb{R}$, for a loop γ with a base point $b \in \mathbb{R}$, one has

$$\overline{(p_{\mathbb{C}}^{-1}(\gamma), C_{\mathbb{C}} \cap p_{\mathbb{C}}^{-1}(\gamma))} = (p_{\mathbb{C}}^{-1}(\bar{\gamma}), C_{\mathbb{C}} \cap p_{\mathbb{C}}^{-1}(\bar{\gamma}))$$

Definition 3.2. *Conjugate of a trivialization* $\Phi : (I \times \mathbb{C}, I \times [d]) \to (p^{-1}(\gamma), C_{\mathbb{C}} \cap p_{\mathbb{C}}^{-1}(\gamma))$ *is the trivialization of* $p_{\mathbb{C}}$ *over the loop* $\bar{\gamma}$ *given by*

$$\bar{\Phi}(t, z) = \overline{\Phi(t, z)} \tag{7}$$

In particular, the monodromy diffeomorphism of $(p_{\mathbb{C}}^{-1}(b), C \cap p_{\mathbb{C}}^{-1}(b))$ *corresponding to trivialization* $\bar{\Phi}$*, in terms of trivialization* Φ *is given by:* $x \to \overline{\Phi(1, \tilde{z})}, x \in p_{\mathbb{C}}^{-1}(b)$ *where* $\tilde{z}$ *is determined by* $\Phi(0, \tilde{z}) = \bar{x} \in p_{\mathbb{C}}^{-1}(b)$

Remark 3.3. In general, it is impossible to trivialize over a loop the pair $(p_{\mathbb{C}}^{-1}(b), C \cap p_{\mathbb{C}}^{-1}(b))$ together with involution given by conjugation. The type of involution (given by the number of fixed points, i.e. the number of real points in the fiber) changes while one moves along γ, but this procedure provides a well-defined diffeomorphism of pairs with involution.

Remark 3.4. An alternative way to define a braid monodromy is to use the so-called coefficient homomorphism defined as the holomorphic map assigning to an affine plane curve given by Weierstrass polynomial $y^d + \sum_{i=0}^{d-1} a_i(x) y^i = 0$ the map $\mathbb{C} \to \mathbb{C}^d$ given by $x \to (a_{d-1}(x), \cdots, a_0(x))$. The restriction of this map to the complement to the set of critical values of projection of this curve onto x-plane takes it to the space of the coefficients of polynomials in one variable without multiple roots i.e. the complement in $\mathbb{C}^d$ to the discriminant hypersurface $Discrim$. This complement is the base of a locally trivial fibration of the complement in $\mathbb{C}^{d+1}$ to the hypersurface given by equation $y^d + a_{d-1} y^{n-1} + \cdots + a_0 \in \mathbb{C}^{d+1}$ onto $\mathbb{C}^d$ with coordinates $(a_{d-1}, \cdots, a_0)$. The action of the fundamental group $\pi_1(\mathbb{C}^d \setminus Discrim, b)$, which is isomorphic to Artin's braid group, is induced by its action on the fundamental group of the fiber of this fibration over the base point b. If $b \in \mathbb{R}^d$ then the complex conjugation acts on the braid group (since discriminant hypersurface is defined over $\mathbb{R}$) but the specific form of this action on generators depends on the choice of b. This complex conjugation on B_d is given by (3) and (4) with $d = N$ and depends on the number of real roots of $y^d + a_{d-1}(b) y^{n-1} + \cdots + a_0(b)$. In particular, if the number of real roots is at most one, the action is given by (6) and if all roots are real then one has $s_i \to s_i^{-1}$ for all i (as in [3]).

Definition 3.2 implies the following relation between the braids that the braid monodromy assigns to conjugate loops:

Proposition 3.5. *Let* $\beta : \pi_1(N_{\mathbb{C}} \setminus Cr) \to B_d$ *be the homomorphism of braid monodromy of a plane curve over* $\mathbb{R}$ *and let* $\bar{\gamma}$ *be the complex conjugate of a loop* $\gamma \in \pi_1(N_{\mathbb{C}} \setminus Cr, b)$*. Then*

$$\beta(\bar{\gamma}) = \overline{\beta(\gamma)}$$

and depending on $\mathrm{Card}(p^{-1}(b) \cap C_{\mathbb{R}})$ *the action of the complex conjugation on a factorization of* $\beta(\gamma)$ *is given by (3),(4),(5). In particular, if* $p_{\mathbb{C}}^{-1}(b)$ *has at most one real point then*

$$\beta(\bar{\gamma}) = (\Re rev(\beta(\gamma)))^{-1} \tag{8}$$

If all points of $p_{\mathbb{C}}^{-1}(b)$ *are real then one has:*

$$\beta(\bar{\gamma}) = (rev(\beta(\gamma)))^{-1} \tag{9}$$

i.e. coincides with the outer automorphism ϵ_d *used in* [7] [3].

Proof. Indeed, the braid $\beta(\bar{\gamma})$ being interpreted as an automorphism of the free group $\pi_1(p_{\mathbb{C}}^{-1}(b)\backslash p_{\mathbb{C}}^{-1}(b)\cap C, b)$ is the composition of conjugations, the automorphism corresponding to β and the conjugation i.e. is $\bar{\beta}$. The equalities (8) and (9) follow from the identities (6) (or (3)) and (4). □

Next, we will find a conjugation invariant form of the braid monodromy factorization of a real curve. Singularities of such a curve $C_{\mathbb{C}}$ are either the points with real coordinates or come in complex conjugate pairs. So are the critical values of projections on the complex locus of a $\mathbb{R}$- line: they are real values at real critical points or come as complex conjugate pairs. Recall that the image of $c \in C$ of projection on x-axis N from a point in $\mathbb{RP}^2 \subset \mathbb{CP}^2$ is a critical value if the line through the center of projection and c intersects C in fewer than $\deg C$ points and a projection from a center $p \in \mathbb{CP}^2$ is called generic if each line in the pencil of lines through p contains at most one singular or critical point on C and each critical point in the smooth locus of C is the simple tangency. We call a point in $\mathbb{RP}^2$ *generic* if it is an arbitrary point in a Zariski dense subset of $\mathbb{RP}^2$.

