Math 210 Calculus II **December 12, 1997**

Final Exam

1. A particle moves through space with position function $\mathbf{r}(t) = \langle \cos(\pi t^2), \sin(\pi t^2), 3t \rangle$. Find the velocity, acceleration and speed of the particle.

2. Find an equation for the tangent plane to the graph of the function $f(x, y) = xy - x^3$ at the point where $(x, y) = (2, 5)$.

3. Suppose that $f(x, y)$ is a differentiable function and that $h(s, t)$ is the composite function $h(s, t) = f(s^2 + t^2, s^2t^2)$. Suppose that the first partial derivatives of f satisfy

$$\frac{\partial f}{\partial x}(13, 36) = \frac{\partial f}{\partial y}(13, 36) = 3.$$

Compute $\dfrac{\partial h}{\partial s}(2, 3)$ and $\dfrac{\partial h}{\partial t}(2, 3)$.

4. A robot equipped with a recording thermometer is exploring a new planet. Assume that the region being explored has a coordinate system in which the unit of distance is 1 meter. Assume that the temperature in Celsius degrees at the point with coordinates (x, y) is $T(x, y) = \dfrac{xy}{2x^2 + y^2}$.

(a) Find the directional derivative of T at the point $(0, 3)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Include the units in your answer.

(b) Suppose that the robot is located at the point $(0, 3)$. In which direction should it move in order to make the temperature increase most rapidly.

(c) Suppose that the robot is moving on an elliptical path with position function $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$. At the moment when the robot passes the point $(0, 3)$ what is the rate of change (in degrees/sec) of the temperature measured by its thermometer?

5. Find the critical points of the function $f(x, y) = x^3 - 3xy + y^3$ and determine which are maxima, minima or saddles.

6. Find the maximum value of the function $f(x, y, z) = 2x - y + z$ on the ellipsoid defined by the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Express your answer in terms of of a, b, and c.

7. Find the volume of the bounded region lying above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 9$.

8. Find the surface area of the part of the paraboloid $z = 1 - x^2 - y^2$ which lies above the plane $z = -3$.

9. Suppose that the vector field $\mathbf{F} = x^2\,\mathbf{i} + yx\,\mathbf{j}$ represents a force field in the plane.

   (a) Compute the work done against $\mathbf{F}$ by moving a unit mass in the counter-clockwise direction around the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$.

   (b) Is the vector field $\mathbf{F}$ conservative? Why or why not?

10. One of the two vector fields below is the gradient of a function $h(x,y)$, and the other cannot be the gradient of a function.

$$\mathbf{F} = (x^2 + y)\,\mathbf{i} + x^2\,\mathbf{j} \qquad \mathbf{G} = 3x^2y^4\,\mathbf{i} + (1 + 4x^3y^3)\,\mathbf{j}$$

   (a) Use partial derivatives to determine which of the fields cannot be a gradient.

   (b) Find a function whose gradient is the other field.