April 19

Definition 1 *Let $\mathcal{P}$ denote the collection of parabolic subgroups of the finite group G with split BN-pair. That is, define*

$$\mathcal{P} = \{{}^{x}P_I : I \subset S, x \in N\}$$

where $P_I \leq G$ denotes the standard parabolic subgroup BW_IB corresponding with $I \subset S$ and $W_I = \langle I \rangle \leq W$.

Proposition 1 *If $P, Q \in \mathcal{P}$ are such that $P \cap Q \in \mathcal{P}$, then*

$${}^{g}P \leq Q \qquad \textit{iff} \qquad P \leq Q \textit{ and } g \in Q.$$

Corollary 1 *Each parabolic $P \in \mathcal{P}$ is conjugate to a unique standard parabolic P_I for some $I \subset S$.*

Let $P \in \mathcal{P}$. Then by Corollary 1, we have $P = {}^{x}P_I$ for unique $I \subset S$ and some $x \in N$. If

$$P_I = L_I V_I$$

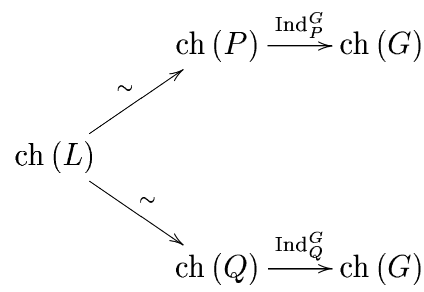
is the Levi decompositions of P_I, then we have

$$P = {}^{x}P_I = {}^{x}L_I {}^{x}V_I$$

is the Levi decomposition of P. Write $L_P = {}^xL_I$ and $V_P = {}^xV_I$.

Suppose $Q \in \mathcal{P}$ is another parabolic subgroup. Then by Corollary 1, we have that $Q = {}^yP_J$ for some unique $J \subset S$ and some $y \in N$. Moreover, P is *not* conjugate to Q unless $I = J$. However, L_P and L_Q can be conjugate even if $I \neq J$.

Suppose, indeed, that $I \neq J$, but P and Q have a common Levi subgroup $L = L_P = L_Q$. Then the operator $R_L^G : \mathrm{ch}\,(L) \to \mathrm{ch}\,(G)$ can be defined in this situation in two ways.



Evidently R depends on the parabolic subgroup, so we will write $R_{L,P}^G$ and $T_{L,P}^G$ instead of R_L^G and T_L^G for the remainder of this lecture. We will show that $R_{L,P}^G = R_{L,Q}^G$, justifying the abbreviated notation.

Recall from last week the following version of the Mackey Intertwining Number Theorem for generalized induction.

Theorem 1 *Let $P, Q \in \mathcal{P}$ with standard Levi subgroups L_P and L_Q and Weyl groups W_L and W_M. Let D be a set of double coset representatives of $W_I \backslash W / W_J$. Let $\xi \in \mathrm{ch}\,(L)$ and $\eta \in \mathrm{ch}\,(M)$. Then*

$$\left\langle R_{L,P}^G \xi, R_{M,Q}^G \eta \right\rangle_G = \sum_{w \in D} \left\langle T_{L \cap {}^wM, L \cap {}^wQ}^L \xi, {}^wT_{L^w \cap M, P^w \cap M}^M \eta \right\rangle_{L \cap {}^wM}$$

Theorem 2 *Let $P, Q \in \mathcal{P}$ be parabolic subgroups having a common Levi subgroup L. Then*

$$R_{L,P}^G \lambda = R_{L,Q}^G \lambda$$

for $\lambda \in \mathrm{ch}\,(L)$.

Proof. The claim is true for groups G with BN-rank 0 since in this case, $G = B$ so that $L = G$ is the only parabolic subgroup. Assume now that the claim is true for groups G with BN-rank $< n$. We have from Theorem 1 the following

$$\left\langle R_{L,P}^G \lambda, R_{L,P}^G \lambda \right\rangle_G = \sum_{x \in D} \left\langle T_{L \cap {}^xL, L \cap {}^xP}^L \lambda, {}^xT_{L^x \cap L, P^x \cap L}^L \lambda \right\rangle_{L \cap {}^xL}$$

$$\left\langle R_{L,P}^G \lambda, R_{L,Q}^G \lambda \right\rangle_G = \sum_{x \in D} \left\langle T_{L \cap {}^xL, L \cap {}^xQ}^L \lambda, {}^xT_{L^x \cap L, P^x \cap L}^L \lambda \right\rangle_{L \cap {}^xL}$$

$$\left\langle R_{L,Q}^G \lambda, R_{L,Q}^G \lambda \right\rangle_G = \sum_{x \in D} \left\langle T_{L \cap {}^xL, L \cap {}^xQ}^L \lambda, {}^xT_{L^x \cap L, Q^x \cap L}^L \lambda \right\rangle_{L \cap {}^xL}.$$

But all three are the same by the induction hypothesis. Therefore, we have

$$\left\langle R_{L,P}^G \lambda - R_{L,Q}^G \lambda, R_{L,P}^G \lambda - R_{L,Q}^G \lambda \right\rangle_G = 0$$

so that $R_{L,P}^G \lambda = R_{L,Q}^G \lambda$. ■

Corollary 2 *(Strong Conjugacy Theorem) Let $I, J \subset S$ and assume $L_I = {}^xL_J$ for some $x \in W$. Let $\psi \in \mathrm{ch}\,(L_J)$ and let $\varphi = {}^x\psi \in \mathrm{ch}\,(L_I)$. Then*

$$R^G_{L_I,P_I}\varphi = R^G_{L_J,P_J}\psi.$$

Proof. We have that $L_I = {}^xL_J$ is the Levi subgroup for ${}^xP_J$ as well as for P_I so that by Corollary 2,

$$R^G_{L_I,P_I}\varphi = R^G_{L_I,{}^xP_J}\varphi = R^G_{{}^xL_J,{}^xP_J}{}^x\psi = \mathrm{Ind}^G_{{}^xP_J}\widetilde{{}^x\psi} = \mathrm{Ind}^G_{{}^xP_J}{}^x\widetilde{\psi} = \mathrm{Ind}^G_{P_J}\widetilde{\psi} = R^G_{L_J,P_J}\psi.$$

■

Recall the following definition.

Definition 2 *A virtual character $\xi \in \mathrm{ch}\,(G)$ is called* cuspidal *if $T^G_{L_I}\xi = 0$ for all $I \subsetneq S$.*

Theorem 3 *(Harish-Chandra) Let $I, J \subset S$ and let φ and ψ be irreducible cuspidal characters of L_I and L_J respectively. Then*

1. *$\left\langle R^G_{L_I}\varphi, R^G_{L_J}\psi\right\rangle = 0$ unless there exists $x \in W$ with $L_I = {}^xL_J$ and $\varphi = {}^x\psi$.*

2. *In the situation that $\left\langle R^G_{L_I}\varphi, R^G_{L_J}\psi\right\rangle \neq 0$, we have $R^G_{L_I}\varphi = R^G_{L_J}\psi$.*