1. Evaluate, by changing the order of integration, the following integral.

$$\int_0^1 \int_{y^2}^1 \sin\left(x^{\frac{3}{2}}\right)\ dx\ dy$$

Answer:

$$\int_0^1 \int_{y^2}^1 \sin\left(x^{\frac{3}{2}}\right)\ dx\ dy = \int_0^1 \int_0^{\sqrt{x}} \sin\left(x^{\frac{3}{2}}\right)\ dy\ dx$$

$$= \int_0^1 \sin\left(x^{\frac{3}{2}}\right)\sqrt{x}\ dx = -\frac{2}{3}\cos\left(x^{\frac{3}{2}}\right)\Big|_0^1 = -\frac{2}{3}\left(\cos(1) - 1\right)$$

2. Find the center of gravity of the semi-circular lamina $D = \{(x, y) : x^2 + y^2 \leq 1, y \geq 0\}$ with density $\rho(x, y) = 1$. You may assume that the area of a disk of radius r is πr^2.

Answer: We compute the mass m of the lamina to be

$$m = \int_D \rho(x, y)\ dA = \int_D 1\ dA = \text{Area}(D) = \frac{\pi}{2}$$

and the moments

$$M_x = \int_D x\rho(x, y)\ dA = \int_0^\pi \int_0^1 r\cos(\theta)\, r\ dr\ d\theta = \int_0^\pi \frac{1}{3}r^3\cos(\theta)\Big|_0^1\ d\theta$$

$$= \int_0^\pi \frac{1}{3}\cos(\theta)\ d\theta = \frac{1}{3}\sin(\theta)\Big|_0^\pi = 0$$

and

$$M_y = \int_D y\rho(x, y)\ dA = \int_0^\pi \int_0^1 r\sin(\theta)\, r\ dr\ d\theta = \int_0^\pi \frac{1}{3}r^3\sin(\theta)\Big|_0^1\ d\theta$$

$$= \int_0^\pi \frac{1}{3}\sin(\theta)\ d\theta = -\frac{1}{3}\cos(\theta)\Big|_0^\pi = \frac{2}{3}$$

so that the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_x}{m}, \frac{M_y}{m}\right) = \left(0, \frac{4}{3\pi}\right).$$

3. Find the volume of the solid bounded from above by the cone $z = \sqrt{x^2 + y^2}$ and below by the paraboloid $z = x^2 + y^2$.

Answer: The surfaces intersect in the circle $x^2 + y^2 = 1$. Let $D = \{x^2 + y^2 \leq 1\}$. Then

$$\text{Volume} = \int_D \left(\sqrt{x^2 - y^2} - x^2 - y^2\right) dA = \int_0^{2\pi} \int_0^1 \left(r - r^2\right) r dr\ d\theta = \frac{\pi}{6}$$

4. Using Lagrange Multipliers, find the maximum and minimum values of $f(x, y) = x^2 y$ subject to the constraint $x^2 + 2y^2 = 6$.

Answer: We must solve the following system

$$2xy = 2x\lambda \quad (1)$$
$$x^2 = 4y\lambda \quad (2)$$
$$x^2 + 2y^2 = 6 \quad (3)$$

If $\lambda = 0$, we have from (2) that $x = 0$ so that by (3) we have $y = \pm\sqrt{3}$. Assume now that $\lambda \neq 0$. Then by (2), if $x = 0$, we have $y = 0$, which contradicts (3) so that $x \neq 0$. Then from (1), we have $y = \lambda$ and then from (2), we have $x = \pm 2\lambda$. This means that $\lambda = \pm 1$ by (3). Therefore, our candidates for extreme points are

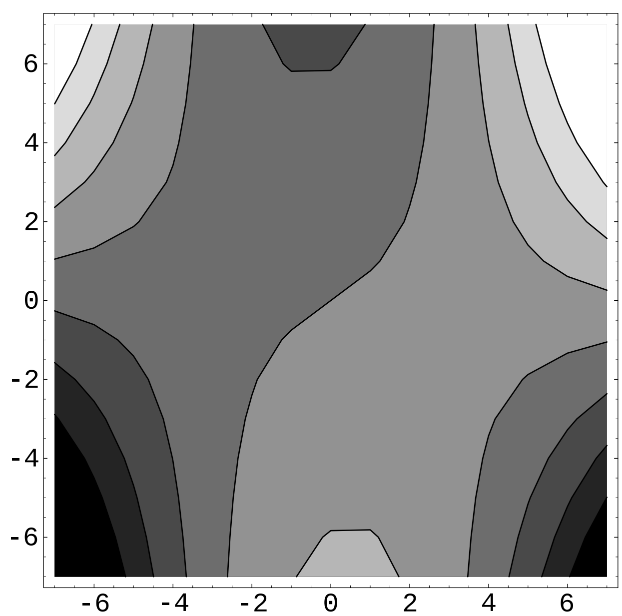
$$\left(0, \sqrt{3}\right) \quad \left(0, -\sqrt{3}\right) \quad (2, 1) \quad (-2, 1) \quad (2, -1) \quad \text{and} \quad (-2, -1).$$

Evaluating f at these points, we have

$$f\left(0, \sqrt{3}\right) = 0$$
$$f\left(0, -\sqrt{3}\right) = 0$$
$$f(2, 1) = 4$$
$$f(-2, 1) = 4$$
$$f(2, -1) = -4$$
$$f(-2, -1) = -4$$

so that $(2, 1)$ and $(-2, 1)$ are maxima and $(2, -1)$ and $(-2, -1)$ are minima.

5. Find and classify the critical points of the function $x^2 y + 6x - 9y$. A contour plot of this function is given below.



Answer: We compute

$$
\begin{aligned}
f_x(x,y) &= 2xy+6 && (4)\\
f_y(x,y) &= x^2-9 && (5)\\
f_{xx}(x,y) &= 2y\\
f_{yy}(x,y) &= 0\\
f_{xy}(x,y) = f_{yx}(x,y) &= 2x\\
D(x,y) &= -4x^2 && (6)
\end{aligned}
$$

From (4) and (5) we see that the only critical points are $(3,-1)$ and $(-3,1)$. From (6), we have $D(3,-1) = D(-3,1) = -36 < 0$ so that both points are saddles.

6. Find the maximum rate of change of $F(x,y,z) = x^2z + xyz$ at the point $(1,1,1)$ and the direction in which it occurs.

Answer: We compute

$$
\begin{aligned}
\nabla f(x,y,z) &= \left\langle 2xz+yz, xz, x^2+xy\right\rangle\\
\nabla f(1,1,1) &= \langle 3,1,2\rangle
\end{aligned}
$$

so that $\langle 3,1,2\rangle$ is the direction in which f is increasing fastest. Then the derivative in this direction is $|\langle 3,1,2\rangle| = \sqrt{14}$.