PROBLEMS IN TOPOLOGY OF THE COMPLEMENTS TO PLANE SINGULAR CURVES

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The paper discusses the statements and the background of several problems in the topology of plane singular algebraic curves.

1. Introduction

The purpose of this note is to discuss what it seems to me presently are the main issues which have to be resolved in the topology of plane singular curves. This area was quite active during the last 25 years after 40 years of hibernation which followed its birth in the works of Zariski and Enriques in 20's and 30's. The state of affairs in the end of 70's is beautifully presented by M.Artin and B.Mazur in introduction to Zariski's Collected papers (cf. [48], section 7. The Fundamental Group p.6-1) and also by D.Mumford in the appendix to [47]. Artin and Masur's survey contains numerous problems. The central issue, as stated by the authors then, still remains the same now: "What can be said about the Poincare group G, that is, the fundamental group of the complement of C (i.e. an algebraic curve) in $\mathbf{P}^2$" However the status of many concrete problems completely changed and others can be stated in a much more specific way. For example the problem of "irreducibility of the family of plane curves with nodes and commutativity of Poincare group" was solved by J.Harris (cf. [20]) which followed the proofs of commutativity by Fulton [18] and Deligne [10]. Subsequent works by Z.Ran (cf. [41]) and M.Nori (cf [35]) provided alternative views on these issues. Also the problems described in the section on "Cyclic multiple planes and knot

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theory" in [48] completely addressed by the theory of Alexander invariants with its topological ([23]) and algebro-geometric aspects ([24], [31],[34]).

My list of problems is very much incomplete and has many important omissions but I hope that it addresses at least one of the goals which I had in mind in selecting particular problems which was to show the relations between the topology of plane singular curves and other areas such as topology of arrangements, symplectic geometry etc. Before stating each problem I often presented background and in principle much of this can be read without extensive preparation. Some additional details of the background are presented in my lectures in Lumini (cf. [30]). In fact these notes can be viewed as a supplement to the latter. The book [11] also contains much of the needed background. It is important to note that a large part of the issues raised in Zariski's work in the 30s and discussed by Artin and Masur were generalized to higher dimensions with the fundamental group being replaced by the higher homotopy groups (an aspect totally absent in the late 70s). I refer to [30] for discussion of these developments and related questions.

Many of the problems from our list were discussed during the workshop in Trieste and I want to thank its participants for lively discussions. I also am grateful to referee for careful reading of the manuscript and the comments. Finally, I want to thank M.Oka for his encouragement to write a contribution to these proceedings. Without him, these notes would not have been written.

2. Alexander Invariants

We shall start by describing problems related to Alexander invariants of plane algebraic curves. While a characterization of the fundamental groups is very difficult, if not impossible, (cf. section 3), the Alexander polynomial of the fundamental group is much better understood [a]. Recall a definition of this invariant. Let $C \subset \mathbf{C}^2$ be an algebraic curve. The fundamental group $\pi_1(\mathbf{C}^2 - C)$ has a canonical surjection onto $\mathbf{Z}$ sending a loop into its linking number with C. Let $\widetilde{\mathbf{C}^2 - C} \to \mathbf{C}^2 - C$ be the corresponding cover and let $K = \mathrm{Ker}(\pi_1(\mathbf{C}^2 - C) \to \mathbf{Z})$. The group $\mathbf{Z}$ acts on the abelianization of K/K' of K which makes $K/K' \otimes \mathbf{Q}$ into a $\mathbf{Q}[\mathbf{Z}] = \mathbf{Q}[t, t^{-1}]$ module. It has a cyclic decomposition $K/K' \otimes \mathbf{Q} = \oplus_i \mathbf{Q}[t, t^{-1}]/(\lambda_i(t))$ for some

[a]Though it does not give complete information on the type of equisingular deformation of a curve cf. [45], [38]

Laurent polynomials $\lambda_i(t)$ defined up to a unit in $\mathbf{Q}[t,t^{-1}]$. The Alexander polynomial of C is defined as

$$\Delta_C(t) = \Pi_i \lambda_i(t).$$

The polynomial $\Delta_C(t)$ is affected by the local type of the singularities of C as follows. For each singular point P of the curve C let us consider the Alexander polynomial $\Delta_P(t)$ of the link in a small 3-sphere in $\mathbf{C}^2$ centered at P. If the closure of C in $\mathbf{P}^2$ is transversal to the line at infinity then one has the following divisibility property:

$$\Delta_C(t) \mid \Pi_{P \in \mathrm{Sing} C} \Delta_P(t). \tag{1}$$

In particular if C is irreducible and all its singularities are either nodes or cusps (i.e. locally given by $x^2 + y^2 =$ and $x^2 + y^3 = 0$ resp.) then $\Delta_C(t) = (t^2 - t + 1)^s$ for some non negative integer s. One can also define the Alexander polynomial at infinity $\Delta_{C,\infty}(t)$ as the Alexander polynomial of the link $C \cap S \subset S$ where S is a sphere of sufficiently large radius in $\mathbf{C}^2$. The global Alexander polynomial $\Delta_C(t)$ divides $\Delta_{C,\infty}(t)$. If C is transversal to the line at infinity and has degree d this is equivalent to

$$\Delta_C(t) \quad | \quad (t^d - 1)^{d-2}. \tag{2}$$

2.1. *Realization Problems*

Problem 2.1. *For which polynomials $P(t) \in \mathbf{Z}[t,t^{-1}]$ does an algebraic curve C exist such that $P(t) = \Delta_C(t)$.*

In particular

(i) what is the maximum $s(d)$ of integers s such that there exists a curve C of degree d transversal to the line at infinity, having nodes and cusps as the only singularities and such that $\Delta_C(t) = (t^2 - t + 1)^s$?

