

Math 502 Metamathematics I
Problem Set 5

Due: Friday October 8

1) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Let T_0 be the axioms for equivalence relations. Suppose $T \supseteq T_0$ is an \mathcal{L} -theory such that for all n there is $\mathcal{M} \models T$ with an equivalence class of size at least n . Then there is $\mathcal{M} \models T$ with an infinite equivalence class. Conclude that the class of all equivalence relations where every class is finite is not an elementary class.

2) Let $\mathcal{L} = \{R\}$ where R is a binary relation. Recall that a *graph* is an \mathcal{L} -structure \mathcal{M} where $R^{\mathcal{M}}$ is symmetric and irreflexive. We say that a graph is *connected* if for each $x \neq y$ we can find a path from x to y . Prove that the class of connected graphs is not an elementary class.

3) Let \mathcal{L} be the language with one binary relation symbol $<$. Let T be an \mathcal{L} -theory extending the theory of linear orders such that T has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $\sigma : \mathbb{Q} \rightarrow M$ of the rational numbers into M . For example there is $\mathcal{M} \models \text{Th}(\mathbb{Z}, <)$ in which the rational order embeds. [Hint: Add constants c_q for all $q \in \mathbb{Q}$ and let $T^* = T \cup \{c_q < c_r : q, r \in \mathbb{Q}, q < r\}$.]

4) We say that a group $(G, +, <)$ is *archimedean* if for all $x, y \in G$ with $x, y > 0$ there is a positive integer m such that $x < my$. Show that there are non-archimedean models of $\text{Th}(\mathbb{Q}, +, <)$.

5) Let $\mathcal{L} = \{s\}$, where s is a unary function symbol. Let T be the \mathcal{L} -theory that asserts that s is a bijection with no cycles (i.e., $s^{(n)}(x) \neq x$ for $n = 1, 2, \dots$). For which cardinals κ is T κ -categorical? Is T complete?