DIVERGENCE OF TEICHMÜLLER GEODESICS

ANNA LENZHEN AND HOWARD MASUR

Abstract. We answer the question of when two Teichmüller geodesic rays stay bounded distance apart and when they diverge.

1. Introduction

Let S be an oriented surface of genus g with n punctures. We assume $3g-3+n \geq 1$. Let $\mathcal{T}(S)$ denote the Teichmüller space of S with the Teichmüller metric $d(\cdot,\cdot)$. A basic question in geometry is to study the long term behavor of geodesics. In this paper we study the question of when a pair of geodesic rays $r_1(t), r_2(t)$ stay bounded distance apart, and when they diverge in the sense that $d(r_1(t), r_2(t)) \to \infty$ as $t \to \infty$. Teichmüller's theorem implies that a Teichmüller geodesic ray is determined by a quadratic differential q at the base point and that the surfaces along the ray are found by stretching along the horizontal trajectories of q and contracting along the vertical trajectories. It is a general principle that the asymptotic behavior of the ray is determined by the properties of the vertical foliation of q.

Many cases of the question of divergence of rays are already known. The first instance was if the quadratic differentials q_1, q_2 defining r_1, r_2 are Strebel differentials. This means that their vertical trajectories are closed and decompose the surface into cylinders. In [5] it was shown that if the homotopy classes of the cylinders for q_1 coincide with those of q_2, then r_1, r_2 stay bounded distance apart. In particular this showed that the Teichmüller metric was not negatively curved in the sense of Busemann. A second known case was if the vertical foliations of q_1, q_2 are minimal, topologically equivalent and uniquely ergodic. In that case the rays also stay bounded distance apart ([6]). On the other hand if the vertical foliations of q_1, q_2 are not topologically equivalent, then the rays diverge ([2]).

The last possibility not covered by the above results is if the vertical foliations of q_1, q_2 are topologically equivalent, have a minimal component and yet are not uniquely ergodic. In that case in each minimal component there exist a finite number of mutually singular ergodic measures, and any transverse measure is a convex combination of the ergodic measures. Ivanov ([2]) showed that if the transverse measures of q_1, q_2 in these minimal components are absolutely continuous with respect to each other, then the rays stay bounded distance apart.

In this paper we prove the converse which then together with the results mentioned above completes the picture. We will prove (Theorem 1) that *the rays r_1, r_2 diverge if the*

Date: February 27, 2008.

The second author is supported in part by the NSF..

vertical foliation of q_1 *is topologically equivalent to the vertical foliation of* q_2*, and there is a minimal complimentary component* Ω *of the vertical foliations such that the transverse invariant measures are not absolutely continuous with respect to each other*

The outline of the proof of the Theorem is as follows. We will find (Proposition 1) for any time t, a subsurface $Y(t) \subset \Omega$ with short boundary such that the area of $Y(t)$ is small on the surface on one ray, and bounded away from 0 on the surface along the other. This is where we use the assumption that the measures are not absolutely continuous with respect to each other. We will also find (Lemma 3) a bounded length curve $\gamma(t) \subset Y(t)$ on one surface which is "mostly vertical" with respect to the flat metric of the quadratic differential. Then on the other surface it has comparable length. The fact that the lengths are comparable and the areas have large ratio will allow us (Lemma 4) to show that the ratio of the extremal length of $\gamma(t)$ on a surface along one ray to the extremal length on the other is large. We then apply Kerckhoff's formula to conclude that the surfaces are far apart in Teichmuller space.

We will also prove (Theorem 2) that given any pair of ergodic measures ν_1, ν_2 on a minimal nonuniquely ergodic foliation F there is a sequence of multicurves $\{\gamma_i(n)\}_{i=1}^p$ such that for a pair of sequences of weights $\{a_i(n)\}$ and $\{b_i(n)\}$ we have $\sum_{i=1}^p a_i(n)\gamma_i(n) \to \nu_1$ and $\sum_{i=1}^p b_i(n)\gamma_i(n) \to \nu_2$ in the topology of PMF. This problem was suggested to us by Moon Duchin.

Acknowledgements. We would like to thank Moon Duchin and Ursula Hamenstadt for useful discussions and their interest in this project. We are also grateful to the Mathematical Sciences Research Institute for the support and hospitality during the time this research was conducted.

2. Measured foliations

Recall a measured foliation on a surface S consists of a finite set Σ of singular points and a covering of $S \setminus \Sigma$ by open sets $\{U_\alpha\}$ with charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2$ such that the overlap maps are of the form

$$(x, y) \to (f(x, y), \pm y + c).$$

The leaves of the foliation are the lines $y =$ constant. The points Σ are $p \geq 3$ pronged singularities. One allows single pronged singularities at the punctures. A measured foliation comes equipped with a transverse invariant measure which in the above coordinates is given by $\mu = |dy|$. Henceforth we will denote measured foliations by $[F, \mu]$.

For any homotopy class of simple closed curves β let

$$i([F, \mu], \beta) = \inf_{\beta' \sim \beta} \int_{\beta'} d\mu.$$

Let Γ_F denote the compact leaves of $[F, \mu]$ joining singularities. It is well-known that each component Ω of $S \setminus \Gamma_F$ is either an annulus swept out by closed leaves or a minimal domain in which every leaf is dense.

