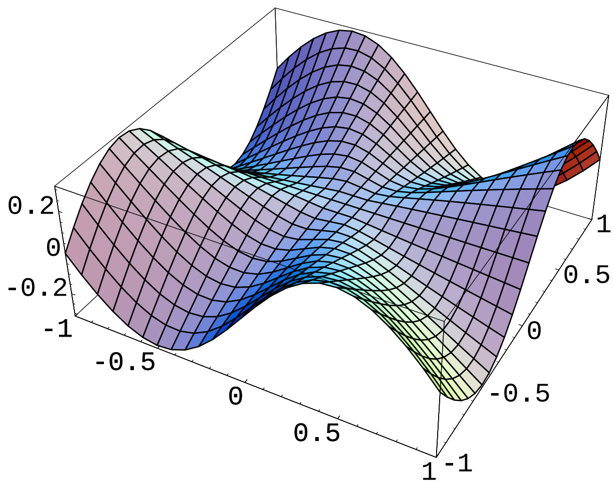
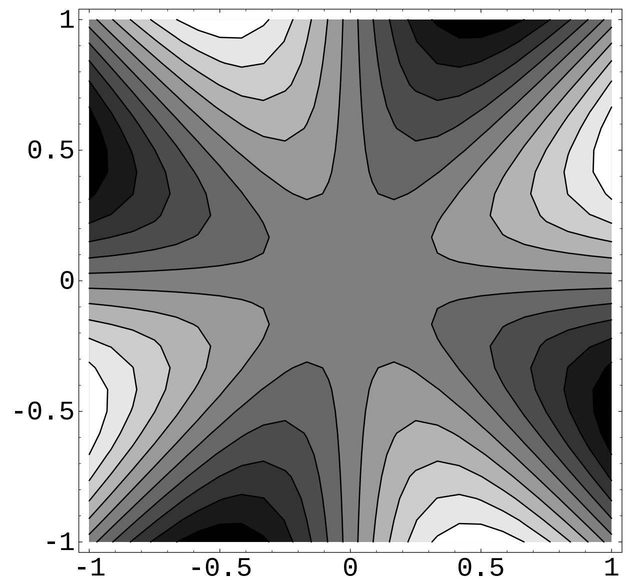
Quiz 2

Let

$$f(x,y)=\begin{cases}\frac{xy\left(x^2-y^2\right)}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

This function is plotted below.





1. Do the partial derivatives f_x and f_y exist and are they continuous at all points $(x,y)\neq(0,0)$?
2. Is f differentiable at all points $(x,y)\neq(0,0)$?
3. Is f continuous at $(0,0)$?
4. Do the partial derivatives f_x and f_y exist and are they continuous at $(0,0)$?
5. Is f differentiable at $(0,0)$?

Answer:

1. We have by the quotient rule that

$$f_x(x,y)=\frac{\left[y\left(x^2-y^2\right)-xy\left(-2x\right)\right]\left(x^2+y^2\right)-xy\left(x^2+y^2\right)(2x)}{\left(x^2+y^2\right)^2}$$

   which exists and is continuous for all $(x,y)\neq(0,0)$. See the answers to the Mock Quiz for an explanation of why this is sufficient. We compute f_y similarly.

2. Yes, because by part (1) we have that f_x and f_y exist and are continuous.

3. Let $\epsilon>0$ and define $\delta=\sqrt{\epsilon}$. Suppose $\sqrt{x^2+y^2}<\delta$. Then

$$|f(x,y)-f(0)|=\left|\frac{xy\left(x^2-y^2\right)}{x^2+y^2}\right|=\frac{|x|\,|y|\,|x^2-y^2|}{|x^2+y^2|}=\frac{|x|\,|y|\,(x^2-y^2)}{x^2+y^2}=|x|\,|y|<\delta^2=\epsilon.$$

   This shows that $\lim_{(x,y)\to(0,0)} f(x,y)=0$ so that f is continuous at 0.

4. Yes because

$$\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \lim_{h \to 0} \frac{h \cdot 0 \left(h^2 - y^2\right)}{h^2 + 0^2} = 0$$

and similarly for f_y.