Newsletter

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Connecting Mathematics Education Research and Classroom Practice

The four articles by mathematics educators in this issue are an outgrowth of the panel discussion of the same name at the special session on Mathematics and Education Reform at the 2002 January Joint Mathematics Meeting in San Diego. In turn, each author— Deborah Loewenberg Ball, Rina Zazkis, Marilyn P. Carlson, and Chris L. Rasmussen— reflects on the dynamical interplay of mathematics education research and teaching mathematics within his or her own classroom. Individual case studies, self-reflections— whatever you choose to call them— these analyses reveal surprising complexities and richness of experience in the classroom .

Knowing Mathematics for Teaching: Relations between Research and Practice

by Deborah Loewenberg Ball, University of Michigan

This special MER panel probed a fundamental question, “How can research in mathematics education relate to practice?” One way to answer this question is to consider how they might be related, to envision possibilities and promise. Another, the one taken by members of this panel, was to examine specific cases in their own work where research and practice intermingle in generative ways. As a case of an area in which disciplined inquiry and practice might inform one another, research and practice on issues of mathematics teacher knowledge offers a useful site to probe how those connections might work productively. To provide a glimpse of these, I trace here my own journey— in the context of the explorations of many others— in pursuit of questions about the mathematics knowledge for teaching. Having begun my work on this problem as a classroom teacher, practice has often been the starting point for me in my work, the site for genesis of problems and ideas. It has also been the site where, in early use, ideas are tested, refined, and sometimes abandoned.

I can trace my first questions about the mathematics knowledge needed for teaching to early in my elementary school teaching career. After about five years in the classroom, I grew aware that my teaching of mathematics was lacking. Although I tried to make the mathematical ideas make sense to my fifth graders, I was frustrated by the fact that they seemed to forget what we worked on as

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MER Forum, Department of Mathematics, Statistics, and Computer Science, The University of Illinois at Chicago,
851 S. Morgan, Chicago, IL 60607

fast as I could teach it. Without any clear sense of what was missing, I thought it possible that my own knowledge of mathematics might be a factor. Unlike other areas of the elementary curriculum, mathematics was an area in which I lacked depth and experience. I liked mathematics and felt myself reasonably competent, but I had the sense that I lacked the sort of perspective and connected knowledge that enabled me to move flexibly in other subjects. I had gone through high school at a time when students were allowed to make many decisions about what to take, and I had consequently opted to take many courses in French, Spanish, and German, as well as English and humanities, and only a little mathematics. I suspected that studying some mathematics might help me.

Unwittingly, I was entering as a young teacher into an arena that had already occupied many others. Many had worked hard to develop courses for teachers, courses that were successful within themselves, but disappointing with respect to what teachers were able to *do* with mathematics in their own practice. Edward Begle (1979) had conducted analyses across studies of teacher effects, attempting to identify the trends in how teachers' mathematics study impacted their effectiveness with students. Begle's analysis produced a surprise: Advanced mathematics course-taking produced positive main effects on students' achievement in only 10% of the cases, and— more startling— negative main effects in 8%.* Begle concluded from his analyses that the belief that "the more a teacher knows about his subject matter, the more effective he will be as a teacher" demanded "drastic modification" (Begle, 1979, p. 51). He noted that, although "it seems to be taken for granted that it is important for a teacher to have a thorough understanding of the subject matter being taught," never in these studies was the question of what should be meant by "thorough" examined (p. 28).

Not yet knowing of these startling results, I decided to take some courses at the university. Because I had never taken any mathematics courses in college other than an independent study that substituted for the standard mathematics course for elementary teachers (I placed out of this course by virtue of scoring very high on a placement exam!), I launched in with a review course in algebra, and then continued with four terms of calculus and number theory. I was teaching fifth grade, and then first grade during this time. Although none of these courses dealt with the mathematics of the primary grades, I noticed that I was learning things that affected how I worked on mathematics with my students. For example, one day in first grade we were using graph paper to measure shapes to provide some early experience with the concept of area measure. Some shapes were regular, some irregular. The book suggested asking the children to trace their hands, and to come up with a measurement for the area. When some children, dissatisfied with the inexactness of the one-inch grid graph paper we were using, asked to go get some of the finer-grid graph paper that the fifth graders used, I heard this as a significant question. I realized that their interest in using the smaller grid graph paper represented an early sensibility for precision and an instinct about "getting closer" without being exact, a notion that would later underlie an appreciation of limits, and specifically the approach to area in integral calculus. Had I not recently studied calculus, I might have heard their desire to get the finer graph paper as unimportant. Put more accurately, I would not really have "heard" it. I surely did not have "thorough" knowledge of mathematics. But I was also clearly learning something useful to my work with children.

Over the ensuing years, I continued to teach elementary school and to take university mathematics courses. I enjoyed them for the most part. A significant course was one on number theory, taught by Joe Adney, then the chair of the Michigan State mathematics department. The course was different from anything I had taken up to that point, because for the first time we were being expected to prove claims, not just engage as spectators. My induction into the world of conjectures and proofs, of lemmas and theorems, fascinated me, and sparked my imagination as a teacher. I suddenly saw the connections to ways in which my students pursued questions and investigations in science, developed interpretations of poems and books, and examined artifacts in social studies. I also saw ways in which proof in mathematics was unlike producing conviction in these other subjects. For example, I saw that examples did not constitute sufficient support for an argument.

I began to draw my students out a little more. When they noticed patterns, or had tentative ideas, I asked them to explain their reasoning, and to give evidence for their assertions. I asked students to respond to other's ideas, as I had seen Professor Adney do with us. The more I pulled them into reasoning, the better they seemed to understand the ideas we were working on. They also surprised me more and more often by noticing things I would not have expected elementary students to think about. For example, they wondered whether 0 was even or not, and they developed a method of subtracting multi-digit numbers that seemed more efficient than the one we knew

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*A positive main effect would mean that more credits of mathematics were associated with greater student performance; a negative main effect meant that more credits of mathematics were associated with lower levels of achievement.

and had always taught.

Teaching demanded a kind of mathematical flexibility and appreciation, as well as knowledge and skill. I wanted to learn more, but was finding it difficult to identify precisely what I needed to learn as well as where I could learn it. I did see that I was learning a great deal from listening closely to my students, and to seeing what they did with interesting mathematics tasks. I tried my hand at making other tasks, and was often rewarded by my own learning as I explored a set of ideas and developed problems for the children to do.

I also realized, however, that my capacity plummeted when I myself was unclear about the mathematics, or when I had mistaken ideas. My ability to steer such tasks well, to hear what the students were thinking, and bring a discussion to a clear forward motion, was slowed. I could see that my improvement as a mathematics teacher was interactive with my mathematical knowledge, but I was unsure about how to describe to others the sort of mathematical knowledge that seemed to help me navigate the complexities of my work.

To explore the nature of this mathematics knowledge, I designed some questions that involved mathematical insight and understanding. And I used these to interview prospective teachers. I wanted to see what beginning teachers– including some who had already studied a lot of college-level mathematics and others who had not– brought with them to their formal preparation for teaching mathematics. In one question, I posed a common error that upper elementary students make when they multiply multi-digit numbers:

> Suppose you are trying to help some of your students learn to multiply large numbers. You notice that when they try to calculate
>
> $$\begin{array}{r} 123 \\ \underline{\text{x } 645} \end{array}$$
>
> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line.
> They are doing this instead:
>
> $$\begin{array}{r} 123 \\ \underline{\text{x } 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$

I was astonished that only about one out of four prospective teachers talked explicitly about place value when they explained the difficulty that the children might be having, or how they might try to "clear up" the problem. For example, Rachel said:

"You would take the last number and multiply it by all three of the top numbers and you put those underneath and then you start with the next one. You'd want to put it underneath the number that you are using. They aren't understanding that they need to be underneath of that instead of just down in one straight row."

And Pam said she would show pupils to "*physically* put a zero every time you moved down a line." She explained that "zero doesn't add anything more to the problem. It's just empty. But instead of having an empty space, you have something to fill in the space so that you can use it as a guideline."