Definition 1. *We say that two foliations $[F_1, \mu_1], [F_2, \mu_2]$ are topologically equivalent if there is a homeomorphism of $S \setminus \Gamma_{F_1} \to S \setminus \Gamma_{F_2}$ isotopic to the identity which takes the leaves of F_1 to the leaves of F_2.*

Definition 2. *A foliation $[F, \mu]$ in a minimal domain Ω is said to be* uniquely ergodic *if the transverse measure μ restricted to Ω is the unique transverse measure of the foliation F up to scalar multiplication.*

More generally suppose Ω is a minimal component of F. There exist invariant transverse measures $\nu_1 = \nu_1(\Omega), \ldots, \nu_p = \nu_p(\Omega)$ such that

- p is bounded in terms of the topology of Ω.
- ν_i is ergodic for each $1 \leq i \leq p$
- any transverse invariant measure ν on Ω can be written as $\nu = \sum_{i=1}^{p} a_i \nu_i$ for $a_i \geq 0$.

Two foliations $[F, \mu_1]$ and $[F, \mu_2]$ are absolutely continuous with respect to each other if when the measures are expressed as a convex combination as above, the indices with positive coefficients are identical.

3. Quadratic differentials and Teichmüller rays

A meromorphic quadratic differential q on a closed Riemann surface X with a finite number of punctures removed is a tensor of the form $q(z)dz^2$ where q is a holomorphic function and $q(z)dz^2$ is invariant under change of coordinates. We allow q to have at most simple poles at the punctures.

As such there is a metric defined by $|q(z)|^{1/2}|dz|$. The length of an arc β with respect to the metric will be denoted by $|\beta|_q$. There is an area element defined by $|q(z)||dz|^2$. We will denote by $\mathrm{Area}_q Y$ the area of a subsurface $Y \subseteq X$.

Away from the zeroes and poles of q there are *natural* holomorphic coordinates $z = x+iy$ such that in these coordinates $q = dz^2$. The lines $x =$ constant with transverse measure $|dx|$ define the vertical foliation $[F_q^v, |dx|]$. The lines $y =$ constant with transverse measure $|dy|$ define the horizontal measured foliation $[F_q^h, |dy|]$. The transverse measure of an arc β with respect to $|dy|$ will be denoted by $v_q(\beta)$ and called the vertical length of β. Similarly we have the horizontal length denoted by $h_q(\beta)$.

We denote by Γ_q the vertical critical graph of q. This is the union of the vertical leaves joining the zeroes of q.

The Teichmüller space of S denoted by $\mathcal{T}(S)$ is the set of equivalence classes of homeomorphisms $f : S \to X$ where $f_i : S \to X_i, i = 1, 2$ are equivalent if there is a conformal map $h : X_1 \to X_2$ with f_2 homotopic to $h \circ f_1$. The *Teichmüller metric* on $\mathcal{T}(S)$ is the metric defined by

$$d((X, g), (Y, h)) := \frac{1}{2} \inf\{\log K(f) : f : X \to Y \text{ is homotopic to } h \circ g^{-1}\}$$

where f is quasiconformal and

$$K(f) := ||K_x(f)||_\infty \geq 1$$

is the *quasiconformal dilatation* of f, where

$$K_x(f) := \frac{|f_z(x)| + |f_{\bar{z}}(x)|}{|f_z(x)| - |f_{\bar{z}}(x)|}$$

is the *pointwise quasiconformal dilatation* at x.

Teichmüller's Theorem states that, given any $X, Y \in \mathcal{T}(S)$, there exists a unique (up to translation in the case when S is a torus) quasiconformal map f, called the *Teichmüller map*, realizing $d(X,Y)$. The Beltrami coefficient $\mu := \frac{\bar{\partial} f}{\partial f}$ is of the form $\mu = k\frac{\bar{q}}{|q|}$ for a unique unit area quadratic differential q on X and some k with $0 \leq k < 1$. Define t by $e^{2t} = \frac{1+k}{1-k}$. There is a quadratic differential $q(t)$ on Y such that in the natural local coordinates $w = u + iv$ of $q(t)$ and $z = x + iy$ of q the map f is given by

$$u = e^t x \quad v = e^{-t} y.$$

Thus f expands along the horizontal leaves of q by e^t, and contracts along the vertical leaves by e^{-t}.

Conversely, any unit area q on X determines a 1-parameter family of Teichmüller maps and a geodesic ray $r(t) = r_{(X,q)}(t)$ in $\mathcal{T}(S)$, called the *Teichmüller ray* based at X in the direction of q. For each $t \geq 0$ one forms the Teichmüller map f_t with Beltrami differential $\mu = k\frac{\bar{q}}{|q|}$ with $e^{2t} = \frac{1+k}{1-k}$. The image surface is denoted by $r(t)$ and the corresponding quadratic differential on $r(t)$ by $q(t)$.

Now let q_1, q_2 quadratic differentials on X_1 and X_2 with vertical foliations $[F^v_{q_1}, |dx_1|]$ and $[F^v_{q_2}, |dx_2|]$ and determining rays r_1, r_2. Our main result is then

Theorem 1. *Suppose $[F^v_{q_1}, |dx_1|]$ is topologically equivalent to $[F^v_{q_2}, |dx_2|]$. The rays r_1, r_2 diverge if there is a minimal complimentary component Ω of the foliation $F^v_{q_i}$ with ergodic measures $\nu_1, \ldots, \nu_p$ and so that restricted to Ω, $|dx_1| = \sum_{i=1}^p a_i \nu_i$, $|dx_2| = \sum_{i=1}^p b_i \nu_i$ and there is some index i so that either $a_i = 0$ and $b_i > 0$, or $a_i > 0$ and $b_i = 0$.*

3.1. **Extremal length.** We recall the notion of extremal length. Suppose X is a Riemann surface and Γ is a family of arcs on X. Suppose ρ is a conformal metric on X. For an arc γ, denote by $\rho(\gamma)$ its length and by $A(\rho)$ the area of ρ.

