Math 215
Some practice problems and solutions

1. Prove by induction that the sum of the squares of the odd integers satisfies:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

for all $n \geq 2$.

Base case ($n = 2$): The left hand side is $1^2 + 3^2 = 10$, while the right hand side is $2(3)(5)/3 = 10$, so the formula holds. For the inductive step, assume the formula holds for $n = k$, where k is an integer ≥ 2. Then

$$\begin{aligned} 1^2 + 3^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= \frac{(2k+1)(k(2k-1) + 3(2k+1))}{3} \\ &= \frac{(2k+1)(2k^2+5k+3)}{3} \\ &= \frac{(2k+1)(k+1)(2k+3)}{3} \\ &= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3} \end{aligned}$$

so the formula holds for all $n \geq 2$.

2. Prove or disprove the following statements:
(a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x^2 - y = 2$.

True. Fix $x \in \mathbb{R}$. Let $y = x^2 - 2$. Then $x^2 - y = 2$.

(b) $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $x^2 - y = 2$.

False. Consider $y = -3$, so $y + 2 = -1$. There is no $x \in \mathbb{R}$ such that $x^2 = -1$, so we cannot solve the equation $x^2 - y = 2$.

(c) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x^2 - y = 2$.

False. For any $x \in \mathbb{R}$, take $y = x^2$. Then $x^2 - y = 0 \neq 2$.

3. There are 30 people in a class, and they need to choose 3 officers: a president, vice president, and secretary. The positions must be held by 3 different people.
(a) If the entire class runs, how many possible outcomes are there to the election of the 3 officers?

The number of outcomes is equal to the number of injective functions from the set of 3 officers to the set of 30 people. Thus, there are $30 \cdot 29 \cdot 28 = 24360$ possible outcomes to the election.

(b) If the set of 3 officers is called the Executive Board (where we forget the individual titles), how many possible Executive Boards can be formed from the class?

The number of possible Executive Boards is the number of subsets of 3 from the set of 30. Thus, there are

$$\binom{30}{3} = \frac{30 \cdot 29 \cdot 28}{3 \cdot 2 \cdot 1} = 4060$$

possibilities.

(c) Suppose Laura is in the class, and she must be on the Executive Board (as either president, vice president, or secretary). How many possible outcomes are there for the election of the 3 officers?

Laura must be on the Executive Board, and there are $\binom{29}{2}$ ways to choose the other members of the board. Then there are 3! ways to permute their officer titles, so the answer is

$$\binom{29}{2} \cdot 3 \cdot 2 = \frac{29 \cdot 28}{2} \cdot 6 = 2436.$$

(d) Suppose Laura is in the class, and she is not allowed to be president, but she can hold any other office (or no office at all). How many possible outcomes are there for the election of the 3 officers?

Here we use inclusion-exclusion. As computed above, there are 2436 outcomes with Laura on the Executive Board. There are exactly $29 \cdot 28 = 812$ outcomes with Laura as president (injective functions from the set of non-presidential offices to the group of 29), so there are $2436 - 812 = 1624$ outcomes with Laura on the Executive Board but not as president. On the other hand, there are $29 \cdot 28 \cdot 27 = 21924$ outcomes where Laura is not on the Executive Board. Therefore, there are $21924 + 1624 = 23548$ outcomes to the election banning Laura from the presidency.

4. Let A, B, and C be sets. Prove that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$. Give an example of three sets where equality (rather than simply $\subseteq$) holds. Give an example where equality does *not* hold.

Suppose $x \in A \setminus (B \setminus C)$. Then $x \in A$ but not in $B \setminus C$. If $x \in B$, then x must also be in C, so x is contained in $(A \setminus B) \cup C$. On the other hand, if $x \notin B$, then $x \in A \setminus B$, so $x \in (A \setminus B) \cup C$. This proves the inclusion.

Consider the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{3\}$. Then $B \setminus C = \{2, 4\}$ so $A \setminus (B \setminus C) = \{1, 3\}$. On the other hand $A \setminus B = \{1\}$, so $(A \setminus B) \cup C = \{1, 3\} = A \setminus (B \setminus C)$.

For the second example, take $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{4\}$. Then $B \setminus C = \{2, 3\}$ so $A \setminus (B \setminus C) = \{1\}$. On the other hand $A \setminus B = \{1\}$, so $(A \setminus B) \cup C = \{1, 4\} \neq \{1\} = A \setminus (B \setminus C)$.

5. Let $X = \{0, e, \pi, 8\}$. What is $\mathcal{P}(X)$? Which subsets of X have cardinality 0? Which subsets have cardinality 3?

The power set $\mathcal{P}(X)$ is the set $\{\emptyset, \{0\}, \{e\}, \{\pi\}, \{8\}, \{0, e\}, \{0, \pi\}, \{0, 8\}, \{e, \pi\}, \{e, 8\}, \{\pi, 8\}, \{0, e, \pi\}, \{0, e, 8\}, \{0, \pi, 8\}, \{e, \pi, 8\}, \{0, e, \pi, 8\}\}$. There is only one subset of cardinality 0, namely $\emptyset$. There are 4 subsets of cardinality 3, $\{0, e, \pi\}$, $\{0, e, 8\}$, $\{0, \pi, 8\}$, and $\{e, \pi, 8\}$.

6. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$.
(a) Find a function $g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f(x) = x$ for all $x \in \mathbb{N}$.

Consider the function g defined by

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Since the image of f is the set of even natural numbers, we see that $g(f(n)) = g(2n) = n$ for all $n \in \mathbb{N}$.

(b) Does there exist a function $g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g(x) = x$ for all $x \in \mathbb{N}$?

No. Indeed, suppose such a g exists. Then $f(g(1)) = 2g(1)$ is even, which contradicts $f(g(1)) = 1$.

7. Let A be the set of even integers larger than 17. Prove, directly from the definition, that A is denumerable.

Define $f : \mathbb{N} \to A$ by $f(n) = 2(n+8)$. The function f is injective, since $2(n_1+8) = 2(n_2+8) \implies 2n_1 = 2n_2 \implies n_1 = n_2$. To see surjectivity, let $a \in A$. Then a is even so $a = 2k$ for some integer k. As $a \geq 18$, we see that $k \geq 9$ so $k = n+8$ for some integer $n \in N$. Therefore, $a = f(n)$. Consequently, f is bijective, and therefore A is denumerable.

8. Write the negations of the following statements. (Do not just place a "no" or a "not" in the sentence.)
(a) Every boy in town owns a green shirt and a pair of green shoes.

There is a boy in town who either has no green shirt or at most one green shoe.

(b) There is a boy in town who has either a deaf brother or a mute sister.

The brother of any boy in town can hear, and the sister of any boy in town can speak.