Math 181 **CALCULUS II** **Final Exam Solution**

December 11, 1997

1. a) (10pts) Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$. Thus

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C.$$

b) (10pts) With $u = x^2$ and $dv = xe^{x^2}\,dx$, $du = 2x\,dx$ and $v = (1/2)e^{x^2}$, so

$$\int x^3 e^{x^2}\,dx = \int u\,dv = uv - \int v\,du = (1/2)x^2 e^{x^2} - \int xe^{x^2}\,dx$$

and thus

$$\int x^3 e^{x^2}\,dx = (1/2)x^2 e^{x^2} - (1/2)e^{x^2} + C.$$

2. a) (10pts) Dividing x^2+1 into x^2 we have

$$\int \frac{x^2}{x^2+1}\,dx = \int (1 - \frac{1}{x^2+1})\,dx = x - \tan^{-1} x + C.$$

b) (10pts) Since $x^2 + x - 6 = (x-2)(x+3) = (x-2)(x-(-3))$ we have

$$\int \frac{1}{x^2+x-6}\,dx = \int \frac{1}{(x-2)(x-(-3))}\,dx = \frac{1}{5}[\ln|x-2| - \ln|x+3|] + C$$

from the table.

3. (15pts) Since the only place $1/x^2$ is not defined is at $x = 0$ the improper integral exists if and only if

$$\int_{-1}^{0} \frac{1}{x^2}\,dx \quad \text{and} \quad \int_{0}^{1} \frac{1}{x^2}\,dx$$

exist. But

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \int_b^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+}(-\frac{1}{x})|_b^1 = \lim_{b\to 0^+}(\frac{1}{b} - 1)$$

does not exist. Therefore the improper integral does not exist.

4. a) (14pts) Since $v'(t) = -1.6$, $v(t) = -1.6t + C$. Thus $0 = v(0) = C$, so $v(t) = -1.6t$. Since $h'(t) = v(t)$, $h(t) = -1.6(t^2/2) + C$. Thus $12 = h(0) = C$, so $h(t) = -.8t^2 + 12$. b) (6pts) $h(t) = 0$ so $-.8t^2 + 12 = 0$ or $t^2 = 15$ and hence $t = \sqrt{15} \approx 3.872\,seconds$.

5. (20pts) At depth $5 \le x \le 12$ feet form the top, a cross section of water of thickness Δx weighs $(62.4)((\pi 14^2)\Delta x)$ pounds. Thus the work done to raise this cross section to the top of the tank is $Force \times Distance = ((62.4)(\pi 14^2)\Delta x)x$. The total work done is

$$\begin{aligned}\int_5^{12} (62.4)(\pi 14^2)x\Delta x &= (62.4)(\pi 14^2)(x^2/2)|_5^{12} \\ &= (62.4)(\pi 14^2)(1/2)(12^2 - 5^2) = 2,286,164.62\,ft\text{–}lbs.\end{aligned}$$

6. a) (4pts) sketch. b) (16pts)

$$\begin{aligned}
\int_0^1 \pi(4-x^2)^2\,dx - \int_0^1 \pi(2+x)^2\,dx &= \pi\int_0^1 (16-8x^2+x^4)\,dx - \pi\int_0^1 (4+4x+x^2)\,dx \\
&= \pi(16x-(8/3)x^3+(1/5)x^5)|_0^1 - \pi(4x+2x^2+(1/3)x^3)|_0^1 \\
&= \pi(16-(8/3)+(1/5)) - \pi(4+2+(1/3)) \\
&= \pi(7+(1/5)) \approx 22.619\ \textit{cubic units}.
\end{aligned}$$

7. (20pts) At distance x from the end of the rod the total mass in a small section of length Δx is $e^{-2x}\Delta x$. Adding these terms to form a Riemann sum and passing to the limit we have the mass of the rod is

$$\int_0^L e^{-2x}\,dx = -(1/2)e^{-2x}|_0^L = (1/2)(1-e^{-2L}).$$

Thus $1/3 = (1/2)(1-e^{-2L})$ which means $2/3 = 1-e^{-2L}$. Thus $e^{-2L} = 1/3$ or $e^{2L} = 3$. Therefore $L = (\ln 3)/2 \approx 0.549\,\textit{units}$.

8. (20pts) Let $p(t)$ denote the income stream and r the interest rate. The present value is

$$\begin{aligned}
\int_0^{10} p(t)e^{-rt}\,dt &= \int_0^5 p(t)e^{-rt}\,dt + \int_5^{10} p(t)e^{-rt}\,dt \\
&= \int_0^5 1000e^{-.05t}\,dt + \int_5^{10} 500e^{-.05t}\,dt \\
&= 1000(e^{-.05t}/(-.05))|_0^5 + 500(e^{-.05t}/(-.05))|_5^{10} \\
&= 1000(20)(1-e^{-.25}) + 500(20)(e^{-.25}-e^{-.50}) \\
&= 1000(20) - 500(20)(e^{-.25}+e^{-.50}) \approx 6,146.686\,\textit{dollars}.
\end{aligned}$$

9. (7pts) a) $p_3(t) = 1-t+t^2-t^3$. b) (8pts) $1/(1-x) = 1+x+x^2+x^3+x^4+\cdots$ so $1/(1-t^4) = 1+t^4+t^8+t^{12}+\cdots$. c) (10pts)

$$\begin{aligned}
\int_0^x \frac{1}{1-t^4}\,dt &= \int_0^x (1+t^4+t^8+t^{12}+\cdots\ )\,dt \\
&= (t+t^5/5+t^9/9+t^{13}/13+\cdots\ )|_0^x \\
&= x+x^5/5+x^9/9+x^{13}/13+\cdots
\end{aligned}$$

10. a) (5pts) $12(5/8)^n$. b) (10pts) $12+2(12)(5/8)+\cdots+2(12)(5/8)^{n-1} = -12+[2(12)+2(12)(5/8)+\cdots+2(12)(5/8)^{n-1}] = -12+[2(12)(1-(5/8)^n)/(1-(5/8))]$. c) (5pts) $-12+[2(12)(1)/(1-(5/8))] = -12+[2(12)(8/3)] = -12+64 = 52\,\textit{feet}$.