Richy's Second Talk

This talk is about the characters in $\mathrm{Irr}_{p'}\left(N_{S(\Omega)}(D)\right)$ and is based on a paper by Paul Fong in J Algebra.

Let Ω be a set of size n with $n = n_0 + n_1 p + n_2 p^2 + \dots$. Then $\Omega = \Omega_- \cup \Omega_+$ where $|\Omega_-| = n_- = n_0$ and $|\Omega_+| = n_+ = n_1 p + n_2 p^2 + \dots$. Let Δ_k be disjoint sets of size n_k for $k \geq 1$. Define $I = \{1, 2, \dots, p\}$ and $\Omega_k = \left(I^k\right)^{\Delta_k}$ so that $|\Omega_k| = n_k p^k$.

Note that $S\left(I^k\right)^{\Delta_k}$ and $\prod_{k \geq 1} S\left(I^k\right)^{\Delta_k}$ act componentwise on Ω_k and on Ω_+ respectively.

Let X_1 be the Sylow p-subgroup of $S(I)$, so $X_1 = \langle (123\dots p)\rangle$.

Recall the definition of the Wreath product $G \wr H$ where G is a finite group and $H \leq S_n$. Look this up in Kerber and James, Chapter 4 for more details. Define

$$G^n = \{f : f : \bar{n} \to G\}$$

where $\bar{n} = \{1, 2, \dots, n\}$. Then

$$G \wr H = G^n \ltimes H = \{(f, \pi) : f : \bar{n} \to G, \pi \in H \leq S_n\}.$$

The Wreath product has a group structure by $(f, \pi)(f', \pi') = (f f'_\pi, \pi\pi')$ where $f'_\pi = \pi \cdot f'$ or internally,

$$f\pi f'\pi' = f\left(\pi f' \pi^{-1}\right)\pi\pi'.$$

Observe that $|G \wr H| = |G|^n |H|$.

Next, we consider the k-fold Wreath product $X_1 \wr X_1 \wr \dots \wr X_1$ k-times. This group is denoted $X_k \subset S\left(I^k\right)$. Associativity holds by an argument in Kerber and James.

Then

$$|X_1| = p = p^{p_0}$$
$$|X_1 \wr X_2| = p^p \cdot p = p^{p^1 + p^0}$$
$$\dots$$
$$|X_1 \wr X_1 \wr X_1| = |X_k| = p^{p^{k-1}} + \dots + 1$$

Consider the exponential valuation $\nu(p) = 1$ and $\nu(n)$ the highest power of p dividing n. Then if $n = \sum_{i=1}^k n_i p^i$, then from a paper by MacDonald from 1971, we have that $\nu(n!) = \frac{n - \sum n_i}{p-1}$. Then $\nu\left(p^k!\right) = \frac{p^k - 1}{p-1} = p^{k-1} + p^{k-2} + \dots + 1$ so that $X_k \in \mathrm{Syl}_k\left(S\left(I^k\right)\right)$. By extension of this argument, we have that $D = X_1^{\Delta_1} \times \dots \times X_k^{\Delta_k} \in \mathrm{Syl}_p\left(S\left(\Omega_+\right)\right)$. This Sylow subgroup is a direct product of iterated Wreath products.

Let $Y_k = N_{S\left(I^k\right)}(X_k)$. If $k = 1$, then $Y_1 = XE$, a Frobenius group with cyclic compliment E as in James and Liebeck, example 2.5.11. Frobenius groups are described in isaacs, Chapter 4. If $1 < H < G$, then $H \cap H^g = 1$ for $g \in G \backslash H$, then H is a Frobenisu compliment and G is a Frobeinus group with compliment H.

Then $N_{S(\Omega)}(D) = S(\Omega_-) \times \prod_{k \geq 1} Y_k \wr S(\Delta_k)$. The irreducible characters of this are of the form $\omega_k \times \phi$ where $\omega_k \in \mathrm{Irr}\left(S\left(\Omega_-\right)\right)$ and $\phi = \prod \phi_k$ for $\phi_k \in \mathrm{Irr}\left(Y_k \wr S(\Delta_k)\right)$ and the p' characters are those with $p \not| (\omega_k \times \phi)(1)$.

Using Clifford theory, since $D \lhd N_{S(\Omega)}(D)$, we have $p \not| (\phi_k)|_D(1)$ so that $\phi_k|_D = \sum a_k \lambda_i$ for $\lambda_i \in \mathrm{Irr}(D), \lambda_i(1) = 1$. Hence, $\omega_k \times \phi \in \mathrm{Irr}\left(N_{S(\Omega)}(D)/D\right)$.

By a theorem of Olsson, we have $Y_k/X_k' = Y_1^k$ so that in our case, $N_{S(\Omega)}(D)/D' \cong S(\Omega) \times \prod_{k\geq 1} Y_1^k \wr S(\Delta_k)$. so $\phi_k \in \mathrm{Irr}\left(Y_1^k \wr S(\Delta_k)\right)^V$.

Recall that $\mathrm{Irr}(Y_1) = \{\xi_1, \xi_2, \ldots, \xi_p\}$ where $\xi_1, \ldots \xi_{p-1}$ are linear and ξ_p has degree $p-1$. Then $\mathrm{Irr}\left(Y_1^k\right)$ are k-tuples of the ξ_i. We write this as $\xi_{\bar{i}} = (\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k})$ for $\bar{i} \in \mathrm{I}^k$.

Using this language, we can label ϕ_k by a function f_k mapping $\mathrm{Irr} Y_1^k \to \mathrm{Irr}(S(\Delta_k))$ We do this as follows.

Partition Δ_k into disjoint subsets $\Delta_{k,\bar{i}}$ of size $|f_k(\xi_{\bar{i}})|$. Let $\zeta_k \in \mathrm{Irr}\left(\left(Y_1^k\right)^{\Delta_k}\right)$. have component $\xi_{\bar{i}}$ is positions indexed by elements of $\Delta_{k,\bar{i}}$.

Then the stabilizer of ξ_k in $s(\Delta_k)$ is $\prod_{\bar{i}\in I^k} S\left(\Delta_{k,\bar{i}}\right) = S(\Delta_k)_{\xi_k}$ so that $Y_k^{\Delta_k} S(\Delta_k)_{\xi_k}$ is the inertia roup of ξ_k in $Y_k \wr S(\Delta_k)$. We have $E(\xi_k)$ extends ξ_k so $Y_k^{\Delta_k} S(\Delta_k)_{\xi_k}$.

Now let ω_k be the character of $S(\Delta_k)_{\xi_k}$ with component $\omega_{f_k(\xi_{\bar{i}})}$ on $S\left(\Delta_{k,\bar{i}}\right)$. Define

$$\phi_k = \mathrm{Ind}_{Y_k^{\Delta_k} S(\Delta_k)_{\xi_k}}^{Y_k^{\Delta_k} S(\Delta_k)} E(\xi_k)\,\omega_k.$$

We have by Clifford theory then $\varphi_k \in \mathrm{Irr}(Y_k \wr S(\Delta_k))$.