Math 181 Final, Spring 2004 - Hints, Answers, and Solutions

Q1 Find the following derivative: $\frac{d}{dx}\int_1^{x^2} arcsin(t)dt$. **Answer:** $2x\arcsin(x^2)$.
Hint: Apply the Second Fundamental Theorem of Calculus and the Chain Rule.
Q2 Evaluate the following integrals.

$$(\mathbf{a})\ \int_0^3 x\sqrt{x^2+16}\,dx \qquad (\mathbf{b})\ \int \frac{dy}{y^2+y-2} \qquad (\mathbf{c})\ \int \arctan(y)\,dy$$

$$(\mathbf{d})\ \int \frac{dx}{x^2+2x+2} \qquad (\mathbf{e})\ \int x^2\sin(3x)\,dx \qquad (\mathbf{f})\ \int_0^1 \frac{dx}{\sqrt{x}}$$

Hints: **(a)** by substitition, **(b)** by partial fractions, **(c)** by parts, **(d)** by substitution, complete the square, **(e)** by parts, twice, **(f)** an improper integral, convergent.
Q3 Show that the volume of a sphere of radius R is $\frac{4\pi R^3}{3}$.
Hint: Compute the volume of a solid of revolution: by rotating the graph $y(x)=\sqrt{R^2-x^2}$, $|x|\le R$, around the x axis.
Q4 A storage tank in the form of the right circular cylinder with height 20 ft and radius 6 ft is standing on the circular base. The tank is half full of a liquid which weights 105 lb/ft^3. Find the amount work done by pumping the liquid to the top of the tank.
Hint: Slice the volume of a liquid horizontally and compute the total pumping work which corresponds to lifting the slices to the top of the tank. $W=105\int_{10}^{20}\pi 6^2 h\,dh$ [ft-lb].
Q5 Determine if the following series converge or diverge.

$$(\mathbf{a})\ \sum_{n=1}^{\infty}\frac{1}{n^2+2} \qquad (\mathbf{b})\ \sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{2^n} \qquad (\mathbf{c})\ \sum_{n=2}^{\infty}\left(\frac{1}{n(\ln(n))^2}\right)$$

Answers:(a) convergent, Comp.Test, **(b)** divergent, Ratio Test, **(c)** convergent, Int. Test.
Q6 (a) Write down the Taylor series about $x=0$ for the following functions.

(1) $\cos(x)$ **(2)** e^x **(3)** $\ln(1+x)$

Answers: $\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.
(b) Write down the first four nonzero terms of the Taylor series about $x=0$ for the function $\frac{e^{2x}}{1-x}$. **Hint:** $e^{2x}=(1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{3!}+...)$, $\frac{1}{1-x}=(1+x+x^2+x^3+...)$, multiply and arrange. **Answer:** $\frac{e^{2x}}{1-x}=1+3x+5x^2+\frac{19}{3}x^3+...$.
(c) Find an expression for the general term in the Taylor series about $x=0$ for the function $\sqrt{1+x}$. **Hint:** $(1+x)^{1/2}$, a binomial series for $p=1/2$. **Answer:** in general $(1+x)^p=1+\sum_{n=1}^{\infty}p(p-1)(p-2)...(p-(n-1))\frac{x^n}{n!}$.
Q7 Suppose that $f(0)=4$, $f'(0)=-2$, $f''(0)=4$, $f^3(0)=6$, $f^4(0)=3$.
(a) Write down the Taylor polynomial of degree four for f about $x=0$.
(b) Use (a) to approximate $f(0.2)$.
Answers: $P_4(x)=4+(-2)x+4\frac{x^2}{2!}+6\frac{x^3}{3!}+3\frac{x^4}{4!}=4-2x+2x^2+x^3+\frac{x^4}{8}$, $f(0.2)=P_4(0.2)$.
Q8 Find **ALL** x for which the series $\sum_{n=1}^{n=\infty}\frac{4^n(x+5)^n}{(n)3^n}$ converges.
Solution: $a_n=\frac{1}{n}(\frac{4}{3})^n(x+5)^n$, $\frac{|a_{n+1}|}{|a_n|}=\left(\frac{n}{n+1}\right)\frac{4}{3}|x+5|$. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L=\frac{4}{3}|x+5|<1$ if $|x+5|<3/4$. By the Ratio Test the series is convergent for $|x+5|<3/4$, i.e., for $-3/4<x+5<3/4$, and divergent for $|x-1|>3/4$, the radius of convergence is 3/4. At the left endpoint, $x+5=-3/4$, $\sum_{n=1}^{\infty}\frac{1}{n}(\frac{4}{3})^n(x+5)^n=\sum_{n=1}^{\infty}\frac{1}{n}(4/3)^n(-3/4)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is convergent the Alternative Harmonic Series, convergent by the Alternative Series Test. At right endpoint, $x+5=3/4$, $\sum_{n=1}^{\infty}\frac{1}{n}(\frac{4}{3})^n(x+5)^n=\sum_{n=1}^{\infty}\frac{1}{n}(4/3)^n(3/4)^n=\sum_{n=1}^{\infty}\frac{1}{n}$ is the Harmonic Series, divergent by the Integral Test, (p-test, $p\le 1$). The interval of convergence is $-3/4\le x+5<3/4$, i.e., is equal $-23/4\le x<-17/4$.