(ii) Does there exist a bound on $s(d)$ which is independent of d?

(iii) What are inequalities for the degrees of the Alexander polynomials for other classes of curves with fixed type of singularities (inculding more general ones than the nodes and cusps)?

The largest value for s in (i) which I know is 3: it is achieved for the sextic with 9 cusps (i.e. the dual curve to a non-singular cubic). For nodal curves, since the fundamental group $\pi_1(\mathbf{C}^2 - C)$ is abelian, the answer to (iii) is yes ($\Delta(C) = 1$). In [38] it was shown that $s(6) = 3$ (cf. also [9], [36], [37])

There is a reformulation of the problem 2.1 which does not involve the Alexander polynomial but is based on the relationship between the latter and the position of singularities. With each singular point $P \in C \subset \mathbf{P}^2$ in [24] we associated the collection of rational numbers κ_i (called *the local constant of quasiadjunction*) and the ideals $J_{\kappa,P,C}$ in the local ring $O_{P,\mathbf{P}^2}$ depending on the local type of the singularity of C at P. In the case of a node this collection is empty and in the case of a cusp it contains single number $\frac{1}{6}$. The corresponding ideal $J_{\frac{1}{6},P,C}$ is the maximal ideal of the local ring at $P \in \mathbf{P}^2$. These ideals define the ideal sheaf $\mathbf{J}_{\kappa,C}$ on $\mathbf{P}^2$ with the stalk at any $P \in \mathrm{Sing}C$ being the smallest ideal $J_{\kappa',P,C}$ with $\kappa' \leq \kappa$ and the stalk in any other point $P \notin \mathrm{Sing}C$ coinciding with the whole local ring $O_{P,\mathbf{P}^2}$. Then the Alexander polynomial has the following expression (here Ξ is the set of all local constants of quasiadjunction of all singularities of C):

$$\Delta_C(t) = \prod_{\kappa \in \Xi} [(t - exp(2\pi i \kappa))(t - exp(-2\pi i \kappa))]^{\dim H^1(\mathbf{P}^2, \mathbf{J}_\kappa(d-3-\kappa d))}. \quad (3)$$

If C has nodes and cusps as the only singularities then $\dim H^1(\mathbf{P}^2, J_{\frac{1}{6}}(d-3-\frac{d}{6})) = \dim H^0(\mathbf{P}^2, J_{\frac{1}{6}}(d-3-\frac{d}{6})) - \chi(J_{\frac{1}{6}}(d-3-\frac{d}{6}))$. The first term in this difference is the dimension of the space of curves of degree $d-3-\frac{d}{6}$ passing through the cusps of C while the second is the "expected" dimension of this space. Such difference is called the superabundance s of the family of curves. Hence the problem 2.1 is equivalent to the question on possible values of the superabundances of linear systems of curves of degree $d-3-\frac{d}{6}$ passing through the cusps of a curve of degree d. A similar reformulation of the problem 2.1 can be made for curves with arbitrary singularities using results of [24] which we leave to the reader.

An interesting version of the realization problem deals with twisted Alexander polynomials. In [8] the authors extended the study of twisted Alexander polynomials in knot theory and considered Alexander polynomials associated with a plane curves and a linear representation of the fundamental group.

Problem 2.2. *(i) Find an expression for the twisted Alexander polynomial generalizing (3).*

(ii) What is the class of polynomials which can appear as twisted Alexander polynomials of plane curves.

3. Fundamental groups

One of the oldest problem in the study of the topology of complements is the problem of characterization, in some sense, of the fundamental groups of the complements to plane curves. The lack of examples which was acutely felt in the past (cf. [25]) has been addressed by now in many respects though there is still a real possibility that we so far have only been probing very special curves. The cyclotomic property of Alexander polynomials rules out many groups as possible candidates for the fundamental groups of the complements ([27]). Below is one very specific question (mentioned by D.Mumford in the footnote to [47]) on general properties of fundamental groups of the complements.

3.1. *Residually finiteness.*

Problem 3.1. *Are the fundamental groups of plane curve residually finite?*

In other words: does the intersection of subgroup of finite index consist of the identity only. An equivalent formulation: is the map $\pi_1(\mathbf{P}^2 - C) \to \pi_1^{alg}(\mathbf{P}^2 - C)$ of the topological fundamental group into the algebraic one injective. It is known that non-residually finite fundamental groups can appear as the fundamental groups of algebraic surfaces ([46]).