And there were other questions. One of my questions, well known by now, was to ask prospective teachers to calculate 1 3/4 ÷ 1/2 and then to write a story problem that matched the arithmetic calculation. Again, most were able to compute the correct answer, but to produce an appropriate story proved difficult for a majority of the students.

That such questions were challenging for prospective teachers was important. But that they were challenging even for those prospective teachers who had already almost completed majors in mathematics was even more significant. These results helped to provide a clue to the decades-old mystery of why achievement is not positively correlated with higher levels of mathematics study. As I had often wondered, what sort of mathematical knowledge is required in teaching? And how is it related to the mathematical knowledge developed in particular courses? I had myself found the mathematics I learned in calculus, number theory, and probability useful to me as an elementary school teacher. But how had that knowledge been transformed? To what uses was it put to make it usable for teaching? The unprepared reactions of the mathematics majors I interviewed suggested that the connection between advanced mathematics study and knowing mathematics for teaching was not yet well understood. As an elementary teacher who had come to these issues along a different path, I found these insights fascinating, if challenging.

As a research community began to tackle the thorny question of mathematical knowledge for teaching, simple formulations began to recede. No longer did scholars attempt to describe the mathematics that teachers needed to know in terms of the number of courses they should take, and although policymakers yearned for a degree or coursework specification, most realized that the question was more complicated.

In the broader community of researchers on teaching, Lee Shulman and his colleagues introduced the notion of "pedagogical content knowledge" (1986, 1987a, 1987b) to capture the special ways in which teachers need to know subject matter. This knowledge, they claimed, consisted of the important ideas and procedures of a field, but also the aspects most difficult to learn, and the range of representations useful for making the ideas accessible. Their work, conducted in English, physics, biology, history, and mathematics, provided helpful insights into the nature of the subject matter knowledge needed for the work of teaching.

Work on this problem proceeded along a number of avenues. Liping Ma used the same questions that had proved so useful in probing U.S. teachers' mathematical knowledge, and posed them to practicing Chinese elementary teachers (Ma, 1999). Significantly, her results were quite different. Ma used her data to develop a notion of "profound understanding of fundamental mathematics," an argument for a kind of connected, curricularly structured, and longitudinally coherent knowledge of core mathematical ideas. Her notion of "knowledge packages" offered a conceptual structure that could overcome the tendency to make endless disconnected lists of what teachers need to know.

In my own courses for prospective teachers, I sought to select mathematical content that would afford them leverage over the recurrent mathematical issues they would face as teachers. In my teaching of elementary children, I turned increasingly to focus on uncovering these issues: What mathematical questions kept coming up for me as a teacher? What was the nature of the mathematical issues that I confronted? How was the work of teaching third graders mathematical, and not merely, as many assume, pedagogical and rooted in questions of cognitive development?

I noticed that there were several such recurrent problems: One is selecting, adapting, and using representations for mathematical ideas (e.g., Ball, 1993). For example, in teaching arithmetic with integers to my third graders— a topic in my school district's curriculum— I explored several alternative models for negative numbers. I examined the possibilities with money and debt, number lines, checkers that could be used to denote positive and negative values, and a building with many floors above and below the ground floor labeled 0. Each highlighted some key aspects of the mathematics; each offered resources for learning; and each held pitfalls to possible confusions. The problem of selecting and using representations was deeply mathematical, and required the unpacking and close analysis of the mathematical ideas and their relationships to other ideas yet to come in the children's mathematical experience. Another recurrent difficulty is trying to make a judgment about whether an idea is mathematically significant and worth taking up. Still another is sizing up the validity of a child's non-standard procedure: Will it work in all cases, for all numbers? Students are continuously trying their own approaches, inventing their own notation, and trying out their own methods. Not all of these will hold up over time, and some are more good fortune than mathematics. Yet some are potentially robust, and merit analysis, testing, and development. I have several times had students develop methods of subtraction that by using negative numbers avoided the "borrowing" procedure common to standard U.S. curriculum:

$$\begin{array}{r} 3\ 6 \\ -\ 1\ 9 \\ \hline 2\ {-3} \end{array} \longrightarrow \quad 17$$

This algorithm employed here permits the user to work separately on each column, using integers to keep track of the parts of the calculation. Is this method viable with seven-, twenty-, or one hundred-digit numbers? Teaching requires teachers to be able to understand and appraise such a procedure, and to make a decision about both its significance and generality, as well as what to do with it in class.

Close scrutiny reveals the myriad tasks of teaching that are permeated with mathematical considerations, that require mathematical knowledge and sensibility. While studies of what teachers know (and do not know) are useful for exploring the contours of the problems we face in professional education and in the development of curriculum, such studies do not help in probing the mathematical nature and demands of teaching. They do not help advance our collective understanding of what it is that teachers need to know and how they need to use such knowledge in their work.

My collaboration with Hyman Bass has afforded a means to probe the mathematical work of teaching. With an eye that is intensely mathematical, when he watches a class of children at work, he perceives the emergent ideas in their insights and efforts and sees the horizons to which they are headed. Often with a sense of surprise and fascination, he suddenly realizes— and appreciates— the thickets of difficulty they and their teachers meet, even in what many consider to be "elementary" content (Ball, 1999). In our joint work, we have asked the question, what is the mathematical terrain of elementary teaching and learning? What are the mathematical demands of the work of teaching? Where and how do mathematical considerations arise, and what is the mathematics that teachers could profitably use to manage these tasks? (Ball and Bass, 2000)

The work on mathematics knowledge for teaching has come full circle. Originating in questions about how different "amounts" of knowledge affected teachers' effectiveness; researchers soon encountered the surprise that answering this question was far from straightforward. The number of courses a teacher had taken was an inadequate proxy for the mathematical knowledge required for teaching. Yet researchers also began to see up close what many teachers had long felt— that teaching is mathematically demanding work, and that many teachers' mathematical education left them unprepared for its challenge.

While research must continue to contribute to our understanding of what it is that teaching requires from a mathematical perspective, the mathematical education of teachers must continue to develop and test alternative ways to prepare teachers for those mathematical tasks and challenges. The recently released *Mathematical Education of Teachers*, a product of the collaborative work of mathematics educators, mathematicians, and teachers, is an example of this much-needed work (CBMS, 2001). As mathematicians and mathematics educators explore what it takes to be mathematically prepared for teaching mathematics, our understanding of the problem and its solutions will be improved. Working only on the basis of conviction will, however, leave us no better off. Understanding the nature of the mathematical knowledge needed for teaching, and how it can be developed and acquired in usable and useful ways, requires disciplined inquiry. How do different kinds of mathematical experiences affect what teachers know and can do in their classrooms? How do different kinds of mathematical understanding impact their effectiveness? What are the *mathematical* problems they must solve on a daily basis, and how does their mathematical education prepare them for those? The bottom line to this question is the quality and effectiveness of *teaching*, not the improvement of *teachers*. Our favorite notions about teacher preparation require testing and improvement. And when we least expect it, answers we could never have predicted will surprise us if we design ways to intertwine practice and research in the pursuit of these problems.

Ackowledgements
My work on this set of issues owes a debt to many other colleagues, among them especially Hyman Bass, Suzanne Wilson, Magdalene Lampert, and David Cohen.

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Do we know the difficulties of our students? Connecting research and teaching practice

by Rina Zazkis, Simon Fraser University

My short answer to the question in the title, *Do we know the difficulties of our students?* is *No*. A longer answer would be, *No, but we're learning*. The rest of this article is an elaboration on what we could learn from mathematics education research and how it may influence our teaching practice.

What can, or can't, mathematics education research do?

Alan Schoenfeld (2000) stated two main purposes of research in mathematics education:

(a) Pure (Basic Science): To understand the nature of mathematical thinking, teaching and learning.

(b) Applied (Engineering): To use such understandings to improve mathematics instruction.

Mathematicians are often interested to know:

- What works?
- Which approach is better?
- What method works best?