The following Proposition describes which critical points of generic projection are unavoidable on the real part of x axis (the target of the projection map).

Proposition 3.6. *Let* C *be a projective plane curve over* $\mathbb{R}$ *transversal to the line at infinity. Let, as above,* $p_P : \mathbb{CP}^2 \setminus P \to N_{\mathbb{C}}$ *be a projection from a point* $P \in \mathbb{RP}^2$ *onto a fixed line* N*. If* P *is generic then the only critical points of* p_P *on the real locus* $\mathbb{RP}^1 \subset N_{\mathbb{C}}$ *are either singular points of* C *with coordinates both being real or images of critical points of restriction of* p_P *on the real locus* $C_{\mathbb{R}}$ *of* C*.*

Proof. We will work in a coordinate system such that the projection from P is just the projection $(x, y) \to x$ onto the x-axis. First, notice that if a coordinate system in $\mathbb{RP}^2$ is generic then each singular point either has both coordinates real or both coordinates have non-zero imaginary parts. We assume that C is in a such coordinate system. Then note also that the number of real lines through a point $(z, w, 1) \in \mathbb{C}^2 \subset \mathbb{CP}^2$ is either infinite (if $(z, w) \in \mathbb{R}^2$) or is either 1 (z, w are on a line $\mathbb{C} = \mathbb{R}^2$ defined over $\mathbb{R}$) or zero. Consider the incidence correspondence $\mathcal{I} \subset N_{\mathbb{C}} \times \mathbb{RP}^2$ consisting of pair (a, P) such that a is the image of a critical point of projection onto $N_{\mathbb{C}}$ from P. The projection $\mathcal{I} \to \mathbb{RP}^2$ is a finite cover and $\mathcal{I}$ is a real two-dimensional manifold. Each real tangent transversal to the real locus of C either is a bitangent or contains singular points with one of the coordinates being not in $\mathbb{R}$. There are no points in the latter class by genericity assumption and only a finite set of points in the former class. Taking $P \in \mathbb{RP}^2$ which pre-image in $\mathcal{I}$ has an empty intersection with this finite set in $\mathcal{I}$ we get the required projection. □

We select coordinates in $\mathbb{CP}^2$ so that the base point of the pencil is at infinity and the lines of the pencils are the lines $x = c$ (i.e. the center of projection is

$(0,1,0))$ and we have the projection $p : \mathbb{C}^2 \to \mathbb{C}_x$ onto the x axis given by $y = 0$. Moreover, the trivialization of p over $\mathbb{C}_x = \{y = 0, z \neq 0\}$ is given by projection $(x, y) \to y$ onto y axis $\mathbb{C}_y$.

Let $Cr \subset \mathbb{C}_x$ be the critical set of the projection and $N = \mathrm{Card} Cr$. Let us view Cr as a subset $\mathcal{P}_N$ discussed in Section 2, select an order in Cr, a base point b, and a good ordered system of generators in $\pi_1(\mathbb{C} \setminus Cr, b)$ as described there. We call this a complete good ordered system of generators *compatible with the real structure* and denote its elements[7]

$$\gamma_1, \cdots \gamma_h, \gamma_l^r, \cdots, \gamma_1^r, \bar{\gamma}_h^{-1}, ...\bar{\gamma}_1^{-1} \tag{10}$$

Here $\bar{\gamma}$ is the class of the loop containing the image of the conjugation map $H^+ \to H^-$ applied loop to γ.

Finally, we will denote by $Dc \subset \mathbb{C}_x$ be a closed subset bounded by a loop with base point b and such that $Dc \cap Cr$ is the set of *real* critical values.

Definition 3.7. The factorization

$$\beta_1 \cdot \cdots \cdot \beta_N = \Delta^2, \qquad N = 2h + l$$

where the braids β_i are the images of the braid monodromy homomorphism $\pi_1(\mathbb{C}_x \setminus Cr, b) \to B_d$ corresponding to the loops (10) will be called *compatible with the real structure.*

The product of the braids corresponding to the loops $\gamma_1, \cdots, \gamma_h$ will be denoted $\mathcal{B}_{H^+}$, the product of the braids corresponding to $\gamma_l^r, \cdots, \gamma_1^r$ we denote $\mathcal{B}_{\mathbb{R}}$, and the product of the braids corresponding to the remaining loops in this system we denote $\mathcal{B}_{H^-}$ so that

$$\Delta^2 = \mathcal{B}_{H^+} \mathcal{B}_{\mathbb{R}} \mathcal{B}_{H^-}$$

Theorem 3.1. *Let C be a projective plane curve over $\mathbb{R}$. Let $p : \mathbb{C}^2 \to \mathbb{C} = N_{\mathbb{C}}$ be a projection of the affine part of $\mathbb{CP}^2$ with the line at infinity being transversal to $C_{\mathbb{C}}$. Let b be the base point in the real locus of $N_{\mathbb{C}}$ such that $\mathrm{Card} p_{\mathbb{C}}^{-1}(b) \cap C_{\mathbb{R}} \leq 1$. The braid monodromy factorization corresponding to a good ordered system of generators of $\pi_1(N_{\mathbb{C}} \setminus Cr, b)$ compatible with real structure induces decomposition*

$$\Delta^2 = \mathcal{B}_{H^+} \cdot \mathcal{B}_{\mathbb{R}} \cdot \mathfrak{R}(rev(\mathcal{B}_{H^+})) \tag{11}$$

where the braids $\mathcal{B}_{H^+}, \mathcal{B}_{\mathbb{R}}$ are as in Definition 3.7.