Definition 3.

$$Ext_X(\Gamma) = \sup_\rho \frac{\inf_{\gamma \in \Gamma} \rho^2(\gamma)}{A(\rho)},$$

where the sup is over all conformal metrics ρ.

We will apply this definition when Γ consists of all simple closed curves in a free homotopy class of some α. In that case we will write $\mathrm{Ext}_X(\alpha)$.

The following formula due to Kerckhoff ([4]) is extremely useful in estimating Teichmüller distances. For $X, Y \in \mathcal{T}(S)$

$$d(X,Y) = \frac{1}{2} \log \sup_\alpha \frac{\mathrm{Ext}_Y(\alpha)}{\mathrm{Ext}_X(\alpha)}. \tag{1}$$

Lemma 1. *Let X be a Riemann surface. Let q be a unit area quadratic differential on X. Let Y be a subsurface with geodesic boundary γ. If the length $|\gamma|_q$ is small enough, then for any homotopy class of curves $\alpha \subset Y$*

$$Ext_X(\alpha) \geq \frac{|\alpha'|_q^2}{Area_q(Y) + O(|\gamma|_q^2)},$$

where α' is the geodesic in the homotopy class of α.

Proof. Let $\epsilon = |\gamma|_q$. Define a metric ρ on X as folows. Let ρ coincide with the q-metric on $\mathcal{N}_\epsilon(Y)$, the ϵ-neighborhhood of Y and the q metric multiplied by a small δ on $Z = X \backslash N_\epsilon(Y)$. Let α'' be any curve in the homotopy class of α. If α'' is not contained in Y then α'' and a segment of γ bound a disk. The fact that $d_\rho(Z, Y) = \epsilon$ and γ is a geodesic implies that we can replace an arc of α'' with an arc of γ to produce $\alpha''' \subset \bar{Y}$ with smaller length. We conclude that the infimum of the flat length in the metric ρ is realized by the geodesic α' in Y. By definition,

$$\mathrm{Ext}_X(\alpha) \geq \frac{\inf_{\alpha'' \sim \alpha} |\alpha''|_q^2}{A(\rho)} \geq \frac{|\alpha'|_q^2}{\mathrm{Area}_q(Y) + O(\epsilon^2) + \delta \mathrm{Area}_q Z}.$$

Since δ is arbitrary we have the result.

□

Definition 4. *Given a quadratic differential q and $\delta > 0$, a geodesic γ in the q metric is called almost (q, δ)-vertical if $\frac{v_q(\gamma)}{h_q(\gamma)} > \delta$.*

Lemma 2. *Let q be a quadratic differential on X a surface without boundary. For any $\delta > 0$ there is a curve β which is almost (q, δ)-vertical.*

Proof. If $\Gamma_q \neq \emptyset$ we are done. Thus we can assume that the vertical foliation is minimal. Let A be the area of q. The first return map of the foliation to a horizontal transversal I with an endpoint at a singularity defines a generalized interval exchange transformation. The maximal number of intervals n_0 is determined by the genus of the surface and the orders of the singularities of q. Choose a horizontal transversal I of length λ satisfying

$$\lambda^2 < \frac{A}{2n_0\delta}. \tag{2}$$

The transversal I determines a decomposition of the surface into rectangles $\{R_i\}$, with heights h_i and widths λ_i, whose horizontal sides are subsets of I. Each rectangle has two horizontal sides on I. Consequently, if we count each λ_i twice we have