According to Schoenfeld, mathematics education research cannot answer these questions, since what "works" and what "works better" depends on the instructor's values and expectations. However, among the fundamental contributions of research in mathematics education Schoenfeld lists the following:

- theoretical perspectives for understanding thinking, learning and teaching;

- descriptions of aspects of cognition (e.g. thinking mathematically; students' understandings and misunderstandings of concepts);

- existence proofs (evidence of cases in which students can learn problem solving, induction, group theory; evidence of the viability of various kinds of instruction);

- descriptions of (positive and negative) consequences of various forms of instruction.

"The central problem of education is not so much the description and categorization of the processes of development of knowledge, as the intervention into these processes," (Sierpinska, 1994, p.121). I believe that such intervention is more successful if based on the understanding of what students' knowledge is, what is it lacking, what is perceived as difficult. Experienced teachers often believe that they have a "feel" for students' difficulties. However, as the example in the following paragraph shows, those difficulties may appear in most unexpected places.

In my first year calculus course at the University of Haifa I came across something called the *difficulty theorem*. Copying notes from a classmate, I asked her why there was the label *difficulty* on the theorem – the theorem didn't seem to me more difficult than other encountered in the course. The only help the classmate could provide was "that's what was written on the board." (Note: It is not financially beneficial to translate textbooks to Hebrew, therefore the main source of information for undergraduate students in Israel is lecture notes copied in haste from the board). I never dared to ask anyone else, assuming that I hadn't learned yet to appreciate the difficulty presented by the theorem. It was two years later, taking the History of Mathematics course, that I understood that my *difficulty theorem* was actually a reference to a theorem of Cauchy. The Hebrew word k-o-sh-i, meaning difficulty, and the transliteration of the name Cauchy into Hebrew characters give identical Hebrew spellings. Since proper names are not capitalized in Hebrew, *difficulty* and *Cauchy* are distinguishable only by context.

Research to practice: an example with a caution

In what follows I provide examples of research findings and describe how they influenced my teaching and the teaching of some of my colleagues. During the past 10 years, the main content theme of my educational research has been elementary number theory. I studied the understanding of basic number theory concepts (such as divisibility, factors and multiples, and prime and composite numbers) by undergraduate students, mostly preservice elementary school teachers. The main means of data collection has been through a semi-structured clinical interview. "Semi-structured" means that the interview questions are prepared in advance, but the interviewer exercises some flexibility in prompting, based on participant's responses.

Consider the following interview excerpts (previously reported in Zazkis and Campbell, 1996). Bob and Patty are presented with the number M=$3^3 \bullet 5^2 \bullet 7$ and asked to determine divisibility of M by different numbers, such as 7, 5, 2, 11, 15, 63, 81, etc.

Interview with Bob:

Interviewer: *Bob, I'm going to ask you to write down a number please. And that number is* $3^3 \bullet 5^2 \bullet 7$*, and we're going to call this number M. Now, my first question is, is M divisible by 7?*

Bob: Yes, it is.

Interviewer: *And would you explain why?*

Bob: Well if 7 (pause), let's see (laugh), M is, or let's see, so 7 is a factor of M, therefore, it's divisible by M, pardon me, by 7.

Interviewer: *And how about 5?*

Bob: 5 is also a factor of M.

Interviewer: *Okay, and would M be divisible by 2?*

Bob: No, it would not, since 2 is um, (pause) since 2 is not seen here, it's not a factor of M.

Interviewer: *Hmm, okay, and why do you feel that that's the case?*

Bob: Um, explain this clearly (pause), since 2 is not one of the numbers that's being multiplied, the product therefore, can't be divided by 2.

[...]

Interviewer: *Would you think that M is divisible by 81?*

Bob: I'd want to find out what M would be, I guess that's the, the best thing, that's what I'd prefer.

Interviewer: *Um hm.*

Bob: I guess knowing what M would equal, and then from there working backwards, finding which numbers can go into that.

Interviewer: *Um hm.*

Bob: Um, right now I can't see whether or not 81 can go in there.

Interviewer: *Okay. Uh, how about 63?*

Bob: (pause) Once again, um, we have 7 now, 7 can go into 63, well ?? 3 can as well (pause), once again I'd have to solve for M, in order to find out whether 63 can divide M.

Interviewer: *Okay, so when you say solve for M, you mean like multiply it out and then divide by 63?*

Bob: Yeah, exactly, exactly.

Interviewer: *Okay. How about 15?*

Bob: (pause) Um, well since there's 5, 5^2 in this problem, we know that the, that the units digit will be 5, now 15 obviously has a 5 in it as well, therefore quite possibly 15 will go into M, and once again I'd have to solve for that.

Partway into the interview with Patty:

[...]

Interviewer: *Okay. And will it be divisible by 2?*

Patty: I would multiply each one and find out what the total number is. So 3 x 3 is 9 x 3 is 27, and this 25 is x 7, (pause) it's not, 2 doesn't go into it evenly.

Interviewer: *So you computed the number and you got 4,725, and now you are sure that it is not divisible by 2.*

Patty: Right.

Interviewer: *But you were able to conclude about divisibility by 7 , before you knew what was the number.*

Patty: Um hm.

Interviewer: *So how is it?*

Patty: Because 7 is a factor of it, so it's, what is it, the commutative law or associate law, 7 is a factor of it.

Interviewer: *And what about divisibility of M by 11?*

Patty: I would divide 4725 by 11 to find out.

Both Bob and Patty are able to conclude divisibility of M by 7 attending to the prime decomposition. However, when asked about divisibility by primes that are not part of M's prime decomposition, Patty's preferred approach is to "find out what the total number is," and then perform division. Bob makes a clear distinction between prime factors of M and prime numbers that are not factors of M - those are "not seen here," that is, are not listed in prime decomposition. However, when asked about composite numbers, Bob's approach regresses to calculation.

Interviews with other participants further support the claim that divisibility and indivisibility is treated differently by students. This may seem surprising at first, since claims of divisibility and indivisibility have a similar structure. Consider the following two claims:

(a) 7 is a prime factor of M, therefore M is divisible by 7.

(b) 11 is not a prime factor of M, therefore M is not divisible by 11.

Why is claim (a) perceived as easy and intuitive, while claim (b) is seen as presenting a difficulty? The analysis of the data suggests a possible answer: Claim (b) takes into account the Fundamental Theorem of Arithmetic, assuring the existence and uniqueness of prime decomposition. While existence is taken for granted, uniqueness, for many students, presents a challenge. An alternative prime decomposition is often assumed to exist, either explicitly or implicitly.

Following the findings briefly described above, my instruction focused on attending to the prime decomposition of a number and the prime decomposition of its factors. A strategy of "big numbers" (Zazkis, 2001) appeared to be helpful. "Big numbers" are those that are beyond the computational abilities of a hand-held calculator. Working with "big numbers" calls on students to attend to the structure of the number, eliminating the possibility of the "safe" fall back to calculation. The following conversation with Jenny exemplifies this approach. Starting with the above mentioned number M, Jenny calculated M and then performed division in order to decide whether it was divisible by 7 and 15. She considered pencil and paper calculations when prompted about the possibility of dealing with the task without the calculator. She was then invited to consider a big number.

Interviewer: *Could you draw these conclusions without the calculator?*

Jenny I guess, it will just take me forever to do the division.

Interviewer: *Please consider another number, let's call it B, where* $B = 3^{30} \cdot 5^{20} \cdot 7$. *Is B divisible by 15?*

[Jenny attempted to calculate B.]

Jenny: The calculator isn't much help here.

Interviewer: *And why's that?*

Jenny: It gives all kinds of digits that don't help here.

Interviewer: *Do you think it is possible to draw a conclusion without the calculator?*

Jenny: I could spend a day to make this number [pause] Wait. I think 15 will go into it.

Interviewer: *And why is that?*

Jenny: Because 15 is 5 times 3 and we have 3 here and 5 here, so when put together, yeah, when put together, 15 is there, in this number.

Interviewer: *And how about 63?*

Jenny: [pause] Yes, it's also there. It's made of two 3's and a 7, and we have here a 7 and 20 3's. So it goes, I mean it's divisible.

