Math 502 Metamathematics I
Problem Set 8

Due: Friday November 20
Questions marked with a dagger† are optional.

Recall that $\phi_0, \phi_1, \ldots$ is our usual listing of partial recursive functions and $W_e = \text{dom } \phi_e$.

1) Let $A \subseteq \mathbb{N}$ be an infinite recursively enumerable set. Prove that there is a total recursive injective $f : \mathbb{N} \to \mathbb{N}$ such that A is the image of f.

2) Suppose $f : \mathbb{N} \to \mathbb{N}$ is total recursive. Prove that

$$A = \bigcup_{n \in \mathbb{N}} W_{f(n)}$$

is recursively enumerable.

3)† a) (Reduction) Suppose A and B are Σ^0_n. Prove that there are A_0 and B_0 in Σ^0_n such that:

i) $A_0 \subseteq A$ and $B_0 \subseteq B$;

ii) $A_0 \cap B_0 = \emptyset$;

iii) $A_0 \cup B_0 = A \cup B$.

b) (Separation) Suppose A and B are Π^0_n and $A \cap B = \emptyset$. Prove that there is $C \in \Delta^0_n$ such that $A \subseteq C$ and $C \cap B = \emptyset$. [HINT: Use part a)]

4) Let Cof=$\{e : \neg W_e \text{ is finite}\}$. Prove that Cof is Σ^0_3.

5) Prove that $\{e : W_e \neq \emptyset\}$ is Σ^0_1-complete.

Two more Compactness Problems These two problems must be turned in by the end of the semester. For the following problems let $\mathcal{L} = \{+, \cdot, 0, 1\}$

C1) Prove that there is $\mathcal{M} \models \text{Th}(\mathbb{N})$ and $a \in \mathcal{M}$ such that

$$\mathcal{M} \models p \text{ divides } a$$

for every prime $p \in \mathbb{N}$.

C2) a) Let $T = \{\phi : \text{if } F \text{ is a finite field, then } F \models \phi\}$. Prove there is $K \models T$ where K has characteristic zero.

b)†† Prove that if $K \models T$, then for each d there is a unique extension $L \supseteq K$ such that the degree of L/K is d.