Proof. Since $\mathcal{B}_{H^-} = \bar{\gamma}_h^{-1}...\bar{\gamma}_1^{-1}$, using Proposition 3.5 this braid can be written as:

$$(\mathfrak{R}(rev(\beta(\gamma_h)))^{-1})^{-1} \cdots (\mathfrak{R}(rev(\beta(\gamma_1)))^{-1})^{-1} = \mathfrak{R}(rev(\beta(\gamma_h)))....\mathfrak{R}(rev(\beta(\gamma_1)))$$
$$= \mathfrak{R}(rev(\beta(\gamma_h))rev(\beta(\gamma_{h-1})) \cdots rev(\gamma_1)) = \mathfrak{R}(rev(\beta(\gamma_1) \cdots \beta(\gamma_h))) =$$
$$Re(rev(\mathcal{B}_{H^+}))$$

as claimed. □

Remark 3.8. We would like to describe the braid $\mathcal{B}_{H^+}$ (and hence $\mathcal{B}_{H^-}$) in terms of the braid $\mathcal{B}_{\mathbb{R}}$ corresponding to the real part of the critical locus i.e. to solve in the braid group the equation (11). Unfortunately, as pointed out to the author by the referee, the constraint of the Theorem 3.1 is not sufficient for recovering the braids $\mathcal{B}_{H^\pm}$ and $\mathcal{B}_{\mathbb{R}}$ even in the simples cases. Consider the case when $\mathcal{B}_{\mathbb{R}} = 1$,

[7] the order of the loops in this system is from the left to the right but subscripts specify the points in the set Cr with the ordering described above.

corresponding to the case of real curves without real points and for which the projection has no real critical values i.e.

$$\mathcal{B}\mathfrak{R}rev(\mathcal{B}) = \Delta^2 \tag{12}$$

Since $\mathfrak{R}(rev(\Delta)) = \Delta$ (cf. [9] Lemma 3), one expects that in decomposition (11) $\mathcal{B}_{H^+} = \mathcal{B}_{H^-} = \Delta$:

However, in B_4, there is another solution to (11) i.e. $\Gamma\mathfrak{R}rev\Gamma = \Delta^2$.

$$\Delta s_1^{-1}s_3\mathfrak{R}(rev\Delta s_1^{-1}s_3) = \Delta s_1^{-1}s_3s_1s_3^{-1}\Delta = \Delta^2 \tag{13}$$

Note that

$$\Delta s_1^{-1}s_3 = (s_1s_2s_3)^2 \quad \textit{and} \quad s_3(s_1s_2s_3)^2s_3^{-1} = \Delta \tag{14}$$

Nevertheless, we show below that for the algebraic curves in question the braids $\mathcal{B}_{H^\pm}$ are the a conjugates of "obvious" solutions. It would be interesting to find additional constraints on the braids $\mathcal{B}_{H^\pm}, \mathcal{B}_{\mathbb{R}}$ that allow to streamline their calculation and to understand the geometric significance of all solutions to (11). The equation (11) has a certain similarity with the problem of finding roots in elements in the braid group (cf. [10]) but a direct connection is not clear.

In response to the author's query, Juan Gonzalez-Meneses informed the author that he obtained classification results regarding the solutions of the equation (12) for each of the Thurston-Nielsen classes: cf. [11].

4. Presentations of fundamental groups of real curves.

Recall that van Kampen's theorem (cf. [23],[21]) gives the following presentation in terms of the braids β_i described in definition (3.7).

$$\pi_1(\mathbb{C}^2 \setminus C) = \{x_1 \cdots x_d | \beta_i(x_j) = x_j\} \quad j = 1, \cdots d, i = 1, \dots, \mathrm{Card}Cr \tag{15}$$

In this section, we show that for curves over $\mathbb{R}$, the fundamental groups of the complement to $C_{\mathbb{C}}$ are quotients of geometrically defined and depending on real structure groups admitting van Kampen type presentations but requiring fewer relations than in (15).

With notations as in Section 3, let $Cr_{\mathbb{R}} \subset Cr \subset N_{\mathbb{C}}$ be the subset of *real* critical values of projection p. Recall that Dc is a disk in $N_{\mathbb{C}}$ bounded by a loop based at b and such that $Dc \cap Cr = Cr_{\mathbb{R}}$.

Theorem 4.1. *(1) The group $\pi_1(p^{-1}(H^+ \bigcup Dc) \setminus C, b)$ has a presentation:*

$$\pi_1(p^{-1}(H^+ \bigcup Dc) \setminus C, b) = \{x_1, .., x_d | \beta_i(x_j) = x_j\} \qquad j = 1, \cdots d, i = 1, .., h + l \tag{16}$$

(2) Inclusion $p^{-1}(H^+ \bigcup Dc) \setminus C \to \mathbb{C}^2 \setminus C$ induces the surjection:

$$\pi_1(p^{-1}(H^+ \bigcup Dc \setminus C), b) \to \pi_1(\mathbb{C}^2 \setminus C, b) \to 1 \tag{17}$$

Proof. Recall that each loop in a good ordered system of generators bounds a disk containing a single critical value of projection p. The complement $\mathbb{C}^2 \setminus C$ can be retracted onto the union of p-pre-images of disks bounded by the loops in a good ordered system of generators (cf. [16]) and the fundamental group of the preimage of a disk corresponding to critical value j has presentation $\{x_1, \cdots x_d | \beta_j(x_i) =$

$x_i, i = 1 \cdots d\}$. Part (1) follows from the van Kampen theorem about a union of spaces (cf. [12]).