Interviewer: *Let's go back for a moment to a number we considered before,* $M = 3^3 \cdot 5^2 \cdot 7$. *Is M divisible by 63 ?*

Jenny: Yes, you can make exactly the same claim.

Interviewer: *What's exactly the same?*

Jenny: Like 63 is made of 9 times 7, you have 7 here, you don't have 9 here, but 9 is made of two 3's and you have more than two 3's here, so that's why it's the same, I mean 63 can go into this M.

Though Jenny is considering specific numbers, her reasoning can be seen as algebraic. It is about form, structure and relationships. Jenny's starting point is in arithmetic calculation. The instructor's goal is to help Jenny understand that $p^x \cdot q^y \cdot t^z$ is divisible by $p^a \cdot q^b \cdot t^c$ if and only if $x \geq a$, $y \geq b$ and $z \geq c$. In fact, Jenny is rather close to this generalization. She definitely recognizes the factors appearing in prime decomposition and by saying "9 is made of two 3's and you have more than two 3's here" she is clearly considering the exponents of 3. A handful of additional investigations may be necessary before the general conclusion can be drawn and expressed in "proper" mathematical terms, but Jenny is on the right track.

Exposure to "big numbers" helped students draw inferences on divisibility and indivisibility considering the prime decomposition. They applied the criteria correctly and appeared confident in their judgment. Celebration? Not yet. In my most recent study I implemented a small variation. I asked students to consider divisibility of K, where $K = 3^3 \cdot 5^2 \cdot 7 + 11$, (yes this is "plus"), by 7. About half of the participants claimed that K was divisible by 7 since 7 appeared in the prime decomposition. So seemingly successful instructional practice requires an amendment. Emphasis on *prime* factors needs to be supplemented with an emphasis on prime *factors*.

Patience, please

Research in undergraduate mathematics education is a new and rapidly developing field of study. There is some criticism being voiced recently, in the mathematics education community, of research that deals with categorization of errors or misconceptions or presents single case studies. There is a call for more integrated longitudinal research that arises from questions of practice. What is not mentioned is that this advocated integrated research stands on the shoulders of studies that are being criticized. Researchers in mathematics education have probably itemized and classified most of what can be said about learning one or two-step word problems, multi-digit addition, and subtraction of fractions. We do know something, but significantly less, about students' understanding of and difficulties with functions, limits, and groups. But mathematics education research has said very little so far about understanding of compactness, convergence, or transformations, to pick just a few examples. Therefore, I argue for a bit of patience. Research on learning undergraduate mathematics needs time to catch up. And studies on students' misconceptions-- that appear as yesterday's fashion in elementary or secondary mathematics education-- may be a necessary stepping-stone to research in undergraduate mathematics education. In order to consider advances in instructional practices we must strive to understand better the nature of students' difficulties.

But there is good news. The set of researchers in elementary mathematics education and the set of teachers of elementary school mathematics are almost disjoint. (Deborah Ball may be one of the very few examples of someone in the intersection of these two sets.) This causes teaching practitioners to view advice from researchers with a degree of caution or even distrust. What makes me feel very optimistic about connecting research and teaching practice *in undergraduate mathematics education* is the fact that researchers in undergraduate mathematics education form a subset of teachers of undergraduate mathematics. And this subset is constantly growing. This phenomenon influences the types of problems researchers focus on and connects research to teaching practice, at least for some of us, firmly and immediately. It is *our* practice too.

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Connecting Research to Practice at the Undergraduate Level: A Personal Journey

by Marilyn P. Carlson, Arizona State University

Many have begun to discuss the role of *research in undergraduate mathematics education* in practice. Although the community of researchers in undergraduate mathematics education is relatively young, a fairly large body of research exists to inform us about the process of learning concepts of algebra, precalculus and beginning calculus (e.g., RCME volumes, MAA Notes). Even though this knowledge exists, very few post-secondary instructors and faculty would profess awareness of this knowledge, and even fewer would claim to use this knowledge to inform aspects of their instruction. It is reasonable, then, to ask why this gap exists. Inherently, the mathematics education researcher professes a strong belief in the value of educational research and its potential for informing curriculum and instruction. It is also likely that most mathematics education researchers would claim that educational research plays a fairly central role in informing their teaching and instructional decisions. In my own experience, I cannot imagine separating my research from my practice. Perhaps it is time to explore new mechanisms for assisting teachers in acquiring an appreciation of and curiosity for the processes of knowing, learning and doing mathematics. Perhaps it is also time to address concerns about the practicality and applicability of formal research knowledge by exploring specific ways to move it to a more central role in guiding instructional and curricular decisions.

If one assumes that the problem of disseminating research knowledge to practitioners can be remedied, a subsequent question might be whether mere exposure to research knowledge is enough. Will this exposure produce a substantive and lasting effect on an individual's instructional delivery and curricular decisions? In reflecting on my own experience as a researcher, I have found that coming to understand and effectively utilize research knowledge has been and continues to be a slow and tedious process. It may be much like the process of acquiring an understanding of various pieces of a complex mathematical idea and learning to use it in a meaningful way.

My Personal Journey

The central strand of my research has involved investigations of students' understanding of the concept of function (Carlson, 1998; Carlson, Larsen and Jacobs, 2001; Carlson, Jacobs, Coe, Larsen & Hsu, in press). Since a large body of research existed prior to conducting my investigations, much of my initial knowledge was acquired by reviewing the function literature. Monk (1992) reported that students exhibit difficulty interpreting graphical information for intervals of a function's domain and suggested that instruction attempt to move students from a pointwise to an across time view of functions. Breidenbach, Dubinsky, Hawks and Nichols (1992) advocated that students' success in working with and using functions was influenced by whether the student conceives of a function as a process that receives input and produces output. They claimed that students who view a function as a sequence of actions to be performed may lack the conceptual tools to work with and use functions in a meaningful way. Kaput (1992) has called for more emphasis on promoting students' ability to see and understand patterns in data, while Thompson (1994) has called for a covariation approach to teaching functions. Many other researchers have reported common student misconceptions and conceptual obstacles (e.g., Sierpinska, Leinhardt). This illustration, although far from complete, suggests that new and somewhat disjoint information often results in more questions than answers. Although my initial readings of the literature did serve to increase my knowledge of the complexities and processes of working with and using functions, it was only the beginning of a long journey. This knowledge started me down a path to better understand the thinking and reasoning of my students and the mathematics that I wanted my students to learn.

In my initial reading of the literature, I found the diverse perspectives and theoretical positions to be confusing. In addition, I was not sure how I could use this knowledge to improve my teaching. As I recall, I thought that if only I could bring coherence to the knowledge that existed I might then be able to move forward to assure that my students acquired the essential understandings they needed. After numerous re-readings of the literature, I embarked on a new journey to make research-based changes in my instruction. I ventured down a path to drive my instructional decisions by more than just my informal reflections and conversations with my colleagues.

As I moved forward with both my research project and my attempts to improve my instruction, I worked to acquire a clearer understanding of what was involved in acquiring a well-connected understanding of the function concept. Drawing from the research literature and my own experiences I developed a description of what I believed were the major components involved in understanding the concept of function (Carlson, 1998). Even though this exercise was the first step toward developing a framework for my study, I have come to believe that it was a crucial step in shifting my instruction to a more conceptual orientation. It was the first time that I had made an attempt to contemplate, in a serious way, the complexity and process of understanding, what is perhaps the most important concept in the undergraduate mathematics curriculum. I came to appreciate that it was not trivial to reflect on an idea from the perspective of the student. It was even less trivial to figure out what experiences students needed to assure that they emerged from my classes with foundational understandings needed to move forward in their mathematical development.

Concurrent with my efforts to alter my instruction, I completed my first formal investigation of undergraduate students' function knowledge (Carlson, 1998). The results of this study conveyed new information about the function knowledge of three populations of high-performing undergraduate students, with follow-up investigations exploring new questions that emerged from each study (Carlson et al., 2001; Carlson et al., in press).