$$\sum_i \lambda_i = 2\lambda.$$

Since we count each λ_i twice we have

$$\sum_i h_i \lambda_i = 2A.$$

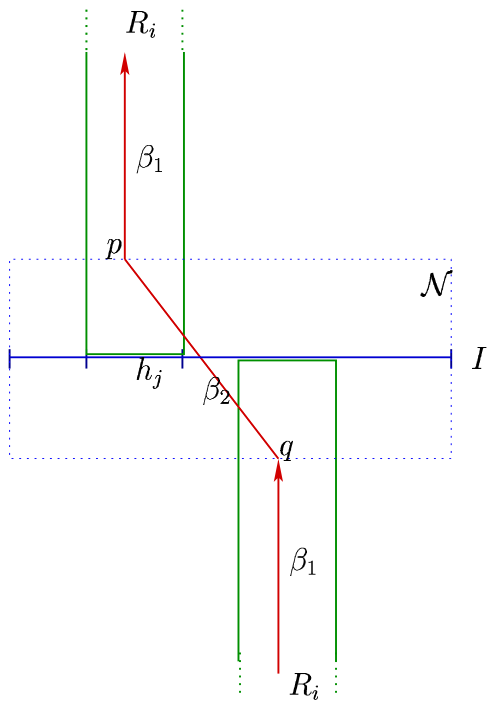

FIGURE 1. β is the union of β_1 and β_2

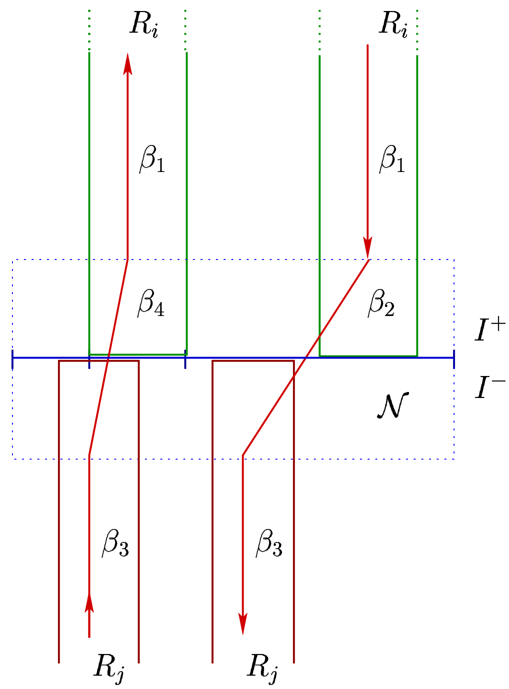

FIGURE 2. β is the union of $\beta_1, \beta_2, \beta_3$ and β_4

We conclude that

$$\max_i h_i > \frac{A}{\lambda n_0} \tag{3}$$

Let h_i realize this maximum. There are two cases. The first case (see Figure 1) is that the top and bottom of R_i are on the opposite sides of I. Fix a small neighborhood $\mathcal{N}$ of I. We form a simple closed curve $\beta = \beta_1 * \beta_2$. Here β_1 is a vertical segment in R_i whose endpoints p and q are on the boundary of $\mathcal{N}$, and β_2 is an arc transverse to the horizontal foliation in $\mathcal{N}$ joining p and q. Then β is also transverse to the horizontal foliation. Its geodesic representative has the same vertical length, namely, h_i. The horizontal length of β is at most λ. Together with (2) and (3) we have that

$$\frac{v_q(\beta)}{h_q(\beta)} \geq \delta.$$

In the second case (Figure 2) the top and the bottom of R_i are on the same side of I (call it I^+). Then there must also be a rectangle R_j with top and bottom on I_-. We may form a simple closed β which consists of a vertical segment in R_i, a vertical segment in R_j and a pair of arcs in $\mathcal{N}$ which are transverse to the horizontal foliation. Similar to the case above, the ratio of vertical and horizontal components of β is at least δ.

$\square$

Definition 5. *Given a unit area quadratic differential q on a surface X, a subsurface $Y \subseteq X$ is said to be (ϵ, ϵ_0)-thick if*

- *either $Y = X$ or $Ext_X(\partial Y) \leq \epsilon$*
- *the shortest non-peripheral curve in Y has q length at least ϵ_0.*

Lemma 3. *For any $B > 0, \epsilon_0 > 0$ there exists $\epsilon > 0, \delta > 0, D > 0, m_0$ such that for any (ϵ, ϵ_0)-thick surface $Y \subset X$ which is not an annulus and such that $Area_q(Y) \geq B$*

(1) there is an almost (q, δ)-vertical geodesic γ whose interior lies in Y and such that $|\gamma|_q < D$.

(2) there is an $m \leq m_0$ and a collection $\omega_1, \dots, \omega_m$ of disjoint vertical leaves so that for every horizontal leaf L

$$|card(L \cap \gamma) - \sum_{i=1}^{m} card(L \cap \omega_i)| \leq 2.$$

Proof. For the proof of the first statement, we argue by contradiction. If it is not true then there is a sequence X_n of surfaces, and unit area quadratic differentials q_n on X_n; a sequence of $(1/n, \epsilon_0)$-thick subsurfaces Y_n with area at least B such that the shortest almost $(q_n, 1/n)$-vertical curve on Y_n has length at least n. Passing to subsequences, we can assume that Y_n converges to some ϵ_0-thick, area at least B punctured surface Y_∞, and that q_n restricted to Y_n converges to a quadratic differential q_∞ on Y_∞, uniformly on compact sets. Specifically this means that for any neighborhood U of the punctures on Y_∞

(1) for large enough t_n there is a conformal map $F_n : Y(\infty) \setminus U \to Y_n$

(2) $F_n^* q_n \to q_\infty$ as $n \to \infty$ uniformly on $Y(\infty) \setminus U$.

By Lemma 2, taking $\delta = 1$, there is a curve β on Y_∞ such that

$$\frac{v_{q_\infty}(\beta)}{h_{q_\infty}(\beta)} \geq 1.$$

By uniform convergence, $v_{q_n}(\beta) \to v_{q_\infty}(\beta)$, and $h_{q_n}(\beta) \to h_{q_\infty}(\beta)$ and thus for large enough n,

$$\frac{v_{q_n}(\beta)}{h_{q_n}(\beta)} \geq 1/2$$

and furthermore, $|\beta|_{q_n} \leq |\beta|_{q_\infty} + 1$. This is a contradiction proving the first statement.

If the second statement is false, then for each m there is a Y_m such that there does not exist a sequence $\omega_1, \dots, \omega_m$ of vertical arcs corresponding to the closed curve γ_m found in the first part. Passing to a subsequence we can assume again that $q_m \to q_\infty$, $Y_m \to Y_\infty$ and $\gamma_m \to \gamma_\infty$, where γ_∞ is almost $(q_\infty, \delta_\infty)$-vertical for some $\delta_\infty > 0$. Now γ_∞ has the property that there is a fixed collection of vertical leaves $\omega_1, \dots, \omega_{m_0}$ with the desired property. If q_∞ has the same singularity pattern as q_m for all large m, then it is clear that there are corresponding vertical segments $\omega_1^m, \dots, \omega_{m_0}^m$ with the required property for γ_m, which is a contradiction. If some zeros of q_m have coalesced to a higher order zero of q_∞ the curve γ_m may include a bounded number of short segments joining zero! s of q_m. In this case we include a bounded number of additional vertical segments. This is again a contradiction proving the lemma. ∎

The following Lemma allows us to find curves with very different extremal length if there are subsurfaces with very different areas.