Part (2) is a corollary of part (1) since the set of relations of $\pi_1(\mathbb{C}^2 \setminus C, b)$ consists of the same relations as the relations for $\pi_1(\pi^{-1}(H^+ \bigcup Dc \setminus C), b)$ and additional relations (corresponding to the critical points of projection of C in the lower half plane. □

Proposition 4.1. *Let $\beta_1, ..., \beta_r \in B_d$ be a finite set of braids and $x_1, \cdots, x_d$ be a system of generators of a free group F_d. Let $\beta = \beta_1 \cdots \beta_r$ be their product, $G(\beta_1, \cdots \beta_r)$ (resp. $G(\beta)$) be the quotients of the free group F_d by the normal subgroup generated by elements $\beta_i(x_j)x_j^{-1}, i = 1, \cdots r, j = 1, \cdots, d$ (resp. the relations $\beta(x_j)x_j^{-1}, j = 1, \cdots, d$). Then one has surjection:*

$$G(\beta) \to G(\beta_1, \cdots, \beta_r) \to 1 \tag{18}$$

Proof. We shall show this by induction over r. Assume that for all $1 \leq j \leq d$ the element $\beta_1 \cdots \beta_{r-1}(x_j)x_j^{-1} \in F_d$ belongs to the normal subgroup N_{r-1} of F_d generated by $\beta_i(x_j)x_j^{-1}, i = 1, \ldots, r-1$. Let $\beta' = \beta_1 \cdots \beta_{r-1}$. Then for any $x_j, j = 1, \ldots, d$

$$\beta(x_j)x_j^{-1} = \beta'(\beta_r(x_j))x_j^{-1} = \beta'(\beta_r(x_j))\beta_r(x_j)^{-1}\beta_r(x_j)x_j^{-1} \tag{19}$$

Let $\beta_r(x_j) = y_1 y_2 \cdots y_s$ where $y_k, k = 1, \ldots, s$ is one of generators $x_1, \ldots, x_d$ or their inverses, with possibly several of y_k corresponding to the same element among x_j. In particular, we have $\beta'(y_k)y_k^{-1} \in N_{r-1}$ by the assumption of induction. Then the right-hand side in (19) can be written as:

$$\beta'(y_1 \cdots y_s)y_s^{-1} \cdots y_1^{-1}\beta_r(x_j)x_j^{-1} = \beta'(y_1) \cdots \beta'(y_s)y_s^{-1} \cdots y_1^{-1}\beta_r(x_j)x_j^{-1} \tag{20}$$

The surjection $F_d \to F_d/N_{r-1}$ takes $\beta'(y_s)y_s^{-1}$ to $1 \in F_d/N_{r-1}$ i.e. the last expression in (20) goes to the same element as does $\beta'(y_1) \cdots \beta'(y_{s-1})y_{s-1}^{-1} \cdots y_1^{-1}\beta_r(x_j)x_j^{-1}$ and the latter goes to the same element as $\beta'(y_1) \cdots \beta'(y_{s-2})y_{s-2}^{-1} \cdots y_1^{-1}\beta_r(x_j)x_j^{-1}$ since $\beta'(y_{s-1})y_{s-1}^{-1}$ goes to $1 \in F_d/N_{r-1}$ and so on. In particular, the last expression in (20) is the normal subgroup $N_r \subset F_d$ generated by N_{r-1} and the element $\beta_r(x_j)x_j^{-1}$ i.e. the subgroup of F_d generated by the relations of $G(\beta_1, \cdots, \beta_r)$ which shows the claim. □

Remark 4.2. It is well known that the fundamental group of the complement to a singular curve is an invariant of equisingular isotopy of complex algebraic curves on surfaces (cf. [17] for references therein). In the case of real curves a natural problem is to understand the *rigid* equisingular isotopy classes or at least the classes of equivariant (with respect to complex conjugation) equisingular isotopy (cf. [22] Sect. 4 for non-singular case). The fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_{\mathbb{C}}, b), b \in \mathbb{RP}^2$ endowed with the involution provides an invariant of classes of such restricted isotopy in the sense that for any two real curves C_1, C_2 isotopic via equivariant equisingular isotopy, there exist an isomorphism of the fundamental groups equivariant with respect to involutions induced by conjugation. For example, if C is a pair of lines in $\mathbb{C}^2$, then $\pi_1(\mathbb{C}^2 \setminus C, b, \mathbb{Z}) = \mathbb{Z}^2, i = 1, 2, b \in \mathbb{R}^2$, and the involution induced by conjugation has the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$) if both lines are defined over $\mathbb{R}$ (resp. both lines are imaginary) i.e. the additional structure distinguishes the classes of rigid isotopy.

For a curve over $\mathbb{R}$, it follows from Prop. 3.5 that van Kampen presentation (15 for the braid monodromy in generators used in Proposition 3.5 the automorphism of the free group given by (3) and (4) passes to an involution of the fundamental group of the complement. Hence we obtained a calculation of this extra structure on the fundamental group.

This involution can be encoded into an exact sequence:

$$0 \to \pi_1(\mathbb{C}^2 \setminus C_{\mathbb{C}}, b) \to \pi_1^{\mathbb{R}}(\mathbb{C}^2 \setminus C_{\mathbb{C}}, b) \to \mathbb{Z}_2 \to 0 \tag{21}$$

in which the action of $\mathbb{Z}_2$ on $\pi_1(\mathbb{C}^2 \setminus C_{\mathbb{C}}, b)$ is given by conjugation. The group in the middle is a topological analog of Grothendieck's fundamental group of a variety over $\mathbb{R}$.

5. Alexander Polynomials

Surjections of the previous section imply the divisibility relations between the Alexander polynomials of the groups considered there.