It is also noteworthy that the research process itself has influenced the nature of my instruction and curricular activities. The creation of thought-revealing tasks, an aspect of my research design, has become common in my curricular activities and assessments. Designing and using these tasks has provided me new information about my students' emerging understandings, something I had not previously attempted to assess. (I had been primarily interested in covering the material and making sure my students could work a large collection of difficult problems. I also paid little attention to whether the many aspects of understanding the function concept were becoming connected and coherent in the minds of my students.) Conducting clinical interviews, one mode of collecting my research data, has also helped me to listen more closely to the verbalizations of my students, and to more carefully probe the understandings and reasoning that support their constructions. These newly acquired habits of interacting with my students have resulted in substantive improvements in the quality of my exchanges with them, both during direct instruction and through the homework that I assign. The knowledge that I have acquired by engaging in the research process has equipped me to access information about students' thinking and understandings that I previously did not consider as important.

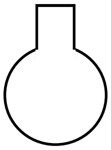
Collectively, my readings of the function literature and subsequent research studies have helped me to realize that acquiring an understanding of the concept of function is much more complex than I had previously imagined. Difficult mathematical concepts appear to develop over a period of years, rather than during brief moments in a particular course, as one might assume from reviewing some textbooks and curricula. This suggests that teachers and curriculum developers should make explicit efforts to develop essential and complex understandings by providing longitudinally coherent curriculum and instruction to realize these goals.

If one sets a goal to prepare students for success in calculus, it would be helpful to know that several research studies have found that students' 'ways of thinking' about the concept of function are important. It has been widely reported in the literature that a view of a function as an entity that receives input and produces output is foundational. This view results in subtle but important differences in the ways that students think about and solve specific problems. When students are asked to solve, say, $f(x) = g(x)$ it is desirable for students to view the task as that of determining the input values for which the output values (height of the graphs) are equal, in addition to learning the procedures for performing the computations. Even though these suggestions may seem minor to the reader, these differences in reasoning have been shown to be critical in the developing understandings of a student.

A 'function machine' conception has been shown to be foundational for developing a coordinated image of two variables changing in tandem (Thompson, 1994; Carlson, 1998). The cognitive activities of coordinating two varying quantities while attending to the ways in which they change in relation to each other (i.e., termed in the literature as *covariational reasoning*) (Carlson et al., 2001) is a 'way of thinking' that allows one to reason about the changing behavior of a dynamic event. It has been found that covariational reasoning is needed to think about and understand the concepts of limit (Cottrill, Dubinsky, Nicols, Schwingendorf, Thomas and Vidakovic, 1996), derivative (Zandieh, 2000), and accumulation (Thompson, 1994), as well as major concepts of differential equations (Rasmussen, 2000). A student who begins calculus with a computational approach to functions will likely not possess the conceptual tools to understand the major ideas of the course.

If a student is given the Bottle Problem below, a meaningful construction may involve the student imagining the volume changing while concurrently attending to how the height is changing, or it may involve the student imaging how the rate-of-change changes while imagining changes in the volume (Carlson et al., in press).

Imagine this bottle filling with water. Sketch a graph of the height of the water as a function of the amount of water that's in the bottle.



Investigations of the various ways that students might reason about dynamic situations have revealed that covariational reasoning is foundational for providing a meaningful discussion of the graphical representation of this and other dynamic events (Carlson et al., in press).

This is only one of various abilities that must be acquired to develop a well-connected understanding of the function concept. Others include the ability to fluently move from one function representation to another, including the ability to recognize and construct algebraic functions from written or verbal situations. Learning about functions also involves understanding the symbols and letters used to express function ideas. Many researchers have found that students have difficulty bringing meaning to symbols and lack flexibility in interpreting the meaning of variable in a variety of contextual situations (Jacobs, 2002). To complicate matters it appears that many (if not most) textbooks and teachers do little to address or promote students' ability to overcome these difficulties.

Another strand of my research has involved investigations of students' evolving problem solving abilities (Carlson, 1999; Carlson and Bloom, under review). Both the readings and my own research have informed me that the process by which students acquire their mathematical understandings is of paramount importance to their emerging mathematical dispositions and reasoning abilities. This knowledge has led to ongoing efforts to engage my students in ways that promote individual growth in various aspects of their evolving problem solving behaviors.

The Interplay Between Research and Practice

I have shared some of my personal struggles to improve my teaching. I have also offered a sampling of the practical knowledge that I have acquired from reading the literature and conducting research, and have directed you to specific articles that may help you to better understand the processes involved in understanding the concept of function from the perspective of the student. As I have attempted to illustrate in my discussions, improving ones teaching involves more than acquiring information about knowing and learning specific mathematical topics. It is an individual journey involving cycles of acquiring new knowledge, incorporating that knowledge into ones instruction and gathering new information to assess one's instructional effectiveness.

Although many decisions of an instructor are acquired from informal knowledge from one's experiences and informal assessments, it seems reasonable that instructors could also benefit from acquiring research-based knowledge related to knowing and learning. Teachers are regularly confronted with many choices about their instruction and curriculum (i.e., the manner in which they interact with their students both during class and through the homework that is assigned). Each choice affects the quality of knowledge that their students acquire and the mathematical dispositions and abilities that they develop. It is therefore important that teachers make informed decisions, and come to recognize that these many decisions do have a profound effect on their students. As is the case in problem solving, decisions about solution paths are most effective if the problem solver is able to draw on well-connected conceptual, factual and procedural knowledge when considering optional solution paths (Carlson and Bloom, under review). It is also the case that good decision-making in teaching should be guided by accurate and well-connected content and pedagogical knowledge (Ball, 1991). Prior to making any changes in one's instruction, I recommend that teachers reflect on the beliefs and knowledge that are driving their many instructional decisions. Acquiring a researcher disposition will allow teachers to begin their personal journey to examine the accurateness of these beliefs and knowledge, relative to how their students are developing. It will also assist them in becoming more reflective about how their many decisions affect the thinking and reasoning of their students.

Concluding Remarks

My explorations have resulted in the evolution of a tight relationship between my research and my instruction. I cannot imagine how to separate research from practice. In my view, they have a symbiotic relationship in that they each feed and form the other. The diverse and complex nature in which teacher- and researcher-knowledge interact, warrants opportunities for sustained professional development that provides opportunity for the practitioner to travel a similar route as that of the researcher. I am not advocating that all teachers become serious educational researchers. Rather, I recommend that new platforms are needed to allow the practitioner to have exposure to the researcher's experience, without enduring the investments involved in the full research process. I suggest that various mathematics education communities devise new mechanisms to start teachers on their own personal journey of investigating and reflecting on the effect of their curricular and instructional decisions on the developing abilities, understandings and reasoning of their students. One suggestion might be to form and sustain learning communities among researchers and teachers with initial workshops that involve video analyses, application of research knowledge, researchers working with "lesson-study groups", etc. Investments to assist teachers in heading down this path may lead teachers toward a life-long journey of intellectual discourse about their instruction.

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RESEARCH and PRACTICE: Collective Learning and Argumentation

by Chris L. Rasmussen, Purdue University Calumet

I am convinced that failure is the only possible outcome for any approach to research in mathematics education in which researchers hand their results to curriculum developers and teachers, who are then expected to apply them in their practices. Education simply does not work that way, either in how other people can be influenced to try out 'ideas', or in how new ideas are located and developed in the first place (Mason, 1998, p. 375).

The above quote reflects the growing awareness that translating research to practice in a top-down manner is unlikely to achieve the goals of improving learning and teaching. This awareness stems in part from developments that highlight the critical role of social and cultural processes of learning and teaching mathematics and from the recognition that students are not lone, passive, computer-like learners but rather part of a complex, self-organizing system in a larger human-constructed system (Kelly & Lesh, 2000). Given such complexity and the limited impact of research prescriptions on teaching and learning, what useful products can we reasonably expect from mathematics education research? Certainly one significant contribution would be stimuli to other researchers and teachers to test out conjectures for themselves in their own context. By a conjecture I do not mean a formal hypothesis in an experimental design, but rather a conjecture is "a means to reconceptualize the ways in which to approach both the content and the pedagogy of a set of mathematical topics" (Confrey & Lachance, 2000, p. 235).