3.2. *Finite groups.*

Problem 3.2. *Which finite groups can appear as the fundamental groups of the complements? What are finite fundamental groups of the complements to the curves with nodes and cusps only?*

Some restrictions on the finite groups in the second class come from the divisibility theorem for the Alexander polynomials over finite field i.e. the relation (1) in which one uses the Alexander polynomial defend using the the homology groups with the coefficients in a finite field. A quartic with 3 cusps in $\mathbf{P}^2$ has finite fundamental group of the complement (the 3 strings braid group of sphere). Interesting examples of finite groups of the complements are due to Oka (cf. [39]).

3.3. *Braid monodromy*

Braid monodromy is a more subtle invariant of singular curves than the fundamental group of the complement. Recall its definition. Let us consider the complement to C in an affine (rather than projective) plane $\mathbf{C}^2$. In the case when the line at infinity is generic, the fundamental group of the complement to projective curve is a quotient of $\pi_1(\mathbf{C}^2 - C \cap \mathbf{C}^2)$ by a subgroup generated by a central element (cf. [27]). Also, the classes of equisingular isotopy of curves in $\mathbf{P}^2$ correspond to the classes of equisingular isotopy of curves transversal to the line at infinity. Let $l : \mathbf{C}^2 \to \mathbf{C}$ be a generic linear projection and $l_C : C \to \mathbf{C}$ be its restriction on C. One has in the target $\mathbf{C}$ of l, a subset R consisting of points for which l_C has fewer than d ($d = \deg C$) preimages. Genericity of l also implies that for any $r \in R$, $l^{-1}(r)$ is not a flex and all but exactly one point in $l_C^{-1}(r) \cap C$ are the transversal intersections at a non singular point of C.

Given a path $\gamma : [0,1] \to \mathbf{C} - R$ we shall restrict the map l to $l^{-1}(\gamma)$ and pick a trivialisation of the locally trivial fibration of pairs $l : (l^{-1}(\gamma), l_C^{-1}(\gamma)) \to \gamma$. If γ is a loop, i.e. $\gamma(0) = \gamma(1)$ then we obtain a diffeomorphism of $\mathbf{C} = l^{-1}(\gamma(0))$ fixing the finite subset $l_C^{-1}(\gamma(0)) = l^{-1}(\gamma(0)) \cap C$. We shall pick this diffeomorphism to be constant outside of a disk of a large radius in $l^{-1}(\gamma(0))$. The group of isotopy classes of such diffeomorphisms is isomorphic to the Artin's braid group $\mathbf{B}_d$ on d strings (cf. [30]). This assignment of a braid to a loop in $\mathbf{C} - R$ is independent of the choice of γ up to homotopy. In fact we obtain the homomorphism:

$$\beta : \ \pi_1(\mathbf{C} - R) \to \mathbf{B}_d \tag{4}$$

called the *braid monodromy* (its composition with the map of the braid group onto the symmetric group gives the classical Σ_d valued monodromy of the multivalued function corresponding to C with the domain being the target of l).

It is useful to describe the homomorphism (4) by its values on special generators of $\pi_1(\mathbf{C} - R)$.

Definition 3.1. An ordered good system of generators of $\pi_1(\mathbf{C} - F)$ where F is a finite subset of $\mathbf{C}$ is defined as follows. For $i = 1, ..., \mathrm{Card}\ F$, let δ_i be the counterclockwise oriented boundary of a sufficiently small disk Δ_i about a point $f_i \subset F$ (small implies that $\Delta_i \cap \Delta_j = \emptyset, i \neq j$). Let p_i be a collection of non-intersecting paths in $\mathbf{C} - \cup \Delta_i$ each having a base point b as the initial point and a point on δ_i as the end point. Then the ordered good system consists of loops $p_i^{-1} \circ \delta_i \circ p_i$ ordered counterclockwise by their intersections with a small disk about b.

It turns out that the product of braids $\beta(\gamma_i)$ in the above ordering does not depend on the choice of generators γ_i of $\pi_1(\mathbf{C} - R)$:

$$\Pi_i \beta(\gamma_i) = \Delta^2 \tag{5}$$

(Δ^2 is the standard generator of the center of Artin's braid group). The set of ordered good systems of generators of $\pi_1(\mathbf{C}-F,b)$ is acted upon by the braid group $\mathbf{B}_{\mathrm{Card}F}$ via ($\sigma_i \in \mathbf{B}_{\mathrm{Card}F}$ are the standard generators):

$$\sigma_i(\gamma_1,...\gamma_i,\gamma_{i+1},....,\gamma_{\mathrm{Card}F}) = (\gamma_1,....,\gamma_{i-1},\gamma_i\gamma_{i+1}\gamma_i^{-1},\gamma_i,\gamma_{i+2},...,\gamma_{\mathrm{Card}F}) \quad (6)$$

(Hurwitz action). The orbit of this action on a vector

$$(\beta(\gamma_1),...,\beta(\gamma_{\mathrm{Card}R}))$$

(where R, as above, is the ramification locus of l_C) is independent of the choice of l, the ordered good system of generators of $\pi_1(\mathbf{C}-R)$, the base point of the latter or a curve within its class of equisingular deformation. So is the case for a simultaneous conjugation:

$$(\beta(\gamma_1),...,\beta(\gamma_{\mathrm{Card}R}) \to (\sigma\beta(\gamma_1)\sigma^{-1},...,\sigma\beta(\gamma_{\mathrm{Card}R})\sigma^{-1}) \quad (\sigma \in \mathbf{B}_d) \quad (7)$$

The fundamental problem is the following:

Problem 3.3. *Characterize factorizations (5) of* Δ^2 *obtained as in the above construction for algebraic curves. Describe the orbits of Hurwitz action and conjugation on the* Card$F$*-fold product* $\mathbf{B}_d \times ... \times \mathbf{B}_d$.