Lemma 4. *For any B, $M > 0$, $\delta > 0$, $\epsilon_0 > 0$ and $C > 0$, there exists an $\epsilon > 0$ so that the following holds. If q_1 and q_2 are quadratic differentials on X_1, X_2 and Y is a subsurface which is not an annulus such that*

(i)

$$Area_{q_1}(Y) \geq B,\ Area_{q_2}(Y) < \epsilon$$

(ii) for any almost (q_1, δ)-vertical curve $\gamma \in Y$ the vertical components satisfy

$$\frac{1}{C} \leq \frac{v_{q_1}(\gamma)}{v_{q_2}(\gamma)} \leq C$$

(iii) $|\delta Y|_{q_i} < \epsilon$ for $i = 1, 2$

(iv) Y is (ϵ, ϵ_0)- thick with respect to q_1

then there exists a curve γ in Y so that

$$\frac{Ext_{q_2}(\gamma)}{Ext_{q_1}(\gamma)} \geq M.$$

Proof. By Lemma 3, for some δ and D there is an almost (q_1, δ)-vertical curve $\gamma \subset Y$ such that

$$\epsilon_0 \leq |\gamma|_{q_1} < D \tag{4}$$

Let σ_i be the hyperbolic metric on X_i and $l_{\sigma_i}(\gamma)$ denote the length of the geodesic in the hyperbolic metric. By Theorems 1 and 4 in Rafi[11],

$$l_{\sigma_1}(\gamma) < C_1 \frac{1}{\sqrt{\text{Area}_{q_1}(Y)}} |\gamma|_{q_1} = C_1 |\gamma|_{q_1}/\sqrt{A} \leq C_1 D/\sqrt{A} \tag{5}$$

the constant C_1 depending only topology of the surface. Also by Maskit's comparison of hyperbolic and extremal lengths in [9],

$$Ext_{q_1}(\gamma) \leq \frac{1}{2} l_{\sigma_1}(\gamma) e^{l_{\sigma_1}(\gamma)/2} \leq C_1 D/\sqrt{A} e^{C_1 D/2\sqrt{A}}. \tag{6}$$

Set $C_2 = C_1 D/\sqrt{A} e^{C_1 D/2\sqrt{A}}$.

On the other hand, by (4), assumption (ii) and the fact that γ is almost (q_1, δ)-vertical

$$|\gamma|_{q_2} \geq v_{q_2}(\gamma) > \frac{1}{C} v_{q_1}(\gamma) > \frac{\delta}{C(1+\delta)} |\gamma|_{q_1} \geq \frac{\epsilon_0 \delta}{C(1+\delta)}. \tag{7}$$

and by Lemma 1,

$$Ext_{q_2}(\gamma) \geq \frac{|\gamma|^2_{q_2}}{\text{Area}_{q_2}(Y) + O(|\delta Y|^2_{q_2})} \tag{8}$$

Putting the inequalities (7), (8) together and using assumptions (i) and (iii), we obtain

$$\frac{Ext_{q_2}(\gamma)}{Ext_{q_1}(\gamma)} \geq \frac{\epsilon_0^2 \delta^2}{C_2 C^2 (1+\delta)^2 (\text{Area}_{q_2}(Y) + O(|\delta Y|^2_{q_2}))} \geq \frac{C_4}{\epsilon} \tag{9}$$

where C_4 does not depend on ϵ if ϵ is small enough. Now setting $\epsilon < \frac{C_4}{M}$ guarantees that the Lemma holds. □

4. Areas of subsurfaces along rays

The proof of the theorem is now based on

Proposition 1. *Suppose q_1, q_2 are quadratic differentials on X_1, X_2 such that the vertical foliations $[F^v_{q_1}, |dx_1|], [F^v_{q_2}, |dx_2|]$ are topologically equivalent and have a minimal nonuniquely ergodic component Ω. Suppose with respect to the invariant ergodic measures $\nu_1, \ldots, \nu_p$ on Ω, $|dx_1| = \sum_{k=1}^p a_k \nu_k$, with $a_1 > 0$, while $|dx_2| = \sum_{k=1}^p b_k \nu_k$ with $b_1 = 0$. Let $q_i(t)$ be the quadratic differentials along the rays $r_i(t)$. Then there exists $\epsilon_0 > 0, M_0 > 0$ so that for small enough ϵ, there is t_0 such that for $t \geq t_0$, there is a subsurface $Y(t) \subset \Omega$ satisfying*

(i) $Y(t)$ is (ϵ, ϵ_0) thick with respect to $q_1(t)$.
(ii) $Area_{q_1(t)}(Y(t)) \geq a_1(1 - M_0\epsilon)$.
(iii) $Area_{q_2(t)}(Y(t)) < \epsilon$.

Proof. For $i = 1, 2$ let $|dy_i|$ denote the transverse measure to the horizontal foliation of q_i. We normalize the measures ν_k so that

$$\int_\Omega d\nu_k |dy_1| = 1. \tag{10}$$

This implies $\sum_k a_k = \text{Area}_{q_1}(\Omega)$. The measures $|dy_1|, |dy_2|$ are uniformly comparable. This implies that there is a constant C such that for any subsurface Z and each k.

$$\frac{1}{C}\int_Z d\nu_k |dy_1| \leq \int_Z d\nu_k |dy_2| \leq C \int_Z d\nu_k |dy_1|. \tag{11}$$

Applying the left hand inequality to $Z = \Omega$ and using the fact that $\text{Area}_{q_2}(\Omega) = \sum_k \int_\Omega b_k d\nu_k |dy_2|$ we find there is a bound C' such that $\sum_{k=1}^p b_k \leq C'$.