Recall (cf. [17] and the references therein) that given a group G and a surjection $\phi : G \to \mathbb{Z}$ one defines the Alexander polynomial as the order of the torsion part of the module over the ring of Laurent polynomials $\mathbb{Q}[t, t^{-1}]$ with underlying $\mathbb{Q}$- vector space being the quotient of Kerϕ by its commutator with constants extended to $\mathbb{Q}$. The module structure is defined by requiring that the action of t be given by the automorphism induced on Kerϕ by the conjugation by a lift to G of the positive generator of the target $\mathbb{Z}$ of ϕ (this action when considered on abelianization of Kerϕ is independent of a lift).

In terms of cyclic decomposition

$$\text{Ker}\phi/(\text{Ker}\phi)' \otimes \mathbb{Q} = \oplus \mathbb{Q}[t, t^{-1}]^a \oplus (\oplus_{i=1}^b \mathbb{Q}[t, t^{-1}]/(\Delta_i(t)) \quad \Delta_i | \Delta_{i+1}, \tag{22}$$

this order is given by

$$\Delta(t) = \Delta_1(t) \cdots \Delta_b(t)$$

if $a = 0$ and $\Delta(t) = 0$ if $a \geq 1$.

Recall also (cf. [1] Th. 2.2) that Artin showed that the fundamental group of the complement to a link in S^3 represented by a closed braid β has the same presentation as the group $G(\beta)$ from Proposition 4.1 [8]

Proposition 5.1. *Let $G(\beta)$ and $G(\beta_1, \cdots, \beta_r)$ be associated with braids β and $\beta_1, \cdots \beta_r$ groups with generators and relations described in Proposition 4.1. Let Δ_β and $\Delta_{\beta_1, \cdots, \beta_r}$ be the Alexander polynomials of these groups relative to surjections ϕ_β and $\phi_{\beta_1, \cdots, \beta_r}$ onto $\mathbb{Z}$ which send each of their generators $x_1, \cdots, x_d$ to the positive generator of $\mathbb{Z}$. Then $\Delta_{\beta_1, \cdots, \beta_r}$ divides Δ_β.*

Proof. The surjection (18) and the commutative diagram

$$\begin{array}{ccc} G(\beta) & \rightarrow & G(\beta_1, \cdots \beta_r) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z} & \end{array}$$

induce the surjections of $\mathbb{Q}[t, t^{-1}]$-modules:

$$\text{Ker}\phi_\beta/(\text{Ker}\phi_\beta)' \otimes \mathbb{Q} \to \text{Ker}\phi_{\beta_1, \cdots, \beta_r}/(\text{Ker}\phi_{\beta_1, \cdots, \beta_r})' \otimes \mathbb{Q} \to 0 \tag{23}$$

---

[8] in fact, [1] shows the presentation of the fundamental group of link in S^3 is the group $G(\beta)$ i.e. $G(\beta) = \{x_1, \cdots, x_d | \beta(x_i)x_i^{-1}, i = 1, \cdots d\}$ with one relation deleted, but there it is also pointed out that this relation is the combination of the remaining $d - 1$ relations.

Since the left group in (18) is the fundamental group of the complement to a link in 3-sphere for which the Alexander module is a torsion module, it follows from (23) that the right group in (18) has as its Alexander module a torsion module and the claim follows. □

Theorem 5.1. *The Alexander polynomial of $\pi_1(\mathbb{C}^2 \setminus C)$ divides the Alexander polynomials of the closed braids in 3-sphere associated with each of the braids $\mathcal{B}_{H^+}, \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{H^+} \cdot \mathcal{B}_{\mathbb{R}}$ in the braid group B_d.*

Proof. The theorem 4.1 identifies the group $\pi_1(p^{-1}(H^+ \bigcup Dc) \setminus C, b)$ with the group $G(\beta_1, \cdots \beta_{h+l})$ and Proposition 5.1 shows that the Alexander polynomial of this group divides the Alexander polynomial of the closed braid $\mathcal{B}_{H^+}\mathcal{B}_{\mathbb{R}}$. The surjection in Theorem 4.1 implies that the Alexander polynomial of $\pi_1(\mathbb{C}^2 \setminus C)$ divides the Alexander polynomials of $\pi_1(p^{-1}(H^+ \bigcup Dc) \setminus C, b)$ as in the proof of Proposition 5.1 which completes the proof. □

6. Examples.

In this section we discuss the braids $\mathcal{B}_{H^+}, \mathcal{B}_{\mathbb{R}}$ for the curves satisfying assumptions on the real part of the critical set and use them to get refined divisibility conditions for the Alexander polynomials in corresponding classes of plane singular curves over $\mathbb{R}$.

The considered extreme cases are the case of the arrangements of real lines and real arrangements of lines with zero-dimensional real locus, the curves of even degree with empty real locus, and related curves of odd degrees. To obtain a non-trivial divisibility relation one makes a different selection of the braid $\mathcal{B}_{H^+}$ or $\mathcal{B}_{\mathbb{R}}$ or their product. In the case of arrangements of real lines, the braid $\mathcal{B}_{H^+}$ is trivial, the Alexander polynomial of the closed braid in $S^1 \times \mathbb{C}$ is zero, and the divisibility relation is empty. On the other hand, $\mathcal{B}_{\mathbb{R}} = \Delta^2$, and one obtains a known divisibility relation mentioned in Section 1. Note that the Alexander module of a link in S^3 (a closed braid) is torsion of the linking number of any two components is non zero (cf. [4]).

On the other hand, for the real curves with no real points, we show that $\mathcal{B}_{H^+} = \Delta$ which leads to refined divisibility constraints (cf. Prop 6.7).