Some restrictions were proposed in [32]. Let, as above, $\sigma_i, i = 1,...,d-1$ be the standard generators of $\mathbf{B}_d$. For the curves with cusps and nodes as the only singularities each $\beta(\gamma_i)$ is conjugate to σ_1 (respectively σ_1^2 or σ_1^3) provided that δ_i is a loop about the point $r \in R$ such that $C \cap l_C^{-1}(r)$ contains the tangency point (respectively the node or the cusp). Some restrictions on the number of factors $\beta(\gamma_i)$ of each type come from the restrictions on the number of nodes and cusps a given curve can have (cf. [33]) and also from Plücker formulas.

Closely related to the problems 3.3 is the problem of constructing invariants of the braid monodromy which are invariants of Hurwitz action and conjugation and hence yield invariants of the curve. One method was proposed in [26] (cf. [2] for other invariants of braid monodromy). Let

$$\rho : \mathbf{B}_d \to GL_k(\Lambda)$$

be a linear representation of the Artin's braid group over a commutative ring Λ. The free module Λ^k can be considered as a $\pi_1(\mathbf{C}-R)$ module using the action of the latter given by composition of braid monodromy and ρ. Then, the Λ-module $H_0(\pi_1(\mathbf{C}-R),\rho(\beta))$ is invariant under Hurwitz moves and hence is an invariant of the curve. The canonical presentation of this Λ-module is given by:

$$\oplus_{i=1}^{\text{Card}R}\Lambda^k \quad \stackrel{\oplus_{i=1}^{\text{Card}R}\rho(\beta(\gamma_i))-\text{Id}}{\longrightarrow} \quad \Lambda^k \to H_0(\pi_1(\mathbf{C}-R),\rho(\beta)) \to 0 \qquad (8)$$

Using this presentation one can calculate such invariant of braid monodromy as the Λ-torsion of $H_0(\pi_1(\mathbf{C}-R),\rho(\beta))$. In the case, when ρ is the reduced Burau representation of the braid group (over $\mathbf{Q}$), i.e. Λ is the ring of Laurent polynomials $\mathbf{Q}[t,t^{-1}]$, the order of the $\mathbf{Q}[t,t^{-1}]$-torsion of $H_0(\pi_1(\mathbf{C}-R),\rho(\beta))$ (which is a Laurent polynomial) is related to the Alexander polynomial of the curve as follows (cf. [26]):

$$\text{Ord}_{\mathbf{Q}[t,t^{-1}]}H_0(\pi_1(\mathbf{C}-R),\rho(\beta)) = (1+t+...+t^{\deg C-1})\Delta_C(t) \qquad (9)$$

Problem 3.4. *Calculate the module $H_0(\pi_1(\mathbf{C}-R),\rho(\beta))$ for non Burau representations of the braid group e.g for Lawrence representations ([22]) and find their geometric interpretation (a counterpart to the above relation with superabundances of linear systems).*

3.4. π_1 *of the complements to generic projections.*

An important class of plane curves with cusps and nodes consists of the branching curves of generic projections. They come up as follows. Let $S \subset \mathbf{P}^N$ be an algebraic surface and π_S be the restriction of a projection on $\mathbf{P}^2$ from a generic subspace $\mathbf{P}^{N-3} \subset \mathbf{P}^N$. The branching curve Br_S of π_S consists of points having fewer than $d = \deg S$ preimages. The equisingular isotopy type of the curve Br_S depends on (the deformation type of) S only. If the embedding of $S \subset \mathbf{P}^N$ is canonically associated with S (i.e. corresponds to the embedding with a fixed multiple of the canonical class) then $\pi_1(\mathbf{P}^2 - C_S)$ is an invariant of the deformation type of S.

In the case when the surface is a non-singular degree n hypersurface $V_n \subset \mathbf{P}^3$ the fundamental group $\pi_1(\mathbf{P}^2 - Br_{V_n})$ is isomorphic to the quotient of the Artin's braid group by its center (cf. [32]). However for surfaces having higher codimension the corresponding groups can be much simpler. We shall state several results due to A.Robb and M.Teicher on the structure of the group $\pi_1(\mathbf{C}^2 - Br_S \cap \mathbf{C}^2)$ where $\mathbf{C}^2$ is the complement to a generic line in $\mathbf{P}^2$. One has a surjection $\sigma_S : \pi_1(\mathbf{C}^2 - Br_S \cap \mathbf{C}^2) \to \Sigma_{\deg S}$ onto the symmetric group $\Sigma_{\deg S}$ and also the surjection ab_S onto its abelianization $\mathbf{Z}$. Let $\Pi_S = \text{Ker}\sigma_S \cap \text{Ker}ab_S$ be the intersection of the kernels of these two surjections. In the case when S is the image of $\mathbf{P}^2$ in $\mathbf{P}^{\frac{n(n+3)}{2}}$ embedded using $O_{\mathbf{P}^2}(n)$ ($n \geq 3$) (Veronese surface) the group Π_S is solvable (cf. [43]). So is Π_S if S is a non-singular complete intersection in $\mathbf{P}^n$ which is not a hypersurface (cf. [42]). In fact in the latter case Π_S has $\mathbf{Z}_2$ as its center and $\Pi/\mathbf{Z}_2$ is free *abelian*). These and other calculations lead to the following:

