April 14

Let P be a parabolic subgroup of G, a finite group with split BN-pair. Let $P = L_I V_I$ be the Levi decomposition where $I \subset S$ is the corresponding set of generators. We defined the truncation

$$T_{L_I}^G \xi = t_{P/V_I}^G \xi$$

to be the character afforded by $\mathrm{Inv}_{V_I} \{x \in X : vx = x\} = X^{V_I}$ and Harish-Chandra induction

$$R_{L_I}^G \lambda = \mathrm{Ind}_P^G \widetilde{\lambda}$$

where $\widetilde{\lambda}$ is the lift of $\lambda \in \mathrm{ch}(L_I)$. We also showed the following.

Proposition 1 *(Reciprocity) Let $\xi \in \mathrm{ch}(G)$ and $\lambda \in \mathrm{ch}(L_I)$. Then*

$$\langle \xi, R_{L_I}^G \lambda \rangle_G = \langle T_{L_I}^G, \lambda \rangle_{L_I}.$$

Recall the following theorem from representation theory.

Theorem 1 *(Mackey) Let $K, K \leq G$, let θ be a character of H, and let η be a character of K. Write θ^x for the character of $H^x = x^{-1}Hx$ defined $\theta^x(h^x) = \theta(x)$. Let D be a set of double coset representatives for $H \backslash G / K$. Then*

$$\left\langle \mathrm{Ind}_H^G \theta, \mathrm{Ind}_K^G \eta \right\rangle_G = \sum_{x \in D} \left\langle \theta^x|_{H^x \cap K}, \eta|_{H^x \cap K} \right\rangle_{H^x \cap K}.$$

This gives us the following formula.

Theorem 2 *(Mackey's Formula) Let $I, J \subset S$ and let $P_I = L_I V_I$ be the Levi decomposition. Let $\lambda \in \mathrm{ch}(P_I)$ and $\mu \in \mathrm{ch}(P_J)$. Then*

$$\left\langle R_{L_I}^G \lambda, R_{L_J}^G \mu \right\rangle = \sum_{x \in D_{I,J}} \left\langle T_{L_K}^{L_I} \lambda, \ {}^x\left(T_{L_{K^x}}^{L_J}\right) \mu \right\rangle$$

where $D_{I,J}$ is the set of distinguished double coset representative of $P_I \backslash G / D_J$, $K = I \cap {}^xJ$ and $K' = {}^xK$.

Proof. (Sketch)

$$\text{LHS} = \left\langle \mathrm{Ind}_{L_I}^G \widetilde{\lambda}, \mathrm{Ind}_{L_J}^G \widetilde{\mu} \right\rangle = \sum_{x \in D} \left\langle \widetilde{\lambda}^x|_{P_I^x \cap P_J}, \widetilde{\mu}|_{P_I^x \cap P_J} \right\rangle_{P_I^x \cap P_J} = \sum_{x \in D} \left\langle \widetilde{\lambda}|_{P_I \cap {}^xP_J}, {}^x\widetilde{\mu}|_{P_I \cap {}^xP_J} \right\rangle_{P_I \cap {}^xP_J}.$$

We have from the third structure theorem that

$$P_I \cap {}^xP_J = L_K (L_I \cap {}^xV_J)(V_I \cap {}^xL_J)(V_I \cap {}^xV_J)$$

and

$$O_p(L_I \cap P_J).$$

Now for each fixed x, we have

$$\left\langle \widetilde{\lambda}, {}^x\widetilde{\mu} \right\rangle = \frac{1}{|P_I \cap {}^xP_J|} \sum \widetilde{\lambda}(lvyz) \ \overline{{}^x\widetilde{\mu}(lvyz)}$$

for $l \in L_K$, $v \in L^I \cap {}^xV_J$, $y \in V_I \cap {}^xL_J$, and $z \in V_I \cap {}^xV_J$ so the sum comes out to

$$|L_K|\,|O_p(P_K \cap L_I)|\,|O_p(P_K \cap L_J)|\,|v_I \cap {}^xV_J|\,.$$

We ultimatly have

$$lvyz \equiv lv \pmod{V_I} \equiv l\,{}^xy \pmod{{}^xV_J}$$

so that $\widetilde{\lambda}(lv\overline{y}z) = \lambda(lv)$ and ${}^x\overline{\mu}(lv\overline{y}z) ={}^x \mu(l\,{}^xy)$ and finally

$$\left\langle \widetilde{\lambda},{}^x\widetilde{\mu}\right\rangle = \left\langle T^{L_I}_{L_K}\lambda,{}^xT^{L_J}_{L'_K}\mu\right\rangle.$$

■

Definition 1 *Let $\xi \in \mathrm{Irr}(G)$. We say that ξ is* cuspdial *if $T^G_{L_I}\xi = 0$ for all $I \subsetneq S$.*

Proposition 2 *The following are equivalent.*

1. *ξ is cuspidal.*
2. *For all $I \subsetneq S$, we have $\sum_{v\in V_I}\xi(vx) = 0$ for all $x \in L_I$.*
3. *$\left\langle \xi, \mathrm{Ind}^G_{V_I}(1)\right\rangle = 0$ for all $I \subsetneq S$.*
4. *$\left\langle \xi, R^G_{L_I}\lambda\right\rangle = 0$ for all $I \subsetneq S$ and all $\lambda \in \mathrm{ch}(L_I)$*

The proof follows from recriprocity.

Proposition 3 *Let $\xi \in \mathrm{Irr}(G)$. Then either ξ is cuspidal or there exists $I \subsetneq S$ such that*

$$\left\langle \xi, R^G_{L_I}\lambda\right\rangle > 0$$

for some $\lambda \in \mathrm{Irr}(L_I)$.

Proof. Suppose ξ is *not* cuspidal. Then there exists a proper subset $I \subsetneq S$ such that $T^G_{L_I}\xi \neq 0$. Choose a minimal $I \subsetneq S$ satsifying the above condition. Then

$$T^G_{L_I}\xi = \sum_{\lambda\in\mathrm{Irr}(L_I)} \left\langle T^G_{L_I}\xi,\lambda\right\rangle_{L_I}\lambda = \sum_{\lambda}\left\langle \xi, R^G_{L_I}\lambda\right\rangle_G\lambda$$

but at least summand $\left\langle \xi, R^G_{L_I}\lambda\right\rangle > 0$ for some $\lambda \in \mathrm{Irr}(L_I)$. ■

We claim now that if λ is as in the proposition, then λ is cuspidal. Suppose λ is *not* cuspidal. We have $K \subsetneq I$ and $T^{L_I}_{L_K}\lambda \neq 0$. By transitivity, we have

$$T^G_{L_K}\xi = T^G_{L_K}\left(T^G_{L_I}\xi\right) = T^{L_I}_{L_K}\left(\sum_{\lambda\in\mathrm{Irr}(L_I)}\left\langle \xi, R^G_{L_I}\lambda\right\rangle\lambda\right) = \sum_{\lambda}\left\langle \xi, R^G_{L_I}\lambda\right\rangle T^{L_I}_{L_K}(\lambda)\,.$$