We can find a finite set of horizontal transversals I to the vertical foliation in Ω such that for any $\nu_i, \nu_j, i \neq j$ there is a transversal I from this set and $\delta > 0$ such that

$$|\frac{\nu_i(I)}{\nu_j(I)} - 1| > \delta. \tag{12}$$

Let Λ_i be the set of generic points for ν_i and the transversals I in the set; that is, if $x \in \Lambda_i$ and $l(x)$ is the vertical leaf of q_1 through x of length T, then

$$\lim_{T \to \infty} \frac{1}{T} |l(x) \cap I| \to \nu_i(I). \tag{13}$$

The sets Λ_i are pairwise disjoint; ν_i almost every point belongs to Λ_i and almost every point in Ω belongs to $\cup \Lambda_i$ with respect to the area element defined by q_1.

By [7], as $t \to \infty$ there is a maximal collection of disjoint curves $\gamma_1(t), \ldots, \gamma_m(t) \subset \Omega$ such that

$$\text{Ext}_{r_1(t)}(\gamma_i(t)) \to 0.$$

Consequently we can find $\epsilon_0 > 0$, such that given any $\epsilon > 0$, for t sufficiently large Ω can be decomposed into a collection $\{Y(t)\}$ of (ϵ, ϵ_0) thick subsurface. There is a uniform bound N for the number of these surfaces.

Let $f_t : X_0 \to r_1(t)$ denote the corresponding Teichmuller map. Let $\Lambda_i(t) = f_t(\Lambda_i)$. We now claim that for t big enough, for each subsurface $Y(t)$ in this collection there is an i such that

$$\cup_{j\neq i}\mathrm{Area}_{q_1(t)}(Y(t) \cap \Lambda_j(t)) \leq \epsilon/CC'. \tag{14}$$

We argue by contradiction. If the claim is not true, there exists $\epsilon_1 > 0$, a sequence $t_n \to \infty$, and disjoint subsurfaces $Y(n) \subsetneq \Omega$ such that for at least two distinct values of j,

$$\mathrm{Area}_{q_1(t_n)}(Y(n) \cap \Lambda_k(t_n)) > \epsilon_1. \tag{15}$$

Assume first that $Y(n)$ does not contain a flat cylinder. By passing to a subsequence we can assume that the surfaces $Y(n)$ converge to a limiting punctured surface $Y(\infty)$ and the corresponding $q_1(t_n)$ converges to a limiting $q_1(\infty)$ on $Y(\infty)$. Again this means that for any neighborhood U of the punctures on $Y(\infty)$

(1) for large enough t_n there is a conformal map $h_n : Y(\infty) \setminus U \to Y(t_n)$
(2) $h_n^* q_1(t_n) \to q_1(\infty)$ as $t_n \to \infty$ uniformly on $Y(\infty) \setminus U$.

Now from condition (2) above it follows that for each such U, for t_n large enough,

$$\gamma_i(t_n) \cap h_n(Y(\infty) \setminus U) = \emptyset.$$

Since $Y(t_n)$ does not contain a flat cylinder in the homotopy class of $\partial Y(t_n)$ we may find U so that

$$\mathrm{Area}_{q_1(t_n)}(Y(n) \setminus (h_n(Y(\infty) \setminus U))) \leq \epsilon_1/2. \tag{16}$$

Now consider the collection of rectangles R with respect to the flat structure of $q_1(\infty)$ that are contained in $K := Y(\infty) \setminus U$. We now follow an argument in [7]. For any such rectangle R we argue that there cannot be points $x_i \in \Lambda_i, x_j \in \Lambda_j; i \neq j$ such that

$$y_i = \lim f_{t_n}(x_i) \in R.$$

For suppose there were x_i, x_j with this property. There is a rectangle $R' \subset R$ whose vertical sides L_i, L_j have endpoints at y_i, y_j. For every horizontal segment H,

$$|\mathrm{card}(H \cap L_i) - \mathrm{card}(H \cap L_j)| \leq 2.$$

Let $y_{i,n} = f_{t_n}(x_i)$, $y_{j,n} = f_{t_n}(x_j)$ and let $L_{i,n}, L_{j,n}$ the vertical segments of $q_1(t_n)$ that converge to L_i, L_j. For t_n large enough, for any horizontal segment H_n of $q(t_n)$

$$\left|\frac{\mathrm{card}(L_{i,n} \cap H_n)}{\mathrm{card}(L_{j,n} \cap H_n)} - 1\right| \leq \delta/2.$$

Mapping $L_{i,n}$ back to X_0 by $f_{t_n}^{-1}$ and applying the above to the transversal $H_n = f_{t_n}(I)$, we have a contradiction to (12) and (13).

Thus for each rectangle R there is some $i = i(R)$ such that for all $j \neq i$ and for all x we have

$$\chi_R(f_{t_n}(x))\chi_{\Lambda_j}(x) \to 0.$$

Then by the Lebesgue dominated convergence theorem and the fact that the map f_{t_n} is area preserving,

$$\mathrm{Area}_{q_1(t_n)}(\Lambda_j(t_n)\cap R)=\int_{X_1}\chi_R(f_{t_n}(x))\chi_{\Lambda_j}(x)|dx||dy_1|\to 0.$$

Now we take a covering of K by such rectangles. If any two rectangles R, R' overlap then $i(R)=i(R')$. It follows from the connectedness of K, that there is a single i such that for all R, $i(R)=i$. Thus for t_n large enough, for all $j\neq i$,

$$\mathrm{Area}_{q_1(t_n)}(\Lambda_j(t_n)\cap Y(t_n))\leq \epsilon_1/2.$$

Combining with (16) we have contradicted our hypothesis (15) and proven the claim (14) in this case.