The purpose of this paper is to illustrate *an* approach to mathematics education research that seeks to contribute to our reconceptualization of learning and teaching as stimuli for other researchers and teachers.[1] As an example, I discuss results from an ongoing, multi-year developmental research project in differential equations. I first provide a brief overview of the methodological approach that we[2] are taking in this project and then I describe one of the lines of investigation carried out to date.[3] In particular, I discuss the notion and documentation of *collective learning*, an idea that complements the view of learning as an individual accomplishment. This research is pragmatically and theoretically significant for it offers a different lens for practitioners to think about the role, nature, and structure of classroom discourse and it expands on an analytic approach for researchers to document the learning of a classroom community.

Classroom Teaching Experiments

No one methodological approach is appropriate for all research interests. As developed by Gravemeijer (1994) and Cobb (2000), a classroom teaching experiment is one of several appropriate methodologies for concerns that center on the teaching-learning process, with particular attention to psychological and sociological processes. (Other methodologies are better suited for concerns that center on the historical or sociopolitical context of reform, for example.) Classroom teaching experiments last anywhere from several weeks to an entire school year and typically involves three or more project team members, including the classroom instructor. Members of the project team attend every class session and participate in daily analyses of classroom events where they engage in an iterative process of analyzing students' learning while simultaneously developing, modifying, and refining conjectures regarding the possible paths that students' learning might take and means to support that learning. Thus, unlike research designs where research and practice are viewed as two separate endeavors, in a classroom teaching experiment research and practice are inseparable activities.

Data collected typically include daily classroom videorecordings, fieldnotes, videorecorded individual student interviews, copies of all student work, audiorecordings of project meetings, and the instructor's daily reflective journal. The daily, iterative analysis of student understanding and reasoning, which tends to be pragmatic in nature, is complemented by more theoretically driven retrospective analyses of all data sources.

In order to plan for and make sense of the complexity of classroom events, we use the interpretive framework developed by Cobb and Yackel (1995). Given that learning is both a collective and individual accomplishment, this framework offers a way to conceptualize and coordinate particular social constructs (social norms, sociomathematical norms, and classroom mathematical practices) with their psychological counterparts (general beliefs about learning and teaching, specifically mathematical beliefs and values, and student's mathematical conceptions). In this report I focus on the social construct of a classroom mathematical practice. But what exactly is a classroom mathematical practice?

Classroom Mathematical Practices

Often in mathematics we sharpen our understanding of a particular idea by discussing what it is, how it is related to other ideas, and what it isn't. In keeping with this spirit, a classroom mathematical practice is a way to think about the mathematical learning of a classroom community as a whole. This may seems like an odd notion, so let me offer an analogy. There is a couple I know that I characterize as argumentative, but they do not share (from my perspective) the exact same argumentative characteristics as individuals. When these two people are together, it is normative for them to be given to controversy, but as individuals neither one of them is particularly disputatious. Thus, it may at times make sense to describe a collection or unit in ways that may or may not pertain in the exact same way to every member of that collection. I think that this is particularly true in teaching. In my own teaching and instructional design efforts, I tend to think in terms of the mathematical progress of the class as a whole while recognizing the existence of individual differences.

More formally, classroom mathematical practices entail taken-as-shared purposes, ways of reasoning, symbolizing, and arguing that are established while discussing specific mathematical ideas. The term taken-as-shared is used, rather than the term shared, to indicate that this is a researcher construct about the group and not a statement about every individual (Cobb, Stephan, McClain, & Gravemeijer, 2001). Finally, collective learning does not exist in isolation from individual learning and vice versa. To clarify, as students give explanations they deem viable, they are not only participating in the taken-as-shared normative understanding of ways of reasoning, they are also contributing to the ongoing negotiation of what is taken as normative. Thus, their individual mathematical conceptions develop concomitantly with the classroom mathematical practices. In the same way, student's individual mathematical conceptions, a psychological construct, and the classroom mathematical practices, the sociological correlate, are seen as mutually constitutive.

In order to analytically determine when a practice, from the researcher's perspective, has become established within the classroom community, we adapted Toulmin's (1969) model of argumentation. Figure 1 illustrates that for Toulmin, an argument consists of at least three parts, called the core of an argument: the data, claim, and warrant.

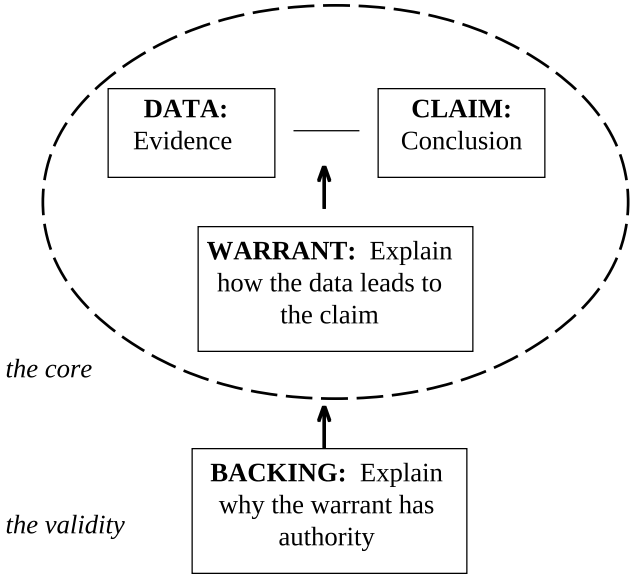


Figure 1. Toulmin's model of argumentation

In any argumentation the speaker makes a claim and if challenged, can present evidence or data to support that claim. The data typically consist of facts that lead to the conclusion that is made. Even so, a listener may not understand what the particular data presented has to do with the conclusion that was drawn. In fact, the listener may challenge the presenter to clarify why the data lead to the conclusion. When this type of challenge is made and a presenter clarifies the role of the data in making his/her claim, the presenter is providing a *warrant*. Another type of challenge can be made to an argument. Perhaps the listener understands why the data supports the conclusion but does not agree with the content of the warrant used. In other words, the authority of the warrant can be challenged and the presenter must provide a backing to justify why the warrant, and therefore the core of the argument, is valid.

We contend that mathematical practices become taken-as-shared when either 1) warrants or backings for an argumentation no longer appear in students' explanations and therefore stands as self-evident, or 2) the warrants and/or backings for one argumentation become the data or warrant for subsequent, more sophisticated arguments. When either of these instances occurs and no member of the community rejects the argumentation, a classroom mathematical practice has become established. This approach builds on Stephan, Bower, Cobb, and Gravemeijer's (accepted) and Yackel's (1997) methodological approach of analyzing argumentation in order to develop criteria for claiming that a community of learners' ways of reasoning (i.e., classroom mathematical practices) become taken as shared.

Discussion

Advances in technology and mathematicians' evolving interests in dynamical systems are currently prompting changes to the first course in differential equations. Traditional approaches in differential equations emphasize analytic techniques for finding closed form expressions for solutions whereas current reform efforts include graphical and numerical approaches for analyzing and understanding the behavior of solutions. How these graphical and numerical approaches are or are not integrated, however, is likely to vary from class to class and from curriculum to curriculum. For example, in Rasmussen (2001) I document how students in one reform approach to differential equations learned graphical and numerical approaches in isolated and disconnected ways to other important ideas and methods in the course.