Problem 3.5. (M.Teicher, cf. [44]) *For which simply connected surfaces of $S \subset \mathbf{P}^N$ is the fundamental group of the complement to the branching curve of generic projection is almost solvable i.e. contains a solvable subgroup of finite index.*

It is expected that this is the case for all simply connected surfaces with exception of small number of classes (e.g. hypersurfaces in $\mathbf{P}^3$).

Another interesting problem concerning the branching curves of generic projections was proposed recently in the work of Auroux, Donaldson, Katzarkov and Yotov. In [5] the authors study the fundamental groups $\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)$ where S_k is the surface S embedded via the complete linear system $H^0(S, O_S(k))$. The stabilized group $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))$ is defined as the quotient of $\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)$ by the normal subgroup generated by the commutators (γ_1, γ_2) where γ_1 and γ_2 are good generators (cf. section 3.3) of the fundamental group having disjoint transpositions as their images $\sigma(\gamma_1), \sigma(\gamma_2)$ in the symmetric group $\Sigma_{\deg S_k}$. If γ_1, γ_2 are the loops about two branches of Br_S intersecting at a node then the corresponding γ_1, γ_2 commute. Therefore, one can view adding such commutation relation to the fundamental group as an algebraic analog of "adding nodes". In general, a degeneration of a curve adding a node can change the fundamental group (see examples of curves with the same number of cusps but different number of nodes in [13]). Nevertheless [5] suggested the following:

Problem 3.6. *For k sufficiently large one has the isomorphism: $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)) = \pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)$*

In fact in [5] the authors propose a conjectural structure theorem of the "stabilized" fundamental group $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))$.

Problem 3.7. *Let Λ_k be the image of the map: $H_2(S, \mathbf{Z}) \to \mathbf{Z}^2$ corresponding to the class $(c_1(O_S(k)), K_S + 2c_1(O_S(k))) \in H^2(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z})$ via the identification of the latter with $Hom(H_2(S, \mathbf{Z}), \mathbf{Z}^2)$. Let $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))^0$ be the intersection of the kernel of the homomorphisms of $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))$ onto $\Sigma_{\deg S_k}$ and the kernel of the abelianization $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)) \to \mathbf{Z}$ ("the reduced stabilized fundamental group"). Then the commutator of $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))^0$ is a quotient of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the* *quotient of* $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))^0$ *by its commutator is isomorphic to* $\mathbf{Z}^2/\Lambda_k \otimes \mathbf{Z}[\Sigma_{\deg S_k}]$.

The analogs of the groups $\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2)$ and $St(\pi_1(\mathbf{C}^2 - Br_{S_k} \cap \mathbf{C}^2))$ can be defined for symplectic 4-manifolds and they play an important role in classification. In fact, problems 3.6 and 3.7 were formulated in [5] in the symplectic setting.

3.5. *Question of Nori.*

This problem does not deal with the fundamental groups of plane curve directly but rather comes up naturally in Nori's theory which gives a clear picture of the reasons for the commutativity of the fundamental groups of plane curves with "mild" singularities. In [35] it is shown that if $\bar{C}$ is a non-singular model of an irreducible nodal curve C with $r(C)$ nodes on a non-singular projective surface X and if $C^2 > 2r(C)$ then the index of the subgroup of $\pi_1(X)$ which is the image of $\pi_1(\bar{C})$ in $\pi_1(X)$ is finite (in fact does not exceed $C^2/C^2 - 2r(C)$). One should compare this with Lefschetz theorem where one makes a weaker assumption that $C^2 > 0$ but the conclusion is that $\pi_1(C) \to \pi_1(X)$ is surjective. Note that the group $\pi_1(C)$ is an amalgamated product of $\pi_1(\bar{C})$ and a free group and hence is much bigger than $\pi_1(\bar{C})$. This prompted the following:

Problem 3.8. (cf. [35]) *Let D be an effective divisor on a complete surface X with $D^2 > 0$. Is it true that the normal subgroup of $\pi_1(X)$ generated by the images of the fundamental groups of normalizations of the components of D has a finite index in $\pi_1(X)$?*

3.6. *Question of Artin and Mazur.*

In [48] the authors raised a question (cf. p. 8) which bears on the relation between the strata of topological equisingularity of the space of plane curves of fixed degree and the topology of the complement. Geometry and topology of these strata is very interesting and little is understood (cf. discussion in section 5.2).