Example 6.1. We consider *maximally componentwise unreal arrangements* which are the arrangements over $\mathbb{R}$ with the minimal number of real points. Note the following:

Proposition 6.2. *Let $\mathcal{A}$ be an arrangement over $\mathbb{R}$. Then the set of its real points is not empty.*

Proof. This is immediate since the intersection point of a pair of conjugate lines is real. □

Let $\mathcal{A}_k$ be an arrangement over $\mathbb{R}$ with $k < \infty$ real multiple points. Such an arrangement has the form

(24)
$$\prod_{j=1}^{r_1}(((x+ay)-n_1)^2+m_j^1y^2)\cdots\prod_{j=1}^{r_i}(((x+ay)-n_i)^2+m_j^iy^2)\cdots\prod_{j=1}^{r_k}(((x+ay-n_k)^2+m_j^{r_k}y^2)$$

where

$$0 \neq a \in \mathbb{R}, n_i \in \mathbb{R}, \quad n_1 > n_2 \cdots > n_k, \quad m_i^j \in \mathbb{R}^+, \pm m_{i'}^{j'} \neq \pm m_i^j \quad \forall (i,j) \neq (i',j')$$

$\mathcal{A}_k$ contains $d = 2\sum_1^k r_i$ lines with k real multiple points $(n_i, 0), i = 1, .., k$ having respective multiplicities $2r_i, i = 1, \cdots, k$. The singular points, which are the intersection points of lines with different n_i's for $a \neq 0$, have x-coordinates with non-zero imaginary parts and come in conjugate pairs i.e. $Cr\mathcal{A}_k = \{x \in \mathbb{C} | x = n_i\}$. The only real points of $\mathcal{A}_k$ are the points $(n_i, 0), i = 1, \cdots, k$.

Let us describe the braid $\mathcal{B}_{\mathbb{R}}$ in this case and more specifically the braids corresponding to (the classes of the) loops $\gamma_i^r \in \pi_1(\mathbb{C}_x \setminus Cr, b)$ (cf. Definition 3.7) i.e. corresponding to paths running from b along real axis to one of the points $(n_i, 0)$ while circumventing points in Cr following a small semi-circle in the upper half plane, then upon reaching the vicinity of $(n_i, 0)$ following the full circle around $(n_i, 0)$ and finally returning to b along the same path.

Proposition 6.3. *There is a collection of non-intersecting segments $\delta_1, \cdots \delta_i \cdots \delta_k$ in $L_{\mathbb{C},b}$ each containing a set A_i or $2r_i$ points belonging to the i-th group of lines (24) in $\mathcal{A}_k$ such that braid $\mathcal{B}_{\mathbb{R}}$ is a product of the conjugates of the full twists $\Delta^2_{A_k}$ about δ_i. In particular Δ^2_i commute.*

Proof. Explicit form of the lines (24) shows that y coordinates of the intersections of lines in this arrangement with the fibers $L_{\mathbb{C},t}$ of projection used to calculate the braid monodromy have the form

$$Re(y) = \lambda^i_j Im(y), \quad i = 1, \cdots, k, j = 1, \cdots, r_i, \lambda^i_j \neq \lambda^{i'}_{j'}, (i, j) \neq (i', j') \tag{25}$$

and the braids corresponding to γ_i^r can de described in terms of the motion of d points $\mathcal{A}_k \cap \mathcal{L}_{\mathbb{C},b}$ along these lines [9]. Hence these d points are split into k groups each containing $2r_i, i = 1, ..., k$ points. Each group moves toward the origin along the respective group of lines in $\mathbb{C}$ and arrives at $0 \in \mathbb{C}$ at k different moments. A group of $2r_i$ points while moving along their respective lines undergoing slight deviations at the moments when other groups reach their critical points (corresponding to γ_i^r deviations from the real axis in x-plane), Just before the time when the i-th group should arrive at the origin the points in this group undergo full twist (the braid corresponding to the the singularity of $2r_i$ pairwise transversal lines) and returning to the the original position of the group along the same path.

With each of k critical points is associated "vanishing segment" containing $2r_i$ points merging into $(n_i, 0)$ which is located at $L_{\mathbb{C},b'}$ where b' is point in the real part of x axis and where γ_i^r starts the circle around $x = n_i$. Transporting this segment along the path of γ_i^r back from b' to b produces the vanishing segment of this group in $L_{\mathbb{C},b}$. While t moves from b' toward n_i and then completes the move along semi-circle, the set of $2r_i$ points merging in $(n_i, 0)$ move from half-plane $Re < 0$ to half-plane $Re > 0$. If t continues to move in a negative direction instead of completing the full circle around $(n_i, 0)$, this segment will remain in the right half plane when t moves to the next critical value. Hence vanishing segments corresponding to the critical values n_i can be selected inductively so that the segment corresponding to n_i does not intersect the segments corresponding to $n_j, j < i$. As a result, we obtain that the braids corresponding to the loops γ_i^r are full twists along the collection of non-intersecting conjugation invariant segments in $L_{\mathbb{C},b}$:

$$\Delta^2_{A_1}, \Delta^2_{A_2} \cdots \Delta^2_{A^k} \tag{26}$$

[9]Explicitly the intersection points of $L_{\mathbb{C},t}$ and the lines corresponding to the i-th factor in (24) are $y = \frac{(n_i - t)(a \pm m_j\sqrt{-1})}{a^2 + m_j^2}$ and $\lambda^i_j = \pm\frac{a}{m^i_j}$

where Δ_{A_i} is a rotation by 180 degrees of the group of points A_i. In particular, these braids commute. The braid $\mathcal{B}_{\mathbb{R}}$ is the product of those twists. □

Next, consider other classes of algebraic curves.

Proposition 6.4. *Let C be a real curve having degree $2d$ without real points and admitting a generic projection $\pi_{\mathbb{C}} : C_{\mathbb{C}} \to \mathbb{C}$ having no critical values on the real axis. Then the braid $\mathcal{B}_{H+}$ is a conjugate of the Garside element Δ_{2d} of the braid group B_{2d}. In particular, the braid monodromy of such curves satisfies the relation:*

$$\beta_1 \cdots \beta_h = \gamma^{-1} \Delta_{2d} \gamma \tag{27}$$

where $\beta_1, \cdots, \beta_h$ are the braids corresponding to the loops in a good ordered system of generators corresponding to the critical values in the upper half-plane and $\gamma \in B_{2d}$.