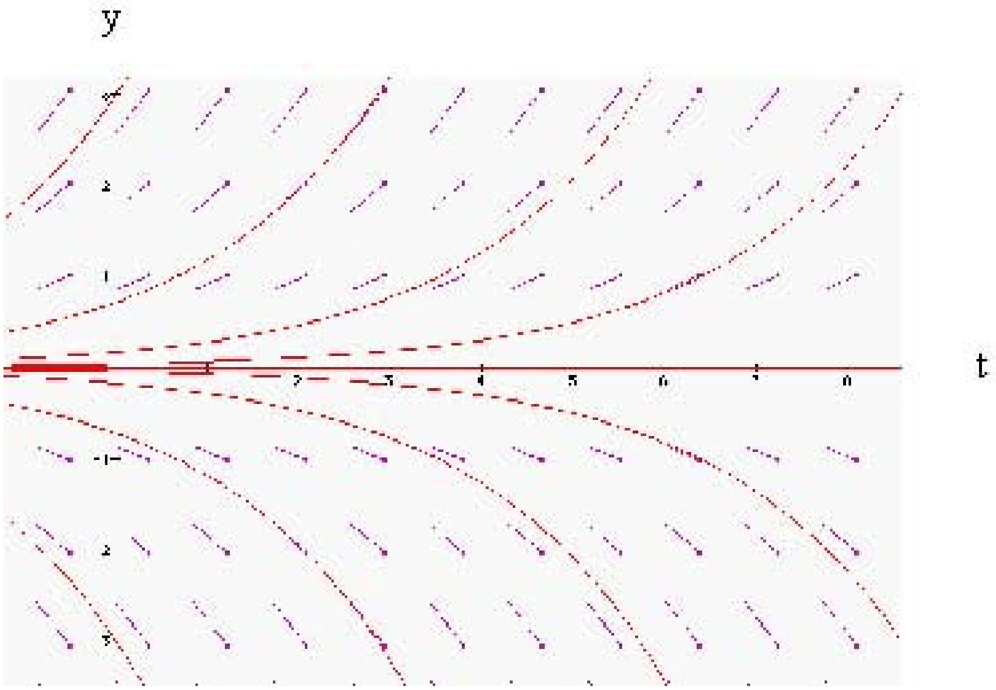


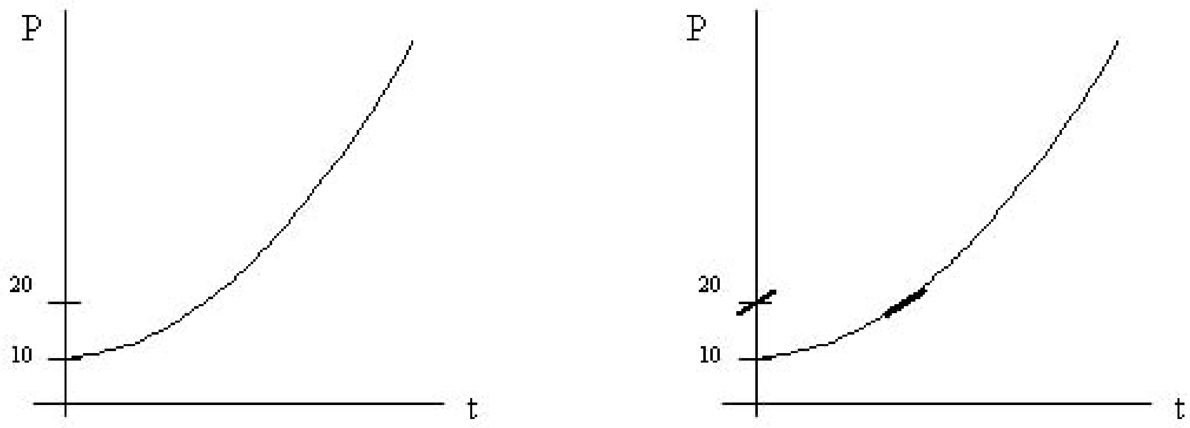
Figure 2. Slope field and representative solution graphs for dy/dt = 0.5y

One of our goals in a 15-week classroom teaching experiment was to explore the prospects and possibilities for student learning about important mathematical ideas and methods in ways that were highly integrated rather than compartmentalized. Rather than learning about graphical, numerical, and analytical techniques as separate methods which then need "connecting," we strove to create situations where these methods actually arose out of students' work in seamless ways. One of our retrospective analyses, therefore, investigated the emergent processes by which students created a structured collection of solution functions rather than on students' use of particular graphical or numerical methods for finding solutions.

Our analysis of whole class discussions across the first few weeks of the semester suggest students' creation of a structured space of solution functions can be understood in terms of what we call "the structuring the space of solution functions practice." This classroom mathematical practice entails the following two co-emergent aspects: taken-as-shared ways of reasoning about patterns in slope fields for autonomous differential equations (i.e., differential equations that do not depend on time) and taken-as-shared ways of reasoning about the implication of these patterns for graphs of solution functions. In this section we elaborate the processes by which this classroom mathematical practice emerged and became established in the classroom.

From an expert's perspective, a slope field for an autonomous differential equation has slope marks that are the same for any fixed value of the dependent quantity. As a result, the graphs of solution functions are shifts of each other along the axis of the independent quantity. For example, Figure 2 shows a slope field and several graphs of solution functions to the autonomous differential equation dy/dt = 0.5y. The reader already familiar with differential equations will most likely “see” the invariance in the slope marks and the way in which graphs of solution functions are structured.

In contrast to conventional approaches where such structuring and organizing might be explained and illustrated by the instructor in a linear progression, we found that the two aspects to the structuring the space of solution function practice actually co-emerged as a result of students’ interest in prediction. In the paragraphs that follow, we illustrate the co-emergence and constitution of the two aspects of this practice by analyzing students’ argumentations in whole class discussions on the 2nd, 4th, 9th, and 12th days of class.



(a) **(b)**

Figure 3. Growth of a rabbit population over time

On the 2nd day of class students made predictions about the growth of a rabbit population over time under the assumptions that there are no predators, resources are unlimited, and the rabbits reproduce continuously (no differential equation was provided). After some discussion students agreed that if there were initially 10 rabbits (where 10 is scaled for say, 10,000) then a graph of the number of rabbits versus time would look like that shown in Figure 3a. The instructor then asked the class, “What if the starting data were 20 rabbits?”

[1] Arthur: It would go up quicker, it would increase quicker.

[2] Instr: It would increase quicker? Say a bit more about that.

[3] Arthur: There’s more rabbits and there would be more babies. It’s like further up the 10 curve.

[4] Randy: Just look at the graph you already have. When you’re at 20 it’s already a more steep increase.

[5] Arthur: Yeah, it would be like the same thing.

As a pedagogical tool, the instructor recorded students’ reasoning with slope mark notation as shown in Figure 3b. These markings are crucial here as they are introduced as a way to notate students’ reasoning and are meant to promote students’ re-invention of the idea of a slope field rather than beginning with a complete slope field constructed by the instructor.

In this episode, Arthur and Randy concurrently initiated reasoning about patterns in rate of change and patterns in graphs of solution functions. First, Arthur claims that [the slope at initial condition 20] would increase quicker. In response to the instructor’s prompt to be more specific, Arthur provides some data to back up his claim. He elaborates that there would be more rabbits, hence more babies at a starting point of 20, and then states that “it’s further up the 10 curve.” Arthur’s last statement, “further up the 10 curve" is data that supports the claim that the slope increases quicker. However, one might ask what "it's further up the 10 curve" has to do with the claim being made. An elaboration of that issue would constitute a warrant that explains what Arthur's data has to do with the claim he made. The teacher, in this instance, did not need to push Arthur to articulate the warrant for this argument because Randy clarifies the data without prompting in line 4. More specifically, Randy explains that one can look at the graph already present (the 10 curve) and see a steeper increase at P=20 on that curve. Implicitly he is saying that the slope at P=20 on the 10 curve is similar to the slope at P=20 on the 20 curve. Arthur further elaborates the warrant by explaining that the slopes are the same (line 5).