Now suppose Y_n contains a flat cylinder C_n with core curve β_n. Then $v_q(\beta_n)\to\infty$ as $n\to\infty$. Again we cannot have images of points of Λ_i, and Λ_j in C_n; otherwise again every horizontal intersecting a vertical segment through one will intersect the vertical segment through the other one, and again we have a contradiction. We have proved the claim (14).

Thus for each $Y(t)$, for all but at most one index k we have

$$\int_{Y(t)} d\nu_k|dy_1|\leq \epsilon/CC'. \tag{17}$$

Let $Z_k(t)$ be the union of those $Y(t)$ such that the above inequality holds for that index. Then $Z_k(t)\neq\Omega$ for otherwise we would have

$$\int_\Omega d\nu_k|dy_1|<N\epsilon/C'C,$$

contradicting (10), for ϵ sufficiently small. Let $Y_1(t)=\Omega\setminus Z_1(t)$, and so we have

$$\mathrm{Area}_{q_1(t)}(Y_1(t))\geq a_1\int_{Y_1(t)}d\nu_1|dy_1|\geq a_1(1-N\epsilon/CC'),$$

proving (ii) with $M_0=N/CC'$. Now conclusion (iii) follows from (17) applied to $k=1$, the right hand inequality in (11), the fact that $b_1=0$, and the sum of b_k is at most C'. □

Proof of Theorem 1. Without loss of generality we can assume that for some minimal component Ω we have $a_1>0$ and $b_1=0$. Fix $M>0$. It suffices to show that there exists t_0 such that for $t\geq t_0$ there is a simple closed curve $\gamma(t)$ with

$$\frac{Ext_{q_2(t)}(\gamma(t))}{Ext_{q_1(t)}(\gamma(t))}>M. \tag{18}$$

Choose $\epsilon>0$ small. Since the measures $|dy_1|$ and $|dy_2|$ are uniformly comparable, the vertical lengths $v_{q_1(t)}(\gamma)$ and $v_{q_2(t)}(\gamma)$ of any curve γ are uniformly comparable. Thus given $\delta>0$, there is a fixed constant C such that condition (ii) of Lemma 4 holds. We may also choose a fixed constant B such that for $t>t_0$, the subsurface $Y(t)$ given by Proposition 1 satisfies

$$\mathrm{Area}_{q_1(t)}(Y(t))>B$$

$$\mathrm{Area}_{q_2(t)}(Y(t)) < \epsilon$$

and

$$\mathrm{Ext}_{r_1(t)}(\partial Y(t)) \leq \epsilon/M.$$

If $|\partial Y(t)|_{q_2(t)} \geq \epsilon^2$, then $\mathrm{Ext}_{r_2(t)}(\partial Y(t)) \geq \epsilon$, and we are done; we may choose $\gamma = \partial Y(t)$. Thus assume

$$|\partial Y(t)|_{q_2(t)} \leq \epsilon^2 \leq \epsilon.$$

If $Y(t)$ is not a flat cylinder, for ϵ small enough, we can apply Lemma 3 and Lemma 4, which provide a curve $\gamma \subset Y(t)$ with the desired property (18).

Thus assume $Y(t)$ is a flat cylinder. Let α be a core curve of $Y(t)$. Then $|\alpha|_{q_1(t)} < \epsilon$. Suppose first that α is $(q_1(t), \delta)$-almost vertical. The reciprocal of the modulus of the cylinder is an upper bound for $\mathrm{Ext}_{r_1(t)}(\alpha)$, and we have

$$\mathrm{Ext}_{r_1(t)}(\alpha) \leq \frac{|\alpha|^2_{q_1(t)}}{\mathrm{Area}_{q_1}(Y(t))} \leq \frac{|\alpha|^2_{q_1(t)}}{B} \leq \frac{\epsilon^2}{B} \tag{19}$$

We now want to estimate $Ext_{r_2(t)}(\alpha)$. By (7), the flat length

$$|\alpha|_{q_2(t)} > \frac{\delta|\alpha|_{q_1(t)}}{C(1+\delta)}.$$

We can assume

$$\mathrm{Ext}_{r_2(t)} \leq \frac{M\epsilon^2}{B},$$

for otherwise we are done. As a consequence of Theorem 5.1 in Minsky([10]), there is an annulus $A(t)$ which is a union of $Y(t)$ and an expanding annulus $Y'(t)$, and a universal constant $c > 0$ such that

$$\mathrm{Ext}_{r_2(t)}(\alpha) \geq \frac{c}{\mathrm{mod}(A(t))} \geq \frac{c}{\mathrm{mod}(Y(t)) + \mathrm{mod}(Y'(t))} \geq$$

$$\frac{c}{-\log|\alpha|_{q_2(t)} + \frac{\mathrm{Area}_{q_2}(Y(t))}{|\alpha|^2_{q_2(t)}}} \geq \frac{c'}{-\log|\alpha|_{q_1(t)} + \frac{\epsilon}{|\alpha|^2_{q_1(t)}}}.$$

Comparing with (19) we see that for ϵ small enough, α is a curve that satisfies (18).

