Schanuel's Conjecture is Π^0_2
Outline–Not for circulation

In his preprint *Turing meets Schanuel* (to appear in the proceeding of Logic Colloquium 2012) Angus Macintyre proves, among other things, that if there is a counterexample to Schanuel's conjecture then we can find a counterexample using recursive complex numbers, i.e. complex numbers of the form $a + bi$ where a and b are recursive real numbers.

Asserting that the transcendence degree of $\mathbb{Q}(x_1, \ldots, x_n)$ is a less than m is a Σ^0_2 condition. We need to say that, after perhaps permuting the variables, we can find a nonzero $p_i \in \mathbb{Q}[X_1, \ldots, X_{m-1}, X_i]$ for $i \le m \le n$ such that

$$p_m(\overline{x}) = p_{m+1}(\overline{x}) = \ldots = p_n(\overline{x}) = 0.$$

The simplest way to assert $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly dependent is also Σ^0_2. Thus Macintyre's result gives the equivalent Π^0_3 statement

$\forall m \forall e_1 \ldots \forall e_m[$ if all e_i code total recursive functions and complex numbers x_i and the x_i are $\mathbb{Q}$-linearly indepedent, then the transcendence degree of $\mathbb{Q}(x_1, \ldots, x_m, \exp(x_1), \ldots, \exp(x_m))$ is at least $m]$.

Suppose $x_1, \ldots x_n$ are complex numbers linearly independent over $\mathbb{Q}$. By a *witness* to the independence of $x_1, \ldots, x_n$ we mean a function $f : \mathbb{Q}^n \setminus \{0\}$ such that $f(\overline{q}) \sum_{i=1}^n q_i x_i = 1$ for all $\overline{q} \neq 0$. There is a unique witness and it is recursive in $x_1, \ldots, x_n$.

Suppose $S \subseteq \mathcal{P}(\mathbb{N})$ is a Turing ideal, i.e., if $x, y \in S$ and $z \le_T x \oplus y$, then $z \in S$ where $x \oplus y$ is the Turing-join of x and y. For S countable an *enumeration* of S is $E \subset \mathbb{N}^2$ such that if $E_n = \{m : (m, n) \in E\}$ then $S = \{E_0, E_1, \ldots\}$. From an enumeration E of S we can find E-computable list $a_0, a_1 \ldots$ of all complex numbers $x + iy$ where x and y are real number recursive in elements of S and E-computable lists $f^m_0, f^m_1, \ldots$ of all functions $f : \mathbb{Q}^m \setminus \{0\} \to \mathbb{C}$ computable in an element of S.

Lemma 0.1 *We can find a Turing ideal S and an enumeration E of S such that E is low (i.e. $E' = 0'$).*

For example we can do this recursively in any completion of Peano Arithmetic and there are low completions of Peano arithmetic.

Fix such an E and S and the enumerations $a_0, a_1, \ldots$, $f_0^m, f_1^m, \ldots$ for $m \geq 1$ as above. Note that if $a_1, \ldots, a_m \in S$ are $\mathbb{Q}$-linealrly independent then we can find a witness in S.

There is a counterexample to Schanuel's conjecture if and only there is a counterexample coded in S and a witness function coded in S.

Thus Schanuel's conjecture holds if $\forall m \forall i_1 \ldots \forall i_m \forall j$ [f_j^m is not a witness to the independence of $a_{i_1}, \ldots, a_{i_m}$ or the transcendence degree of

$$\mathbb{Q}(a_{i_1}, \ldots, a_{i_m}, \exp(a_{i_1}), \ldots, \exp(a_{i_m}))$$

is at least m].

Saying that f_j^m is not a witness is

$$\bigvee_{\overline{q} \neq 0} f_j^m(\overline{q}) \sum_{k=1}^{n} q_k a_{i_k} \neq 0$$

and hence $\Sigma_1^0(E)$. While saying the transcendence degree is at least m is $\Pi_2^0(E)$. Thus the whole statement is $\Pi_2^0(E)$.

But, if x is low, then $\Pi_2^0(x)$ is the same as Π_2^0. Indeed,

$$y \text{ is } \Delta_2^0(x) \Leftrightarrow y \leq_T x' \Leftrightarrow y \leq_T 0' \Leftrightarrow y \text{ is } \Delta_2^0.$$