Let U be the connected component of the equisingular stratum of the space of plane curves of degree $n(n-1)$ containg the branching curves of generic projections of a non singular surface of degree n in $\mathbf{P}^3$. Consider an open subset U_P of U consisting of curves not passing through a point $P \in \mathbf{P}^2$. Then the fundamental group $\pi_1(U_P)$ acts on the group $\pi_1(\mathbf{P}^2 -$

$C_0, P) = \mathbf{B}_n/(\Delta^2)$ via the monodromy ($C_0 \in U_P$, $\mathbf{B}_n$ is the Artin's braid group and Δ^2 is the generator of the center cf. [32] and section 3.3).

Problem 3.9. *Determine the homomorphism*

$$\pi_1(U_P) \to \mathrm{Aut}(\mathbf{B}_n/\Delta^2).$$

The automorphism group of $\mathbf{B}_n$ is calculated in [15].

4. Multivariable Alexander invariants

In the case of reducible plane curves the Alexander invariant discussed in section 2 has a multivariable refinement. From now on let C be a curve in $\mathbf{C}^2$ having r irreducible components all transversal to the line at infinity. The abelianization of the commutator $A(C) = \pi_1(\mathbf{C}^2 - C)'/\pi_1(\mathbf{C}^2 - C)'' \otimes \mathbf{C}$ can be viewed as a module over the group ring $\mathbf{C}[H_1(\mathbf{C}^2 - C, \mathbf{Z})]$ of the homology group. Since $H_1(\mathbf{C}^2 - C, \mathbf{Z}) = \mathbf{Z}^r$ (cf. [30]) this group ring is isomorphic to the ring of Laurent polynomials $\Lambda_r = \mathbf{C}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$. A construction of commutative algebra (cf. [16]) associates with a Λ_r-module M its support which is a subscheme of $\mathrm{Spec}\mathbf{C}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}] = \mathbf{C}^{*r}$ consisting of the prime ideals $\wp \in \mathrm{Spec}\mathbf{C}[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$ whose localization $M_\wp$ at $\wp$ is a non-zero module. This leads to the following

Definition 4.1. The i-th characteristic variety of a curve C is a subvariety of $\mathbf{C}^{*r}$ which is the reduced subscheme of the support of i-th exterior power of $A(C)$:

$$V_i(C) = \mathrm{Supp}\Lambda^i(\pi_1(\mathbf{C}^2 - C)'/\pi_1(\mathbf{C}^2 - C)'' \otimes \mathbf{C}).$$

In the irreducible case ($r = 1$) the characteristic variety $V_1(C)$ is the subset of $\mathbf{C}^*$ which is the collection of roots of the Alexander polynomial $\Delta_C(t)$ (cf. section 2). If in definition 4.1 one replaces $\pi_1(\mathbf{C}^2 - C)$ by the fundamental group of a link L in a 3-sphere S^3 then the corresponding characteristic variety is the zero set of the multivariable Alexander polynomial of $\pi_1(S^3 - L)$. However for a fundamental group of a curve the characteristic variety $V_1(C)$ typically has codimension greater than 1 in $(\mathbf{C}^*)^r$ and hence cannot be the zero set of a multivariable polynomial.

The realization problems mentioned in section 2 have the following multivariable counterparts.

Problem 4.1. *Find a bound on the number of irreducible components of $V_1(C)$. Are the irreducible components of $V_1(C)$ containing the identity of*

$\mathbf{C}^{*r}$ *combinatorially defined i.e. depend only on the isomorphism class of the data consisting of the set of components and local type of singular points each component contains?*

Let us make some comments on the second part of this problem by providing first some more details on the structure of $V_1(C)$. The irreducible components of $V_1(C)$ have a remarkably simple geometric structure: each is a coset of a subgroup of $\mathbf{C}^{*r} = H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ having a finite order in the quotient by this subgroup (cf. [28], [1]; this is in sharp contrast with the fundamental group of links in S^3). Moreover for each irreducible component of $V_1(C)$ there exists a holomorphic map $f : \mathbf{C}^2 - C \to \mathbf{C} - D$ where $\mathrm{Card}D \geq 2$ such that this component coincides with the coset $\rho f^*(H^1(\mathbf{C} - D, \mathbf{C}^*)) \subset H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ where ρ is a point of finite order in $\mathbf{C}^{*r}$. In particular in the case when $V_1(C)$ has components having a positive dimension, $V_1(C)$ also has the *subgroups* of $H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ of the same dimension.

In the case when all irreducible components of $C \subset \mathbf{C}^2$ are lines. i.e. C is an arrangement of lines, the components of $V_1(C)$ can be described as follows. Let $H^*(\mathbf{C}^2 - C, \mathbf{C})$ be the cohomology algebra of the complement. It has a combinatorial description known as the Orlik-Solomon algebra of the arrangement (in a more general case of hyperplanes in $\mathbf{C}^n$) (cf. [40]). Each $\omega \in H^1(\mathbf{C}^2 - C, \mathbf{C})$ defines the complex

$$K^*_\omega : H^0(\mathbf{C}^2 - C, \mathbf{C}) \xrightarrow{\cup\omega} H^1(\mathbf{C}^2 - C, \mathbf{C}) \xrightarrow{\cup\omega} H^2(\mathbf{C}^2 - C, \mathbf{C}) \qquad (10)$$

in which differential is given by the cup product with ω. Let us consider the following set

$$\{\omega \in H^1(\mathbf{C}^2 - C, \mathbf{C}) | \mathrm{dim} H^1(K^*_\omega) \geq 1\}. \qquad (11)$$