Remark 6.5. Consider the following conditions on a curve and its projection:

a) empty (i.e. without real points) over $\mathbb{R}$ which the projection onto the x-axis is generic (as described before Prop.3.6) and has no real critical values.

b) real curves given by an equation $f(x, y) = 0$ where $f(x, y)$ has a constant sign for all $(x, y) \in \mathbb{R}^2$ and projection onto x axis is generic.

c) curves of an even degree admitting generic projection without real critical values.

Clearly a) implies b) and c) implies a). Indeed, a curve of an odd degree does have real points. If c is a real critical value of projection of a curve as in c), then the critical point is either real, i.e. the curve is not empty, or a pair of conjugate critical points i.e. the projection is not generic. A curve of an even degree, admitting generic projection without real critical values does not have to be empty (e.g. projection from an interior point of a real circle).

Proof. (of Prop. 6.4). Let $C_G \subset \mathbb{CP}^2$ be a union of real smooth quadrics given by equation $G(x, y) = \prod_{j=1}^{d}(j(x^2 + N^2) + y^2) = 0$, $(N \in \mathbb{N})$. Projection of C_G onto x axis has two critical values $x = \pm N\sqrt{-1}$ each corresponding to a singular point of C_G (for $d > 1$) each being a union d smooth branches tangent to the fiber of the projection. The intersection index of each two branches at the critical point is 2. The alternative local model of this singularity (at the origin) is $\prod_1^d (j^2 x - y^2)$. The subset of j-th branch of the latter germ over the loop $\gamma(s) = e^{2\pi i s}$ in the x-axis is $(e^{2\pi i s}, j e^{\pi i s}), j = 1, \cdots d, 0 \le s \le 1$. It follows that the corresponding braid of this singularity is the half-twist Δ_{2d}. Since, the loop $\gamma(s)$ (with the base point on the real axis) is homotopic to the loop represented by the real axis in $H^+ \setminus i\sqrt{N}$, the braid of C_G corresponding to the real axis is conjugate to Δ_{2d}.

Let $F = 0$ be the equation of C and let h be the number of critical points of projection of $C_{\mathbb{C}}$ onto x-axis located in H^+. Consider deformation $F_s = (1 - s)F + sG, 0 \le s \le 1$ and the corresponding family of curves C_{F_s}. Critical points of projection of C_{F_s} located in H^+ (resp. H^-) move to $N\sqrt{-1}$ (resp. $-N\sqrt{-1}$) along paths $\gamma_i(s)$ starting at h critical points of F formed by the critical points of $C_{F_s,\mathbb{C}}, s < 0 < 1$ and ending at the respective critical point of G.

As long as the projection of the curve $C_{F_s,\mathbb{C}}$ onto x axis does not have real critical values, this curve remains transversal to $\mathbb{R} \times \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ where the first factor is the real locus of x-axis and the second one is y-axis. In particular, the braid $\mathcal{B}_{H^+}$ corresponding to the curve $C_{F_s,\mathbb{C}}$ does not change. For s such that $\gamma_i(s)$ crosses the real axis from H^+ to H^-, the conjugate $\bar{Q}$ of the corresponding critical point Q of $C_{F_s,\mathbb{C}}$ has the same local type as does Q. The critical value of projection of $\bar{Q}$ crosses from the opposite half-plane of the critical value at Q and moves along $\overline{\gamma(s)}$. In particular, the factor of $\mathcal{B}_{H^+}$ which the braid monodromy associates with the element in the good ordered system of generators corresponding to the critical value on the path $\gamma_i(s)$ moving toward the real axis is the braid corresponding to the same element in $\pi_1(H^+ \setminus Crit(F(s))$ and the same singularity as before the crossing the real axis. So there will be no change in the braid $\mathcal{B}_{H^+}$ and the braid corresponding to C_F coincides with a conjugate of the braid of C_G i.e. $\gamma\Delta_{2d}\gamma^{-1}, \gamma \in B_{2d}$. □

Recall that an acnode of multiplicity l (cf. [24]) is a germ of a curve defined over $\mathbb{R}$ which set of real points consists of a single point and the set of complex points is a union of l transversal smooth branches. Note that this implies that must be even i.e. $l = 2k$ and the branches are pairwise conjugate.

Proposition 6.6. *Let C be a curve of degree $2d$ defined over $\mathbb{R}$ for which the set of real points consists of a single acnode of multiplicity $2k$. Then up, to conjugation in B_{2d}, one has $\mathcal{B}_{\mathbb{R}} = \Delta_{2k}^2$ and $B_{H^+} = \gamma\Delta_{2d}\Delta_{2k}^{-1}\gamma^{-1} = \mathcal{B}_{H^-}$ where Δ_{2k} is a 180 degrees rotation of a complex conjugation invariant subset of $2d$ points.*