Using Toulmin's model of argumentation, the argument that emerges in this instance can be diagrammed as follows:

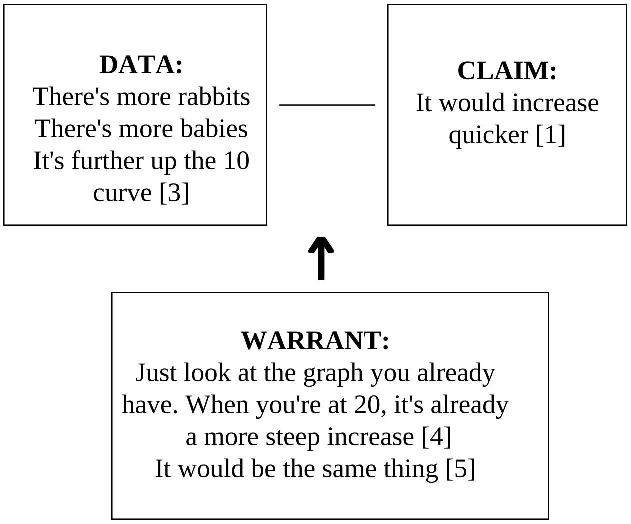


Figure 4. Structure of Arthur's and Randy's argument

Although no backing was present in this argumentation, one might ask why the slopes at P=20 should be the same for two different starting points. In other words, the legitimacy of the warrant can be called into question. The backing in this case might consist of the fact that these graphs describe autonomous situations. Although this type of backing did not emerge in this episode, it was brought in during subsequent discussions. We claim that this episode illustrates the initiation of the structuring the space of solution functions practice. As we will see, this math practice becomes established during subsequent episodes in which students continue to reason about the relation between slopes at particular populations at various times.

During a whole class discussion on day 4, students discussed their reasoning on a task where they were given a partially filled out slope field for the differential equation dP/dt = 3P(1 – P/100) and asked to investigate graphs of solution functions corresponding to different starting scenarios. In support of their ideas, students generated argumentations that suggested that the emergence of patterns in slope fields and the implication of these patterns for graphs of solution functions emerged concurrently. For example, Joaquin argues for patterns in the slope field as follows:

When we started at 10 rabbits (points to population of 10 at time zero on his drawing), and we got to, say the three years or whatever that it went by and we finally got to 30 rabbits (draws in slope mark at population 30 on the solution corresponding to the initial condition of 10), even though we started off with 30 rabbits over here (gestures to the slope at initial condition 30), it had the same slope as that 10 did at time like two years.

Joaquin's explanation is similar to the previous contributions on the second day in that he concludes that one can merely take the slope of the tangent line at P=30 on the 10 curve and indicate the same slope at P=30 for t=0. The support for this is that the slopes will be the same. At this point in the classroom such support was beyond justification; no one challenged the fact that the slopes were the same. Building on Joaquin's reasoning, Arthur contributes to the co-emergence of both aspects of the structuring the space of the solution functions practice by arguing that since the slopes are the same for a set value of the dependent variable, all graphs of solution functions are shifts of each other along the t-axis. Arthur explains:

Another thing we found out that like since all the graphs, slopes are the same, it's just like you're sliding this whole graph over one. Like going over here, towards 15 (gestures to grab the curve for the initial population just above 0) it's like the exact same thing, if you slide it over one more time you get the 30 graph, another time, you get the 45, so if you know the graph, you can kinda predict what happened in the past, a little bit before your time zero cause the graph is the same for all of them, you just pop it back for whatever your time interval was between the different 15 and 30 populations.

In his argumentation, Arthur claims that one can shift not merely individual slopes from one curve to another, but rather the solution function itself. Using Toulmin's model, the data for this argument is that all the slopes are the same. The collective argumentation has evolved from copying individual slopes to shifting the entire solution function itself. In other words, shifting slopes point by point, since the slopes are the same at those points, has dropped out of the argumentations and has been replaced by shifting entire solution curves. Methodologically speaking, we can see that the warrant from the previous argumentations [the slopes are the same] has shifted to serve as data for a more complex argument [shifting functions, not just slopes]. This gives further credence to our claims that shifting slopes along the t-axis was taken-as-shared in this mathematical practice.

During whole class discussions on day 9 and day 12, students give arguments where the two aspects of the structuring the space of solution functions practice are used as evidence for their conclusions. In terms of argumentation, there is a shift in the role played by the statements involving these ideas. This shift in role provides empirical support for the claim that the structuring the space of solution functions practice was taken-as-shared. For example, on day 9 the students debate whether a certain graph of an exact solution function to a particular differential equation is “steeper” than the graph of a certain Euler method approximation. In arguing their positions, students use slope field patterns and implications of these patterns as support for their conclusions. On day 12 students discuss whether the graphs of two solution functions to a Newton’s Law of cooling differential equation will ever touch each other. For example, Paul argues they would not as follows:

I kinda thought of it like this, well we know from some of the stuff that we’ve already done, those two graphs, same graphs just shifted, shifted along the horizontal by however many time increments. So I guess you could kinda think of it as like they’re parallel at every point. Parallel or they have a parallel tangent at like every point because those slope lines are the same across whatever point you’re at along the y, I guess. So if the graphs are just shifted over from each other, why would they ever touch? If they are indeed the same graph.

In Paul’s argumentation, he uses the idea that graphs of solution functions are shifts of each other to draw a new mathematical conclusion (that they never touch). This is significant in that he uses claims that were debated previously (“we know from stuff that we’ve already done”) to draw new conclusions. Further, no one in class rejected his explanation and during subsequent class periods, this type of reasoning was beyond justification. This fits our evidentiary criteria for establishing the structuring the space of solution functions practice. This mathematical practice arises from “re-inventing” the slope field and its associated meaning rather than decoding the meaning of a fully formed slope field from the beginning.

Concluding Remarks

The title of this paper, Research and Practice: Collective Learning and Argumentation, highlights the dual purposes of this paper. On the one hand, I argued that the classroom teaching experiment methodology is an example of an approach where research and practice are inseparable activities. The interested reader should consult the Handbook of Research Design in Mathematics and Science Education (Kelly & Lesh, 2000) for other examples. On the other hand, I illustrated how one might view learning as a collective accomplishment and how argumentation can be used to document this learning. When we are in a classroom situation and students develop arguments to support or refute ideas, there are different ways that an instructor and/or a researcher can view the learning that is occurring. From my own personal experience, it is often untenable for me to keep track of the diverse reasoning of 30 individual students, both in my instructional planning and classroom practice. The idea of a classroom mathematical practice with its attention to argumentation is one way to think about collective learning in a classroom. In terms of my own teaching and work with other instructors, the constructs of warrants and backings continue to be useful ways to understand and interpret classroom discourse as we strive to create learning environments where students routinely explain their thinking and attempt to make sense of others’ reasoning.

The details of this paper also suggest some novel ideas about how students’ learning in differential equations can proceed. In particular, when students are engaged in instruction that supports re-inventing conventional representations, slope fields and graphs of solution functions can and do emerge from their mathematical activity. Further, this research indicates that reasoning about patterns in slope fields and the implications of these patterns for graphs of solution functions emerges simultaneously in students’ argumentations. Theoretically, our research furthers methodological approaches that analyze communal aspects of learning. In particular, we have used Toulmin’s model of argumentation to analyze the emergence of classroom mathematical practices in a differential equations context. Additionally, we have identified a new criterion for determining when mathematical practices are taken-as-shared, namely that the supporting evidence in an argumentation shifts roles [from warrant and backings to data and warrants] in subsequent, more sophisticated arguments. These practical and theoretical findings add to the growing research base in both the teaching and learning of differential equations and methodological aspects of documenting learning in social context.

At the beginning of this paper I supported the stance that a significant product from mathematics education research would be stimuli that offer other researchers and teachers a means to reconceptualize their teaching of a particular mathematical topic. The extent to which others will find the ideas in this paper useful for their own research and/or practice is something that readers will have to judge for themselves.

Footnotes

[1] The approach described in this report is by no means the only approach. For example, in a very different methodological approach, Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000) conducted a large scale comparison study on the effects of the Core Plus Mathematics Project curriculum and more conventional curricula on growth of student understanding, skill, and problem solving ability in algebra. In addition to shedding light on overall achievement patterns, this research invites readers to rethink what algebraic reasoning is and what understandings and skills we value. From my perspective, it is not whether research employs quantitative or qualitative methods, but rather the stance one takes regarding the situatedness of learning and teaching.

[2] Research project team members who have participated in this project at different times include Erna Yackel, Michelle Stephan, and Karen King.

[3] Portions of this analysis were reported in Rasmussen and Stephan (2001).

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