This set is combinatorially defined and in fact is a union of linear subspaces in $H^1(\mathbf{C}^2 - C, \mathbf{C})$ (cf. [28], [29]). Moreover the exponential map $\exp : H^1(\mathbf{C}^2 - C, \mathbf{C}) \to H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ (induced by the map $\mathbf{C} \to \mathbf{C}^*$ having as the source the tangent space to $\mathbf{C}^*$) provides the one to one correspondence between the components of $V_1(C)$ containing the identity of $H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ (i.e. the subgroups) and the irreducible components of the set (11). In particular the set (11) is combinatorially defined i.e. one has a positive answer to the second part of problem 4.1 in the case when C is an arrangement of lines. A generalization of this to arrangements of rational curves in discussed in [7].

In the case of arrangements one has a more precise conjecture than what is suggested by 4.1 and will be discussed in section 5.

Essential components of characteristic varieties having a positive dimension and non coordinate torsion points can be calculated in terms of position of singularities by a formula generalizing the expression (3) (cf. [28]). We refer to [30] for discussion of related interesting invariants (polytopes of quasi-adjunction) and corresponding problems and to [12] for a study of translated components.

5. Complements to Arrangements

Study of the complements to arrangements is a vast subject. Here we shall focus only on few problems, which it seems represent a gateway to more general case of arbitrary reducible curves.

5.1. *Dimensions of components of characteristic varieties*

Let us consider the problem of an estimate on the dimension of the characteristic variety. In order to isolate the central issue we shall review a definition of an essential component of $V_1(C)$ (cf. [28], [3]). Let C be an arrangement of lines in $\mathbf{C}^2$ and let $C \cup L$ be obtained by adding a line L to the arrangement C. The space $\mathrm{Spec}\mathbf{C}[H_1(\mathbf{C}^2 - C, \mathbf{Z})]$ containing the characteristic varieties can be identified with the space of characters of the fundamental group: $\mathrm{Hom}(\pi_1(\mathbf{C} - C, \mathbf{Z}), \mathbf{C}^*) = H^1(\mathbf{C}^2 - C, \mathbf{C}^*)$ and similar identification can be made for $C \cup L$. The (injective) map $H^1(\mathbf{C}^2 - C, \mathbf{C}^*) \to H^1(\mathbf{C} - C \cup L, \mathbf{C}^*)$ induced by inclusion takes a component of $V_1(C)$ into a component of $V_1(C \cup L)$. The corresponding inclusion of components may or may not be strict (cf [3]). We call a component $H_{C \cup L} \subset V_1(C \cup L)$ non essential in the latter case, i.e. if it coincides with the image of a component in $V_1(C)$ and essential is it is not a non essential one. Clearly the key issue is to decide what the dimensions of the *essential* components are and how big these dimensions can be relative to the number of lines in the arrangement (or the degree of the curve in case of an arbitrary reducible curve).

For the arrangement of d lines passing through a point in $\mathbf{C}^2$ the variety V_1 is given by the equation $t_1 .... t_d = 1$ (cf. [28]) i.e. we have a $d-1$-dimensional component of $V_1(C)$ which is large relative to the degree of C. It is surprisingly hard to find other arrangements with large dimension of essential components.

Problem 5.1. *Find a bound $\phi(N)$ on the dimension of an essential component of the characteristic variety for an arrangement of N lines. Can the* *dimension of an essential component of a characteristic variety be greater than 4 for an arrangement different from a pencil of lines?*

There are quite a few arrangements with dimension of essential components equal to 2 but even in these cases the arrangements are related to beautiful and subtle non-linear geometry. The arrangement of 12 lines formed by all the lines in $\mathbf{P}^2$ which contain 3 of the 9 inflection points of a non-singular cubic yields an arrangement with the essential component having dimension 3. A class of arrangements for which one has $\phi(N) \leq 4$ is given in [29] and represents the main evidence for the existence of an interesting answer to the problem 5.1. See [17] and [6] for recent results regarding this problem.

5.2. *Arrangement Strata*

It was mentioned already in section 4 that the existence of a component of characteristic variety having dimension k is equivalent to the existence of a holomorphic map of $\mathbf{C}^2 - C$ onto complement in $\mathbf{C}$ to k distinct points. In other words, for an arrangement C of lines in $\mathbf{P}^2$ the existence of a component of $V_1(C)$ having a positive dimension k yields a pencil of curves in $\mathbf{P}^2$ of fixed degree d with $k+1$ members being the (possibly non reduced) curves having only the lines as their irreducible components. Such a pencil corresponds to a line in the space of plane curves of degree d i.e. the projectivization $\mathbf{P}(H^0(\mathbf{P}^2, O(d)))$ of the space of sections of $O_{\mathbf{P}^2}(d)$. This projectivization contains subvarieties $A_{l_1,...,l_s}$ corresponding to partitions of $d = l_1 + ... + l_s \;\; l_i \geq 1$ and consisting of curves of degree d given by the equations $L_1^{l_1} \cdot ... \cdot L_s^{l_s} = 0$ where L_i are linear forms. If $l_i = 1, i = 1, ..., d$ then the stratum $A_{l_1,...,l_d}$ represents the d-fold symmetric product of $\mathbf{P}^2$ embedded in $\mathbf{P}^{\frac{d(d+3)}{2}}$. Subvarieties corresponding to other partitions of d are the strata of canonical stratification of the symmetric product and all $A_{l_1,...,l_s}$ are the strata of equisingular stratification of the space of plane curves (cf. [30]). A k-secant of $A_{l_1,....,l_s}$ is a line in $\mathbf{P}(H^0(\mathbf{P}^2, O(d+1)))$ intersecting $A_{l_1,...,l_s}$ at k points. A reformulation of the problem 5.1 is the problem of determining for which k the "arrangement stratum" $A_{l_1,...,l_s}$ admits a k-secant.