Proof. Let $F(x, y)$ be an equation of C. As in Prop. 6.4 the assumption of Prop. 6.6 implies that if there exists (x_0, y_0) such that $F(x_0, y_0) > 0$ then $F(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ with the vanishing taking place only at the acnode. As in Prop. 6.4 consider the family of curves C_s with the equation $F_s(x, y) = (1 - s)F + sG = 0$. Then $F_s(x, y) > 0$ for all $(x, y), 0 < s \leq 1$. The local braid of acnode is (a conjugate in B_{2d} of) the full twist Δ_{2k}^2 on a set of $2k$ points. It coincides with the component $B_{\mathbb{R}}(C_0)$ of the braid of the curve C corresponding to the real part of the set of critical values. Moreover, when s moves away from zero, the critical value κ of the projection at the acnode splits for $0 < s << \epsilon$ into a finite set $\mathcal{S}(s)$ of K conjugate pairs of points. We select the base point b (cf. beginning of Section 3) in proximity to κ so that Garside operations on elements of $B_{2k} \subset B_{2d}$ coincide with those induced from B_{2d} i.e. $\mathfrak{R}_{B_{2d}}(rev_{B_{2d}}\Delta_{2k}) = \Delta_{2k}$. A small loop γ_ϵ around κ, also containing the set $\mathcal{S}$ inside it, is taken by the braid monodromy to $\mathcal{B}_{\mathbb{R}}(C_0)$ but the loop γ_ϵ is also split into a product of two conjugate loops $\gamma_\epsilon^+, \gamma_\epsilon^-$ each containing inside the subsets $\mathcal{S} \cap H^+$ and $\mathcal{S} \cap H^-$ respectively. The local version of the argument in Prop.6.4 deforming the germ of the acnode to the special germ having only two conjugate critical values shows that the braid monodromy takes each of $\gamma_\epsilon^\pm$ to the rotation by 180 degrees Δ_k. This implies that the loop $\mathcal{B}_{H^+}(C_0)\beta(\gamma_\epsilon^+)$ is a conjugate of Δ_{2d}. Hence $\mathcal{B}_{H^+}$ is a conjugate of $\Delta_{2d}\Delta_{2k}^{-1}$. □

Now we turn to the Alexander polynomials and explicit divisibility relations.

Proposition 6.7. *Let C be a real curve of even degree d admitting a projection with no real critical values. Then*

$$\Delta_C(t) | (t^d - 1)^{\frac{d}{2}-1}(t^{\frac{d}{2}} + 1)(t - 1) \tag{28}$$

In particular, the multiplicity of the root $exp\frac{2\pi i}{6}$ of the Alexander polynomial of such a curve, having nodes and cusps as the only singularities, is at most $\frac{d}{2} - 1$.

Proof. We need to find the Alexander polynomial of the link which is represented by the closed braid on d strings given by Δ i.e. the rotation by 180 degrees. As was already mentioned, such a link also can be described as the link of plane curve singularity given by local equation $\prod_{j=1}^{\frac{d}{2}}(j^2x - y^2)$, (when preimage of $x = 1$ are integers $\pm j, j \leq \frac{d}{2}$) or equivalently $x^{\frac{d}{2}} - y^d = 0$. The Alexander polynomial of such a link is the characteristic polynomial of its monodromy. This is a weighted homogeneous singularity with weights of x and y being $\frac{d}{2}$ and d. Now the claim follows from Brieskorn-Pham-Milnor classical calculations (cf. [19] Sec. 9). □

For the curves with a single acnode we obtain as an immediate consequence of Proposition 6.6 since $\mathcal{B}_{\mathbb{R}} = \Delta_k^2$:

Corollary 6.8. *The Alexander polynomial of a curve over* $\mathbb{R}$ *with a single real point which is an acnode divides the Alexander polynomial of the link which is the closure of the braid* $\Delta_d\Delta_k^{-1}$.

Example 6.9. The above Proposition gives a sharp "estimate of the degree of Alexander polynomials" of real curves of degree 6 without real points.

Recall that the Alexander polynomial of a plane curve of degree 6 with $k > 6$ cusps and at most nodes as other singularities is given by $(t^2 - t + 1)^{k-6}$. More specifically, for a curve of degree d with cusps and nodes as the only singularities, denoting the zero-dimensional subset of $\mathbb{P}^2$ formed by cusps as Ξ, the Alexander polynomial is equal to 1 if d is not divisible by 6 and otherwise is equal to $(t^2-t+1)^s$ where $s = \dim H^1(\mathbb{P}^2, \mathcal{I}_{\Xi}(d - 3 - \frac{d}{6}))$ with $\mathcal{I}_{\Xi}$ denoting the ideal sheaf of functions vanishing at points of Ξ. For $d = 6$ the latter is equal to $H^0(\mathbb{P}^2, \mathcal{I}_{cusps}(2)) - \chi(\mathbb{P}^2, \mathcal{I}_{cusps}(2)) = k - 6$ since $\dim H^0(\mathbb{P}^2, \mathcal{I}_{\Xi}(2)) = 0$ if the number of cusps is greater than 6 (if the cardinality of Ξ is 6 this dimension can be either 1 or 0 depending on 6 cusps being positioned on a conic or not).

The divisibility theorem of [15] for a complex curve of degree d transversal to the line at infinity tells that the Alexander polynomial of the curve divides $(1 - t)(1 - t^d)^{d-2}$. Indeed, the latter is the Alexander polynomial of the link at infinity which is the link of the closure of the braid Δ^2 (the Hopf link). This bounds the exponent of the Alexander polynomial of sextic by 4. On the other hand by 6.7 the Alexander polynomial of a sextic with no real points should divide $(t^3 - 1)(t + 1)^2(t^2 - t + 1)^2(t - 1)$. There is no sextic with 9 cusps with no real points since the number of cusps that are not real is even. On the other hand, there exist a real sextic with 8 cusps and no real points (cf. [6], [13]) for which the multiplicity of the factor $(t^2 - t + 1)$ is 2.

Remark 6.10. It is not hard to calculate the Alexander polynomial of the link which is the closed braid corresponding to a curve of an odd degree having no real critical values. In this case the projection without critical points yields again rotation of a set with an odd number of points and the corresponding braid is again Δ. Such link appears as the link of singularity $yx^{\frac{d-1}{2}} - y^{d+1}$ and calculation of the characteristic polynomial of this weighted homogeneous singularity (with non-integer weights, in which case one can use [20]) one obtains

$$(t^d - 1)^{\frac{d-1}{2}}(t - 1) \tag{29}$$

This may lead for example to a restriction on the degrees of the Alexander polynomials with only ordinary triple points (in which case d must be divisible by 3). It is unlikely however that this can give a sharp bound on the degree.

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