Math 215
Practice Exam

1. Prove by induction that the sum of the squares of the odd integers satisfies:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

for all $n \geq 2$.

2. Prove or disprove the following statements:
(a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x^2 - y = 2$.
(b) $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $x^2 - y = 2$.
(c) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x^2 - y = 2$.

3. There are 30 people in a class, and they need to choose 3 officers: a president, vice president, and secretary. The positions must be held by 3 different people.
(a) If the entire class runs, how many possible outcomes are there to the election of the 3 officers?
(b) If the set of 3 officers is called the Executive Board (where we forget the individual titles), how many possible Executive Boards can be formed from the class?
(c) Suppose Laura is in the class, and she must be on the Executive Board (as either president, vice president, or secretary). How many possible outcomes are there for the election of the 3 officers?
(d) Suppose Laura is in the class, and she is not allowed to be president, but she can hold any other office (or no office at all). How many possible outcomes are there for the election of the 3 officers?

4. Let A, B, and C be sets. Prove that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$. Give an example of three sets where equality (rather than simply $\subseteq$) holds. Give an example of three sets where equality does *not* hold.

5. Let $X = \{0, e, \pi, 8\}$. What is $\mathcal{P}(X)$? Which subsets of X have cardinality 0? Which subsets have cardinality 3?

6. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$.
(a) Find a function $g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f(x) = x$ for all $x \in \mathbb{N}$.
(b) Does there exist a function $g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g(x) = x$ for all $x \in \mathbb{N}$?

7. Let A be the set of even integers greater than 17. Prove, directly from the definition, that A is denumerable.

8. Write the negations of the following statements. (Do not just place a "no" or a "not" in the sentence.)
(a) Every boy in town owns a green shirt and a pair of green shoes.

(b) There is a boy in town who has either a deaf brother or a mute sister.