Suppose now the core curve α of $Y(t)$ is not $(q_1(t), \delta)$ almost vertical. If $Y(t)$ is nonseparating, choose a nontrivial isotopy class of arcs in the complement of $Y(t)$ joining the top and bottom of $Y(t)$. If $Y(t)$ is separating choose two nontrivial isotopy classes, one that joins the top of $Y(t)$ to itself and the other which joins the bottom to itself. These families can be chosen to lie in the thick part of the surface $r_1(t)$ and as such have extremal length bounded independently of t. In the first case we also take a families of arcs β crossing $Y(t)$ that intersect a fixed perpendicular at most once. The arcs β are $(q_1(t), \frac{1}{2\delta})$ almost vertical. In the second case we take a pair of arcs crossing $Y(t)$ which are $(q_1(t), \frac{1}{2\delta})$ almost vertical. Let $\delta' = \frac{1}{2\delta}$. Now we can form a closed curve γ as a concatentation of an arc outside $Y(t)$

and an arc β. or in the separating case, a pair of arcs outside and a pair of arcs crossing. For some constant c, again by Theorem 5.1 in Minsky([10]).

$$\mathrm{Ext}_{r_1(t)}(\gamma) \leq c\mathrm{Ext}_{r_1(t)}(\beta) \leq c\frac{|\beta|^2_{q_1(t)}}{\mathrm{Area}_{q_1(t)}(Y(t))} \leq c\frac{|\beta|^2_{q_1(t)}}{B}.$$

On the other hand the curves γ formed this way are longer and fewer than the arcs β crossing the cylinder. Since the arcs β are $(q_1(t), \delta')$ almost vertical, we have

$$\mathrm{Ext}_{r_2(t)}(\gamma) \geq \mathrm{Ext}_{r_2(t)}(\beta) = \frac{|\beta|^2_{q_2(t)}}{\mathrm{Area}_{q_2(t)}(Y(t))} \geq \frac{\delta'|\beta|^2_{q_1(t)}}{\epsilon C(1+\delta')},$$

and we are done. for ϵ small enough. □

Theorem 2. *Let $\nu_1, \ldots, \nu_k$ be maximal collection of ergodic measures for a minimal foliation $[F, \mu]$. Then there is a sequence of multicurves $\gamma_n = \{\gamma_n^1, \ldots, \gamma_n^k\}$ such that $\gamma_n^j \to [F, \nu_j]$ in $\mathcal{PMF}$.*

We note that each γ_n^j may itself be a multicurve.

Proof. Choose corresponding unit area quadratic differentials q_i on some surface X_0 whose vertical foliations are $[F, \nu_i]$. Let $r_i(t)$ the corresponding ray. For any sequence of times $t_n \to \infty$ by Proposition 1 there is a collection of disjoint domains $Y_1(t_n), \ldots, Y_p(t_n)$ such that

$$\mathrm{Area}_{q_i(t_n)}(Y_i(t_n)) \geq B > 0.$$

Suppose first that $Y_i(t_n)$ is not a cylinder. By the first part of Lemma 3 we may pick a $(q_i(t_n), \delta)$ almost vertical curve $\gamma_i(t_n) \subset Y_i(t_n)$ of length at most D. We claim that $\gamma_i(t_n) \to [F, \nu_i]$. As before, let $\Lambda_i(n)$ be the image of the generic points inside $Y_i(t_n)$; generic say with respect to the transversal for the set of all simple closed curves α. The generic points are dense, hence we can find a collection $\{\omega_j(n)\}_{j=1}^k$ of vertical segments beginning at generic points satisfying the second conclusion of Lemma 3. By construction of the $\omega_j(n)$, for any fixed α,

$$\frac{i(\gamma_i(t_n), \alpha)}{\sum_{j=1}^k i(\omega_j(n), \alpha)} \to 1.$$

Since $\omega_j(n)$ is a vertical segment through a generic point,

$$\frac{i(\omega_j(n), \alpha)}{|\omega_j(n)|_{q_i}} \to \nu_i(\alpha).$$

Summing over all $1 \leq j \leq k$ we have

$$\frac{i(\gamma_i(t_n), \alpha)}{\sum_{j=1}^k |\omega_j(n)|_{q_i}} \to \nu_i(\alpha).$$

However the sum in the denominator is exactly $v_{q_i}(\gamma_i(t_n))$ and so if we let $s_n = \frac{1}{v_{q_i}(\gamma_i(t_n))}$ then we have

$$\lim_{n\to\infty} s_n i(\gamma_i(t_n), \alpha) \to \nu_i(\alpha)$$

and we are done. □

References

[1] L Ahlfors *Lectures on Quasiconformal Mappings*, University Lecture Series, AMS, 2nd edition, 2006
[2] N.Ivanov
[3] A.Fathi, F.Laudenbach, V.Poenaru, *Travaux de Thurston sur les surfaces*, Asterisque, Vol. 66-67 (1979)
[4] S. Kerckhoff, *The asymptotic geometry of Teichmuller space*, *Topology*, Vol. 19, 23–41.
[5] H. Masur, *On a class of geodesics in Teichmüller space* Annals of Math., Vol. 102, No. 2 (1975), 205–221.
[6] H.Masur *Uniquely ergodic quadratic differentials*, Comment. Math. Helvetici Vol. 55 (1980) 255-266
[7] H.Masur *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential* Duke Math. J. Vol 66 (1992) 387-442
[8] H Masur, S Tabachnikov *Rational Billiards and Flat Structures*, Handbook of Dynamical Systems, B. Hasselblatt, A. Katok eds. Elsevier (2002), 1015-1089
[9] B Maskit *Comparison of hyperbolic and extremal lengths*, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 381-386. MR 87c:30062
[10] Y Minsky *Extremal length estimates and product regions in Teichmüller space*, Duke Math. J. Volume 83, Number 2 (1996), 249-286
[11] K Rafi *Thick-Thin decomposition for quadratic differentials*, Math. Res. Lett. 14 (2007), no. 2, 333-341

Anna Lenzhen:
Dept. of Mathematics
University of Michigan
Ann Arbor, Michigan, 48109
E-mail: alenzhen@umich.edu

Howard Masur:
Dept. of Mathematics, University of Illinois at Chicago
Chicago, IL 60680
E-mail: masur@math.uic.edu