One way to get a bound on possible degrees of multisecants is to use the degrees of defining equations:

Lemma 5.1. *a) If $A_{l_1,...,l_s}$ is a set theoretical intersection of hypersurfaces of degrees not exceeding $k-1$ then any k-secant of $A_{l_1,...,l_s}$ must belong* *to $A_{l_1,...,l_s}$. In particular the dimension of the characteristic variety of an arrangement composed of curves of degree d satisfies*

$$k \leq \max_{\mathbf{K}} M(l_1, ..., l_s)$$

where the maximum is taken over all systems of equations $\mathbf{K}$ for $A_{l_1,..l_s}$ and $M(l_1,..l_s)$ is the maximum of degrees of equations in $\mathbf{K}$.

b)Lines in the space $\mathbf{P}(H^0(\mathbf{P}^2, O(d))$ of plane curves of degree d which belong to $A_{1,...,1}$ are represented by pencils members of which are unions of fixed components and a pencil all members of which are lines containing a fixed point (i.e. the pencils have the form $\lambda F \cdot L_1 \cdot ... \cdot L_k + \mu F \cdot L'_1 \cdot ... \cdot L'_k$ where F is a product of linear forms and L_i, L'_i are linear forms all vanishing at the same point).

The first part follows from Bezout theorem. To see the second, let us consider the movable part of the pencil i.e. the pencil formed by the components of all curves which do not belong to all elements of the pencil. Let $\tilde{\mathbf{P}}^2$ be the blow up of $\mathbf{P}^2$ at the base points of the movable part. The map $\tilde{\mathbf{P}}^2 \to \mathbf{P}^1$ given by the movable part of the pencil yields the composition $\tilde{\mathbf{P}}^2 \xrightarrow{\phi} \Sigma \xrightarrow{\psi} \mathbf{P}^1$ where ϕ has irreducible fibers and ψ is finite (Stein factorization). Σ must be rational since otherwise the pullback of a holomorphic 1-form from Σ will yields a holmorphic 1-form on $\tilde{\mathbf{P}}^2$ and $\mathbf{P}^2$ does not have non-zero holomorphic 1-forms. Hence ϕ is a pencil of lines and therefore it consists of lines passing through a point.

This leads to the following

Problem 5.2. *Find the degrees of defining equations of the arrangement strata $A_{l_1,...,l_s}$.*

H.Brill showed that if $l_i = 1$ then $A_{1,...,1}$ is the zero set of a system of equations of degree $d+1$ (cf. [19]). In the case $d = 3$ this can be seen as follows. A cubic curve C is a union of lines if and only if any point is an inflection point. Therefore the Hessian is vanishing on C and if F is the equation of C then there is a constant γ such that $Hess(F) = \gamma \cdot F$. This is equivalent to saying that the rank of the 2×10 matrix formed by coefficients of $Hess(F)$ and F being equal to 1. Since the degree of $Hess(F)$ in coefficients of F is 3 we obtain $\binom{10}{2} = 45$ equations for the stratum $A_{1,1,1}$ all having degree 4. In fact, the 12 lines containing 9 inflection points of a cubic form a pencil of cubic curves yielding a 4-secant of the arrangement stratum of $A_{1,1,1}$ (and hence the arrangement with 3-dimensional characteristic variety).

Example 5.1. (Next two examples were pointed out by I.Dolgachev) Plane section of the pencil of desmic surfaces (Kummer surfaces corresponding to squares of elliptic curves). Such pencil contains 4 tetrahedra yielding in a plane section a pencil of quartics which is tri-secant of the top stratum of $A_{1,1,1,1}$ (cf. [21]).

Example 5.2. d) Modular configurations (cf. [14]) define arrangements of hyperplanes for which plane sections yield the tri-secants of $A_{1,\dots,1}$. The hyperplanes can be constructed either using Schrödinger representation of $G = (\mathbf{Z}/N\mathbf{Z})^2$ or embedding of modular surfaces parametrising elliptic curves with level N structure. We refer to I.Dolgachev's paper for detailed discussion of this configuration. For $N = 3$ one obtains the Hesse arrangement.

Problem 5.3. *Find new examples of arrangements with characteristic varieties having essential components of positive dimension and classify secants of the arrangement stratum in the space of plane curves having fixed (at least small) degree.*

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